A Unified Geometric Approach to Graph Separators

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Abstract

We propose a class of graphs called k-overlap graphs. Special cases of k-overlap graphs include planar graphs, k-nearest neighbor graphs, and earlier classes of graphs associated with finite element methods. We prove a separator bound of $O(k^{1/d}N^{(d-1)/d})$ for k-overlap graphs embedded in d dimensions. This result unifies several earlier separator results including Lipton and Tarjan's 1979 result for planar graphs. All our arguments are based on geometric properties of embeddings. Our separator bounds come with randomized linear-time and randomized NC algorithms. Moreover, our bounds are the best possible up to the leading term.

1 Introduction

Graph partitioning is a fundamental problem in Computer Science that has many important applications including Numerical Analysis (Lipton, Rose and Tarjan;[LRT79]), VLSI design (Ullman;[Ull84]) and even complexity theory (Paterson;[Pat72]).

Recently several groups of authors (Vavasis;[Vav91], Miller and Thurston;[MT90b], Miller and Vavasis;[MV91]) have proposed classes of graphs that can be embedded in d dimensions and that have $O(N^{(d-1)/d})$ *separators*. "$O(N^{(d-1)/d})$ separators" means that for an N-node graph in the class, there exists a subset of nodes of size $O(N^{(d-1)/d})$ whose removal disconnects the graphs into two roughly equally-sized components. See below for the formal definition. For the applications mentioned in the last paragraph, $d = 2$ and $d = 3$ are the interesting cases, in which case the bounds are $O(N^{1/2})$ and $O(N^{2/3})$ respectively.

All of these earlier classes of graphs have the disadvantage that, when specialized to two dimensions, they apparently do not contain all planar graphs. This is a serious drawback because the earliest and best-known separator result is Lipton and Tarjan's, [LT79], theorem that all planar graphs have $O(N^{1/2})$ separators. Moreover, these classes contained only graphs with bounded degree.

In this paper we propose a new class of graphs, *overlap graphs*. This class enjoys the following properties:

1. In two dimensions, planar graphs are special cases of overlap graphs.

2. In d dimensions for $d \geq 2$, any finite subgraph of the infinite d-dimensional grid graph is an overlap graph.

3. The overlap graphs in d-dimensions have $O(N^{(d-1)/d})$ separators.

To our knowledge, this is the first time that a class of graphs has been proposed with these three very natural properties. In addition, as we argue below, overlap graphs include Miller and Vavasis's density graphs as a

*This work was supported in part by National Science Foundation grant DCR-8713489.

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‡Supported by an NSF Presidential Young Investigator award.

special case. The proof that planar graphs are special cases of overlap graphs relies on recent deep theorems by Andreev and Thurston [And70a, And70b, Thu88] characterizing all planar graphs in a novel geometric fashion.

Our proof techniques are novel and have the potential of being applied to other problems. Our bounds for density graphs are better than Miller and Vavasis's bounds and in fact achieve matching lower bounds except for low-order terms. In order to achieve tight bounds, we use arguments that take advantage of slight differences between the various p-norms when applied to high-dimensional vectors, a technique that appears to be new and is interesting on its own right.

Finally, we will argue that a generalization of overlap graphs, called *k-overlap graphs*, include as a special case k-nearest-neighbor graphs–an important class of graphs from computational geometry, statistic analysis, and image understanding. Moreover, our separator bound is also optimal in terms of k.

2 Definitions

The notion of a separator introduced in the last section has been well-known since 1979. The following definition formalizes this idea and introduces the notation that will be used for the remainder of the paper.

Definition 2.1 (Separators) *A subset of vertices C of a graph G with n vertices is an $f(n)$–***separator** *that* δ**-splits** *if $|C| \leq f(n)$ and the vertices of $G - C$ can be partitioned into two sets A and B such that there are no edges from A to B, $|A|, |B| \leq \delta n$, where f is a function and $0 < \delta < 1$.*

Separator results for families of graphs closed under the subgraph operation immediately lead to divide-and-conquer recursive algorithms for a variety of applications. In general, the efficiency of such algorithms depends on a δ bounded away from 1 and $f(n)$ a slowly-growing function.

Two of the most well-known families of graphs which have small separators are trees and planar graphs. It is known that a tree has a single vertex separator that 2/3–splits. Lipton and Tarjan [LT79] proved that any planar graph has a $\sqrt{8n}$–separators that 2/3–splits. They also give a linear time algorithm for finding such a separator. Many interesting extensions of this work have been made [Dji82, Mil86, Gaz86, GM87] and separator theorems had also been obtained also for graphs with bounded genus [GHT82, HM86]. Very recently, Alon, Seymour and Thomas [AST90] proved the following interesting separator theorem: all graphs with no minor isomorphic to the h–clique have an $O(h^{3/2}\sqrt{n})$–separator. Many applications of separator theorems have been given for VLSI layout [Lei83a, Lei83b], finite element method and numerical analysis [LRT79, PR85].

The development of computational geometry and numerical analysis calls for deeper understanding of separator properties for graphs embedded in fixed dimensional space, especially in 2–space and 3–space. Although, the planar separator theorem is applicable to many interesting families of graphs embedded in 2-space, we shall show that there are some natural graphs in 2–space, e.g., k-nearest neighborhood graphs, which are neither planar, nor with bounded genus, nor with bounded minor. In general, none of the above separator theorems are useful for graphs in 3–space.

Example 2.2 *Let G be graph formed by a $2 \times \sqrt{n} \times \sqrt{n}$ grids in 3–space (see Figure 1). Clearly, G has a $2\sqrt{n}$–separator. However, G is a graph with genus $\Omega(\sqrt{n})$. It also has a minor isomorphic to $\sqrt{n}$–clique.*

Figure 2 shows an 8–nearest neighborhood graph which has an $O(\sqrt{n})$–separator. But it has genus $\Omega(\sqrt{n})$ and a minor isomorphic to the $\sqrt{n}$–clique.

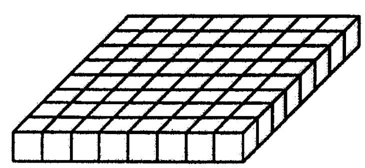

Figure 1: A 3–dimensional graph with large genus and large minor but small separator

The more recent papers mentioned in Section 1 on d-dimensional separators attempt to use geometric information in the existence proofs and separators, as opposed to the combinatorial approaches mentioned in the last few paragraphs. This paper also takes the geometric point of view. However, it introduces many new techniques.

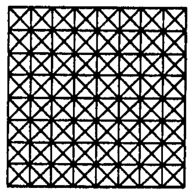

Figure 2: An example of 8–nearest neighborhood graph in 2–space

The family of graphs is defined based on a notion of a *neighborhood system*, which induces a special case of an overlap graph.

Definition 2.3 (Neighborhood System) *Let $P = \{p_1, \ldots, p_n\}$ be points in $\mathbb{R}^d$. A* **k-neighborhood system** *for P is a set, $\{B_1, \ldots, B_n\}$, of closed balls such that (1) B_i is centered at p_i and (2) For each i the interior of B_i contains at most k points from P.*

In the full paper we will discuss more general neighborhood systems.

The *intersection graph* induced by this neighborhood system is the undirected graph with one node for each ball, and an edge when two balls intersect.

We now define the main class of graphs under consideration, α-overlap graphs. For this definition, we introduce the following notation. If B is a ball of radius r in $\mathbb{R}^d$, then $\alpha \cdot B$ denotes the ball with the same center as B but radius αr.

Definition 2.4 (Overlap Graph) *Let $\alpha > 0$ and let $\{B_1, \ldots, B_n\}$ be a k-neighborhood system for $P = \{p_1, \ldots, p_n\}$. The* **(α, k)-overlap graph** *for the k-neighborhood system $\{B_1, \ldots, B_n\}$ is the undirected graph with vertices $V = \{1, \ldots, n\}$ and edges*

$$E = \{(i,j) : (B_i \cap (\alpha \cdot B_j) \neq \emptyset) \text{ and } ((\alpha \cdot B_i) \cap B_j \neq \emptyset)\}.$$

For simplicity, we call a $(1, k)$-overlap graph a *k-intersection graph*. In the case that $\alpha = 1$ and $k = 1$, and no two balls in the neighborhood system have a common point in their interior, we have the family of graphs known as *sphere-packings*; this interesting class of graphs will be discussed later.

Note that given a set P of n points in $\mathbb{R}^d$, we can uniquely define for each point $p \in P$ the largest ball centered at p whose interior contains at most k points of P (provided $n > k$). These balls immediately lead to an instance of a k-neighborhood system, which is called *the k-neighborhood system of P*. The intersection graph and overlap graphs defined in this k-neighborhood system is called respectively the intersection graph and overlap graphs of P.

The following is the main theorem for our report.

Theorem 2.5 (Main) *Let G be an (α, k)–overlap graph for some fixed d. Then G has an $O(\alpha \cdot k^{1/d} \cdot n^{(d-1)/d} + q(\alpha, k, d))$-separator that $\frac{d+1}{d+2}$-splits. Further, such a separator that $\frac{d+1+\epsilon}{d+2}$-splits can be computed in random constant time, using linear number of processors, for any $1/n^{1/2d} < \epsilon < 1$.*

The function $q(\alpha, k, d)$ depends exponentially on d but is independent of n. Since the interesting cases are when $d = 2$ or $d = 3$ and when n is large, this term should be considered low order.

The remainder of the paper is organized as follows. Section 3 presents applications of overlap graphs including a geometric proof of the planar separator theorem. Matching lower bound on the size of separator for overlap graphs is also given. Section 4 presents some important geometric lemmas which are used in the proof of the main separator theorem in Section 5. Section 6 extends the main result to non-Euclidean space. In Section 7 we discuss applications to Computational Geometery. Section 8 lists some open questions.

3 Why Overlap Graphs

In this section, we show that the class of overlap graphs includes many natural classes of graphs as special cases. In particular, it contains the set of all planar graphs, density graphs, and k-nearest neighborhood graphs.

3.1 A Geometric Proof of the Planar Separator Theorem

Let $\{B_1, \ldots, B_n\}$ be a sphere packing, i.e., no two balls share a common points in their interior. The intersection graph of a sphere packing is called a *sphere packing graph*. It is not hard to see that each sphere packing graph in 2-space is a planar graph. But the following remarkable result of Andreev and Thurston indicates that the converse implication also holds.

Theorem 3.1 (Andreev and Thurston) *Each triangulated planar graph G is isomorphic to a sphere packing graph in 2-space.*

As a consequence of our main theorem,

Corollary 3.2 *Every planar graph has an $O(\sqrt{n})$–separator.*

3.2 Separator for k-nearest Neighborhood Graphs

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbf{R}^d$. For each $p_i \in P$ and $k \in N$, let $N_k(p_i)$ be the set of k-nearest neighbors of p_i in P (ties are broken arbitrarily).

Definition 3.3 *A* **k-nearest neighborhood graph** *of $P = \{p_1, \ldots, p_n\}$ in $\mathbf{R}^d$ is a graph with vertices $V = \{1, ..., n\}$, and edges*

$$E = \{(i,j) : p_i \in N_k(p_j) \text{ or } p_j \in N_k(p_i)\}.$$

Immediately from the the definitions, for any set of n points P in $\mathbf{R}^d$, the k-nearest neighborhood graph of P is a subgraph of a intersection graph of a k-neighborhood system. Consequently,

Corollary 3.4 *All k-nearest neighborhood graphs have an $O\left(k^{1/d} n^{\frac{d-1}{d}}\right)$-separator.*

3.3 Separator for Density Graphs

Definition 3.5 (Density Graphs) *Let G be an undirected graph and let π be an embedding of its nodes in $\mathbf{R}^d$. Then we say that π is an embedding of* **density** *α if the following inequality holds for all vertices v in G. Let u be the closest node to v. Let w be the farthest node from v that is connected to v by an edge. Then*

$$\frac{||\pi(w) - \pi(v)||}{||\pi(u) - \pi(v)||} \leq \alpha.$$

In general, G is a **density graph** *if there exist a π and $\alpha > 0$ such that π is an embedding of density α.*

Immediately from the definition, each α–density graphs is a subgraph of $(\alpha, 1)$-overlap graph. Therefore,

Corollary 3.6 *Let G be a density graph in $\mathbf{R}^d$. Then G has an $O\left(\alpha \cdot n^{\frac{d-1}{d}}\right)$-separator.*

It is worthwhile to compare our separator result of density graphs with the one of Miller and Vavasis. In their paper [MV91], Miller and Vavasis showed that each density graphs has an $O\left(\alpha^{d(d-1)} n^{\frac{d-1}{d}}\right)$-separator.

Hence, our result greatly improves the one of Miller and Vavasis in the term of α and our bound is optimal in terms of α. This answers an open problem in [MV91] in the affirmative.

3.4 Lower Bound

We can shown that the separator bound of the main theorem is tight using the following example. Let P be the set of all points of the $m \times m \times \cdots \times m$ regular grid in $\mathbf{R}^d$ and let G be the (α, k)-overlap graph for the points P. Using the methods described in Leighton [Lei83a] we get a lower bound of $\Omega(\alpha k^{1/d} m^{d-1})$.

4 Some Geometric Lemmas

In this section, we state a set of basic geometric lemmas. Their proofs can be found in the full paper.

Define the *kissing number* τ_d to be the maximum number of nonoverlapping unit balls in $\mathbf{R}^d$ that can be arranged so that they all touch a central unit ball [CS88]. It is not hard to see that $\tau_1 = 2$ and $\tau_2 = 6$. The fact $\tau_3 = 12$ was formally proved in nineteenth century after a famous discussion between Isacc Newton and David Gregory. At that time Newton believed the answer was 12, while Gregory thought that 13 might be possible. It is somewhat surprising that we also know the answer in 8 and 24 dimensions ($\tau_8 = 240$ and $\tau_{24} = 196560$), but in no other dimensions above 3. In higher dimensions rather less is known. The best upper bound was given by Kabatiansky and Levenshtein, and a lower bound by Wyner respectively as following [CS88],

$$2^{0.2075..d(1+o(1))} \leq \tau_d \leq 2^{0.401d(1+o(1))}$$

For each positive real δ, let $A_d(\delta)$ be the maximum number of points that can be arranged on a unit sphere, such that the distance between each pair of points is at least δ. Note that $A_d(1) = \tau_d$.

Lemma 4.1 (Point Coverage) *Let $\{B_1, \ldots, B_n\}$ be a k-neighborhood system in d-space. Then for all $p \in \mathbf{R}^d$,*

$$|\{i : p \in B_i\}| \leq \tau_d k.$$

The following lemma can be proved in a similar manner to Lemma 4.1.

Lemma 4.2 (Ball Intersection) *Let $\{B_1, \ldots, B_n\}$ be a k-neighborhood system in $\mathbf{R}^d$. Let p_i be the center of B_i ($1 \leq i \leq n$). Then for all balls $B \subset \mathbf{R}^d$ (say with center p and radius r), for each positive real $\alpha \geq 1$,*

$$|\{i : B_i \cap B \neq \emptyset \text{ and } p_i \in \mathbf{R}^d - \alpha \cdot B\}| \leq A_d\left(\frac{\alpha - 1}{\alpha}\right) k.$$

Lemma 4.3 *Let $\ldots, m_{-1}, m_0, m_1, m_2, \ldots$ be a doubly infinite sequence of nonnegative numbers such that each m_i is bounded above by θ and such that at most a finite number of m_i's are nonzero. Let $d \geq 2$ be an integer. Then*

$$\left(\sum_{k=-\infty}^{\infty} m_k 2^{-k(d-1)}\right)^{d/(d-1)} \leq c_d \theta^{1/(d-1)} \sum_{k=-\infty}^{\infty} m_k 2^{-kd}$$

where c_d is a positive number depending on d.

5 A Geometric Construction of the Main Theorem

By applying Theorem 2.5 with the choice that $\alpha = 1$, we obtain,

Theorem 5.1 *If $\{B_1, \ldots, B_n\}$ is k-neighborhood system in $\mathbf{R}^d$, and G is the intersection graph of $\{B_1, \ldots, B_n\}$, then G has an $O(k^{1/d} n^{\frac{d-1}{d}})$-separator that $\frac{d+1}{d+2}$-splits G.*

For simplicity we shall focus on the proof of Theorem 5.1. Observe that these graphs may still have unbounded degree and are interesting on their own. Also, we shall show that these results are best possible in term of n, α, and k.

The basic idea to prove the Main Theorem is to first construct a real-valued function f based on the structure of the given graph G; then to show that there is a $(d-1)$-sphere S in $\mathbf{R}^d$ splits the vertices of G not on S into two sets, the interior and exterior of S, each of size at most $(\frac{d+1}{d+2})n$, such that the cost of S, denoted as $\text{Cost}_f(S)$, is bounded from above by $O\left(\alpha k^{1/d} n^{\frac{d-1}{d}}\right)$, where

$$\text{Cost}_f(S) = \int_{v \in S} (f(v))^{d-1} (dv)^{d-1};$$

Such a sphere S is called a *continuous separator* of G based on f. We then deduce a vertex separator of the underlying graph from the continuous one, such that the size of the vertex separator is linearly bounded by the cost of the continuous separator.

Notice, however, in order to deduce a vertex separator from the continuous counterpart, the continuous function f must be *faithful* in the sense that the cost of a continuous separator models faithfully the size of a vertex separator of the underlying graph. In other words, the continuous function f encodes faithfully some combinatorial properties related to separators of the underlying graph.

The above basic idea is taken from Miller and Thurston [MT90b], however, our specific construction is quite different and more sophisticated, and it contains many novel ideas. Our construction of the real-valued function is derived from the one used by Miller and Vavasis [MV91] for density graphs, but we shall show that our construction is more 'faithful' to the structure of the underlying graphs. Because of this, our results can be applied to much larger class of graphs. Moreover, when applied to density graphs, we obtain the best possible dependence on the density, improving on the result of Miller and Vavasis.

5.1 Computing a Continuous Separator

Let $f(x)$ be a real valued nonnegative function defined on $\mathbf{R}^d$ such that f^k is integrable for all $k = 1, 2, 3, \ldots$. Such an f is called a *cost function*. The total cost of the system is

$$\text{Total–Cost}(f) = \int_{v \in \mathbf{R}^d} (f(v))^d (dv)^d \leq \infty$$

Similarly, for all $(d-1)$–sphere S, the cost of S is

$$\text{Cost}_f(S) = \int_{v \in S} (f(v))^{d-1} (dv)^{d-1}$$

A $(d-1)$-sphere S is a *δ-splitting sphere* of a set P of n distinct points in $\mathbf{R}^d$ if S splits the points of P not on S into two sets, the interior and exterior of S, each of size at most δn.

Theorem 5.2 (Miller and Thurston) *If f is a cost function on $\mathbf{R}^d$ and P a set of n distinct points in $\mathbf{R}^d$ then there is a $(\frac{d+1}{d+2})$-splitting sphere S of P such that*

$$\text{Cost}_f(S) = O\left((\text{Total–Cost}(f))^{\frac{d-1}{d}}\right)$$

Further, using a result of Miller and Teng [MT90a] for approximating center points in fixed dimension, it can be shown that

Lemma 5.3 *If f is a cost function on $\mathbf{R}^d$ and P a set of n distinct points in $\mathbf{R}^d$ then a $(\frac{d+1+\epsilon}{d+2})$-splitting sphere S of P of cost*

$$\text{Cost}_f(S) = O\left((\text{Total–Cost}(f))^{\frac{d-1}{d}}\right)$$

can be computed in random constant time, using $O(n)$ processors, where $\frac{1}{n^{1/2d}} \le \epsilon \le 1$.

5.2 A Cost Function for Intersection Graphs

We now construct a cost function f for k-intersection graphs such that $\text{Total} - \text{Cost}(f) = O(k^{1/(d-1)}n)$. We shall show that our construction can be generalized for (α, k)-overlap graphs.

Let $\{B_1, \ldots, B_n\}$ be a k-neighborhood system of point P in $\mathbf{R}^d$. Let G be the intersection graph of $\{B_1, \ldots, B_n\}$. To define the cost function for G, we first define n functions, $f_1, \ldots, f_n$, where the function f_i is called the *local cost function* of B_i and is defined based on B_i, as follows: let r_i be the radius of B_i and let $\gamma_i = 2r_i$,

$$f_i(x) = \begin{cases} 1/\gamma_i & \text{if } ||x - p_i|| \le \gamma_i \\ 0 & \text{otherwise} \end{cases}$$

Intuitively, f_i sets up a cost on each $(d-1)$-sphere S such that the closer S is to p_i, the larger p_i contributes to the cost of S.

For each $(d-1)$-sphere S, let

$$M(S) = \{B_i \quad : \quad B_i \cap S \neq \emptyset\},$$

It is not hard to see (see Subsection 5.4) that if S is a sphere separator that δ-splits the k-intersection graph G then $M(S)$ is a vertex separator that also δ-splits G. What we shall show later (see Subsection 5.4) is that there is a constant c such that if S intersects B_i, then $\int_{v \in S}(f_i(v))^{d-1}(dv)^{d-1} \ge c$.

We say a cost function f is *faithful* if for all $(d-1)$-spheres S,

$$\int_{v \in S}(f(v))^{d-1}(dv)^{d-1} \ge \sum_{i=1}^{n}(\int_{v \in S}(f_i(v))^{d-1}(dv)^{d-1})$$

We can conclude from the above discussion that if a cost function f is faithful, then $|M(S)| \le \text{Cost}_f(S)$.

Thus, the cost function should defined to be the minimum function that is faithful. Let us start with some notation.

Let $a_1, \ldots, a_n$ be n nonnegative numbers. Define the *L_p^{th} norm* of $a_1, \ldots, a_n$, denoted as $L_p(a_1, \ldots, a_n)$, to be

$$L_p(a_1, \ldots, a_n) = \left(\sum_{i=1}^{n} |a|^p\right)^{1/p}, \quad \text{where } p > 0$$

The following lemma states the relationship between different norms.

Lemma 5.4 $L_p \le L_{p-1}$.

Proof: Folklore. □

The cost function[1] of the k-overlap graph G is then defined to be the L_{d-1}^{st} norm of $f_1, \ldots, f_n$, i.e.,

$$f(x) = L_{d-1}(f_1, \ldots, f_n) = \left(\sum_{i=1}^{n}(f_i(x))^{d-1}\right)^{1/(d-1)}$$

Now, to prove Theorem 5.1, we first show

Lemma 5.5 *For any set P of n points in $\mathbf{R}^d$, if $f_1, \ldots, f_n$ and f are defined as above, then*

$$\text{Total-Cost}(f) = O(k^{1/(d-1)}n).$$

Proof: Let V_d be the volume of a unit ball in $\mathbf{R}^d$. Clearly, $\int_{x \in \mathbf{R}^d}(f_i(x))^d(dx)^d = V_d$.

[1] Notice that in the construction of Miller and Vavasis for density graph, the cost function is defined to be the L_1 norm of the functions defined over each vertices.

Consequently, letting

$$g(x) = L_d(f_1, \ldots, f_n) = \left(\sum_{i=1}^{n} (f_i(x))^d\right)^{1/d},$$

we have

$$\int_{x \in \mathbf{R}^d} (g(x))^d (dx)^d = V_d n$$

Therefore, Lemma 5.5 follows immediately from the following lemma. □

Lemma 5.6 *For all $x \in \mathbf{R}^d$,*

$$(g(x))^d \leq (f(x))^d \leq c_d 2^d (6^d \tau_d k)^{1/(d-1)} \cdot (g(x))^d.$$

Proof: The first inequality follows immediately from the definitions of f and g and Lemma 5.4.

For the second inequality, we focus on a particular point $p \in \mathbf{R}^d$. Notice that if $g(p) = 0$, then, $f(p) = 0$ as well. The inequality follows.

Now, assume $g(p) > 0$.

Define

$$M_l = \left\{ i \in \{1, \ldots, n\} : 2^{-l} \leq f_i(p) < 2^{-l+1} \right\}$$

for all integers l.

Because that $\cup_{-\infty < l < \infty} M_l = \{i : f_i(p) \neq 0\}$ and M_l's are pairwise disjoint, each index i such that $f_i(p) \neq 0$ occurs in exactly one of M_l's.

Let $m_l = |M_l|$. We claim $m_l \leq 6^d \tau_d k$.

We now prove the claim.

For each $i \in M_l$, by the definition of M_l and f_i, $2^{l-1} \leq \gamma_i < 2^l$, recall γ_i is twice of the radius of B_i.

Let B be a ball centered at p with radius $2^l + 2^{l-1}$. Since $||p - p_i|| \leq \gamma_i$, it follows $B_i \subset B$.

Notice that for all $i \in M_l$, int B_i contains no more than k points from $\{p_j : j \in M_l\}$. By Lemma 4.1, no point from B is covered by more than $\tau_d k$ balls from $\{B_j : j \in M_l\}$. Therefore,

$$\tau_d k \cdot \mathrm{vol}(B) \geq \sum_{j \in M_l} \mathrm{vol}(B_j)$$

Let $V_d(r)$ be the volume of a ball in $\mathbf{R}_d$ of radius r. We have for all $j \in M_l$, $\mathrm{vol}(B_j) \geq V_d(2^{l-2})$. Consequently,

$$\tau_d k \cdot V_d(2^l + 2^{l-1}) \geq |M_l| V_d(2^{l-2}),$$

which implies $|M_l| \leq 6^d \tau_d k$, thus the claim.

Now, we have

$$\begin{aligned}
(f(p))^d &= \left(\sum_{l=-\infty}^{\infty} \sum_{i \in M_l} f_i(p)^{(d-1)}\right)^{d/(d-1)} \\
&\leq \left(\sum_{l=-\infty}^{\infty} m_l (2^{-l+1})^{d-1}\right)^{d/(d-1)} \\
&\leq 2^d \left(\sum_{l=-\infty}^{\infty} m_l (2^{-l})^{d-1}\right)^{d/(d-1)}
\end{aligned}$$

where $m_l \leq 6^d \tau_d k$.

Setting $\theta = 6^d \tau_d k$ and applying Lemma 4.3, we obtain

$$f(p)^d \leq c_d 2^d (6^d \tau_d k)^{1/(d-1)} \sum_{l=-\infty}^{\infty} m_l 2^{-ld}$$

This summation is a lower bound on $g(p)^d$ because for each $i \in M_l$, $f_i(p)^d \geq 2^{-ld}$. This completes the proof of the lemma. □

The following lemma follows directly form Lemma 5.5 and Theorem 5.2.

Lemma 5.7 *There exists a $(\frac{d+1}{d+2})$-splitting sphere S of P with $Cost_f(S) = O\left(6\xi k^{1/d} n^{\frac{d-1}{d}}\right)$, where $\xi = 2^{d-1} (\tau_d)^{1/d} V_d$.*

5.3 A Cost Function for (α, k)–overlap Graphs

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in $\mathbf{R}^d$ and G be the (α, k)–overlap graph defined by P. Let r_i be the radius of B_i. Let $\gamma_i = 2\alpha r_i$. We define a function f_i as

$$f_i(x) = \begin{cases} 1/\gamma_i & \text{if } x \in (2\alpha) \cdot B_i, \text{ i.e., } ||x - p_i|| \leq \gamma_i \\ 0 & \text{otherwise} \end{cases}$$

The cost function of the (α, k)–overlap graph G is then defined to be

$$f(x) = L_{(d-1)}(f_1, \ldots, f_n)$$

The following are two lemmas which can be proved similarly to Lemma 5.6 and 5.7, respectively.

Lemma 5.8 $(g(x))^d \leq (f(x))^d \leq c_d 2^d (6^d \alpha^d k)^{1/(d-1)} \cdot (g(x))^d$.

Lemma 5.9

$$\int_{x \in \mathbf{R}^d} (f(x))^d (dx)^d \leq \xi_d \alpha k^{1/d} n$$

Consequently,

Lemma 5.10 *There exists a $(\frac{d+1}{d+2})$-splitting sphere S of P with $Cost_f(S) = O\left(\xi \alpha k^{1/d} n^{\frac{d-1}{d}}\right)$.*

5.4 A Vertex Separator From A Continuous One

To complete the proof Theorem 5.1, we shall construct a vertex separator C of G from a continuous separator S obtained from Lemma 5.10. We then show that $|C| = O(\mathrm{Cost}_f(S))$.

The vertices in C are those points $q \in P$ such that its ball, B_q intersects S. If the edge $(p, q) \in E$ is such that p is interior to S and q is exterior then either B_q or B_p must intersect S and thus either p or q is in C. Therefore, C $(\frac{d+1}{d+2})$-splits the points interior to S from those exterior to S, excluding those points in C. The following lemma bounds the size of $|C|$

Lemma 5.11

$$|C| \leq 3A_d(1/2)k + \left(\frac{4\sqrt{7}}{7}\right)^{d-1} \frac{1}{V_{d-1}} Cost_f(S).$$

Proof: We write C as the union of three subsets $C = C_1 \cup C_2 \cup C_3$, where

$$\begin{aligned} C_2 &= \{i \in C : p_i \in \mathbf{R}^d - (2 \cdot S)\} \\ C_3 &= \{i \in C : p_i \in (2 \cdot S) : \gamma_i \geq \text{radius}(S)\} \\ C_1 &= C - C_2 - C_3 \end{aligned}$$

It simply follows from Lemma 4.2, that $|C_2| \leq A_d(1/2)k$. Similarly, $|C_3| \leq 2A_d(1/2)k$.

We now bound the size of C_1. First notice that

$$\begin{aligned} \mathrm{Cost}_f(S) &= \int_{v \in S} \sum_{i=1}^{n} f_i(v)^{d-1} (dv)^{d-1} \\ &\geq \sum_{i \in C_1} \int_{v \in S} f_i(v))^{d-1} (dv)^{d-1}. \end{aligned}$$

By the definition of C_1, for each $i \in C_1$, S has a common point with B_i. Further, the *radius*$(S) \geq \gamma_i$. This implies that the area of $S \cap (2 \cdot B_i)$ is at least $\left(\frac{\sqrt{7}}{4}\gamma_i\right)^{d-1} V_{d-1}$.

Therefore,

$$\begin{aligned} \int_{v \in S} (f_i(v))^{d-1} (dv)^{d-1} &\geq Area(A \cap (2 \cdot B_i)) \left(\frac{1}{\gamma_i}\right)^{d-1} \\ &\geq \left(\frac{\sqrt{7}}{4}\right)^{d-1} V_{d-1}. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathrm{Cost}_f(S) &\geq \sum_{i \in C_1} \int_{v \in S} f_i(v))^{d-1} (dv)^{d-1} \\ &\geq |C_1| \left(\frac{\sqrt{7}}{4}\right)^{d-1} V_{d-1} \end{aligned}$$

Thus, $|C_1| \leq \left(\frac{4\sqrt{7}}{7}\right)^{d-1} \frac{1}{V_{d-1}} \mathrm{Cost}_f(S)$. Detailed proof will be given in the full paper. □

Notice that $\mathrm{Cost}_f(S) = O\left(k^{1/d} n^{\frac{d-1}{d}}\right)$ and $k \leq n$ implies $k \leq k^{1/d} n^{\frac{d-1}{d}}$. It follows that $|C_1| = O\left(k^{1/d} n^{\frac{d-1}{d}}\right)$. This complete the proof of Theorem 5.1.

Similarly, we can show that a sphere separator of an (α, k)-overlap graph can be pulled back to a vertex separator of the graph, whose size is linear in the cost of the sphere separator.

6 Non-Euclidean Neighborhood Systems

In this section, we shall show that our separator theorems for the overlap graphs of Euclidean neighborhood system can be generalized to neighborhoods determined by any norm. Thus, a **neighborhood system for a norm** $||v||$ is a collection of ball $B_i = \{p \ : \ ||p - p_i|| \leq r_i\}$.

Observe, that any norm determines a unique symmetric closed convex body, i.e., the unit ball centered at the origin. A body B about the origin is symmetric if $p \in B$ implies that $-p \in B$. It is also true that any symmetric convex body B determines a unique norm, namely, that norm with unit ball B.

A more general neighborhood system can be obtained from translation and dilations of any closed convex body. Due to space constraints we only discuss this more general case. In a body that is not necessarily symmetric, the first issue becomes that of defining the center of the body.

Let Γ be a bounded convex body in $\mathbf{R}^d$. Let E' be the largest ellipsoid that is contained in Γ and E the smallest one that contains Γ. It is proven by Lowner and John [Lov86] that E' and E are concentric. Moreover, E arises from E' by enlarging by a factor at most d. (E', E) is called a *Lowner-John pair* and the center O_Γ of both E' and E is *Lowner-John* center.

Theorem 6.1 *For each convex body Γ in $\mathbf{R}^d$, if G is an (α, k)-overlap graph of in $S(\Gamma)$, then G has an $O(\alpha \cdot d \cdot k^{1/d} \cdot n^{(d-1)/d} + q(\alpha, k, d))$-separator that $\frac{d+1}{d+2}$-splits.*

7 Applications

Our separator results have applications in both Numerical Analysis and Computational Geometery. In case of numerical analysis the use of separators is well documented [LRT79, GT87, PR85] at least for direct methods. We discuss some of the applications that we have found for the latter case. In this paper we will just outline some of the results we have so far.

We can compute the intersection graph for a k-neighborhood system for a fixed dimension d in $O(\log n)$ time using n processors on a randomized PRAM with unit time prefix sum. The algorithm is a straight forward divide-and-conquer algorithm. Let $B_1, \ldots, B_n$ be the balls and $P_1, \ldots, P_n$ their centers. For simplicity we assume that $d = 2$, the general case is similar. We will simply pick the continuous separator, a circle S, as in Section 5.1. By Section 5.4 expect only $O(n^{3/4})$ balls to intersect S with probability of failure at most $n^{-1/4}$. If we assume that a prefix sum can be preformed in constant time, [Ble90], then we can check to see if the circle S gives a small separator in constant time and divide the balls into two sets, those balls that intersect the interior of the circle and those that intersect the exterior. It is not hard to see that the algorithm runs in randomized $(\log n)$ time with total work at most $O(n \log n)$.

If we do not assume constant time prefix sums then the solution is much messier and requires the use of many known tricks. In this case we get an $O(\log n \log\log n)$ time algorithm using n processors.

We can also find the solution to the all-nearest neighbor problem in fixed dimension d that runs in $O(\log^2 n)$ time using n processors on a randomized PRAM with unit time prefix sum.

The full writeup with algorithms for these problems and several others will appear subsequent papers, see [Ten91].

8 Open Questions

1. What is the computational complexity for deciding whether a graph G is k–embeddable or (α, k)–embeddable?
2. Is there a polynomial time algorithm for computing the disk packing of a planar graph?

Acknowledgements We would like to thank David Applegate, Bruce Hendrickson, Ravi Kannan, Tom Leighton, Mike Luby, Oded Schramm, Bill Thurston, Doug Tygar, and Kim Wagner for invaluable help and discussions.

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