Computation in Markoff-Hurwitz Equations

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***Abstract*—In the more than 100 years since Markoff-Hurwitz Equations, they play a decisive role, have turned up in an astounding variety of different settings, from number theory to combinatorics, from classical groups and geometry to the world of graphs, from discrete mathematics to scientific computation. We will first introduce other people's work in this area. Then we present general properties of solutions. We prove solution does not exist when $n = 3$ and $k = 2$. Integer sequences and solutions are reported for n ≤ 10. Conjectures are posted.**

Keywords: Markoff and Hurwitz equations, solution trees, search solution space, solution generator, integer sequences

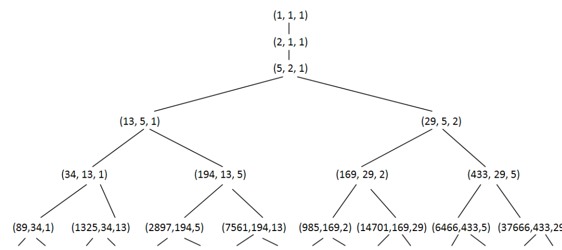
I. Introduction

The Diophantine equation
$x_1^2 + x_2^2 + ... + x_n^2 = kx_1x_2...x_n$ (1)
with k a positive integer and n$\geq$3 is known as a Hurwitz or Markoff-Hurwitz equation or generalized Markoff equation. Such equations were first studied by Hurwitz [1] who thought of them as generalizations of the Markoff equation.

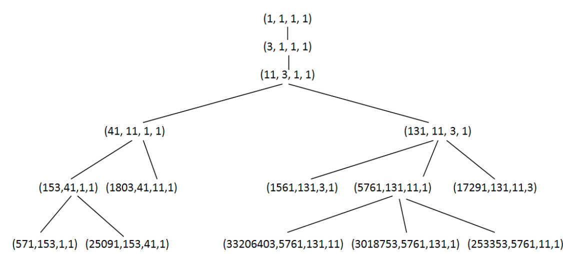
The Markoff equation $x^2 + y^2 + z^2 = 3xyz$ as first studied by Markoff in 1879 [2] , [3]. He made it famous when he noted the connection between its integral solutions, classes of quadratic forms, and Diophantine approximation. Using a descent argument, he showed all the positive integral solutions can be generated by the fundamental solution (1,1,1) and a group of automorphisms. Its set of integer solutions is infinite and nontrivial, yet is easy to describe [4], [5], [6], [7]. In [8], equation (1) was tackled by using computational assist approach. They discussed general properties of solutions and presented an efficient systematic solution space search algorithm, and reported the search finding. They gave the theorems on non-existence of solutions on some n & k combinations suggested from search results for x_i values up to 1,000,000. They also present an extremely fast algorithm for generating the solutions, which match exactly the result from the systematic searching/checking. Some conjectures were proposed based on their finding and observation.

A. Markoff Tree

The Markoff equation is quadratic in each variable, so given a solution (x, y, z), we can find solutions $(3yz - x, y, z)$, $(x, 3xz - y, y)$, $(x, y, 3xy - z)$. Using this map, permutations of the variables, and the fundamental solution $(1, 1, 1)$, we can construct a Markoff tree of positive ordered solutions. We let the three coordinates in a solution be in decreasing order. The following is such a tree.



We also construct a tree for
$$x^2 + y^2 + z^2 + w^2 = 4xyzw.$$



B. Other People's Work

The solution triples (x,y,z) to equation (1) with x, y, z > 0, are called Markov triples, and the numbers that appear in such a triple are called Markov numbers. The first 4 Markov triples are: (1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13).

The particular interest of the Markoff equation lies in the fact that it is a quadratic equation in each of x, y and z, and hence new solutions can be obtained by a simple process from any given one. In [2] Markoff demonstrated that every Markoff triple can be obtained from (1, 1, 1) by repeatedly generating new neighbors.

Uniqueness conjecture:

Every Markov number appears exactly once as the maximum in a Markov triple.

Many people have studied this conjecture. This conjecture turns up in an amazing number of different variants, from numbers and matrices to geometry and matchings of graphs. The conjecture has become widely known when Cassels mentioned it in [13]. It has been proved only for some rather special subsets of the Markoff numbers. The following result for Markoff numbers which are prime powers or 2 times prime powers was first proved independently and partly by A. Baragar [14] (for primes and 2 times primes), Button [15]

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(for primes but can be easily extended to prime powers) and Schmutz [16] (for prime powers but the proof works also for 2 times prime powers) using either algebraic number theory or hyperbolic geometry. A Markoff number is unique if it is a prime power or 2 times a prime power. In [17] Zhang claimed that if c is an even Markoff number then c $\equiv$ 2 (mod 32). A Markoff number c is unique if one of 3c + 2 and 3c − 2 is a prime power, 4 times a prime power, or 8 times a prime power.

A. Baragar described the Markov equations and their orbits of integer solutions in [10]. He showed that the number of orbits of integer solutions is finite, and he described a sequence of equations for which this number goes to infinity. This is described in more detail in [7]. He also described the asymptotic growth of Markov numbers in each orbit, and sketched a proof which requires an assumption which he later removed in [11], and improved in [12].

In [18], Aigner showed that the triples (1, 1, 1) and (2, 1, 1) are the only Markov triples with repeated numbers.

D. Zagier [19] investigated the asymptotic growth for the number of solutions to the Markov equation (k=n=3) below a given bound and Baragar [12] investigated the cases n≥4. Several other researchers also studied the asymptotic growth or the ratio of the numbers from the generalized equation (1).

C. Some Well Known Theorems

As a foundation, we recall the following important related theorems.

Theorem 1. When n=2, $x_1^2 + x_2^2 = kx_1x_2$ has solution iff k=2.

For any positive integer c, $x_1 = x_2 = c$ is a solution.

Theorem 2. If $n = k$, x_1=x_2=...=x_n=1 is a solution to equation (1).

Theorem 3. If x_1, x_2,..., x_n is a solution to equation (1), then x_1, x_2,..., x_{i-1}, $kx_1x_2...x_{i-1}x_{i+1}...x_n - x_i$, x_{i+1}, ..., x_n is also a solution (for each 1≤i≤n).

Theorem 4. Equation (1) does not have solutions when k>n for n≥2.

Proof. [1], [9].

A Famous Theorem: Equation (1) does not have solution when $k > n$ for $n \geq 2$. [**?**],[**?**]

We will give an outline of a constructive proof of this theorem in the following.

(1) Claim: $(n-1)+u^2 \leq nu$ when $n > 3$ and $1 \leq u \leq n-1$.

To prove this claim, we let $f(u) = u^2 + (n-1) - nu$. Then $\frac{d}{du}(u^2 + (n-1) - nu) = 2u - n$.

We need to consider:

(1) $u = 1$ or $u = n - 1$.

(2) $1 < u < n/2$.

(3) $n/2 \leq u < n - 1$.

Case 1: $x_1 = x_2 = ... = x_n$

Solution does not exist.

Case 2: $x_1 = x_2 = ... = x_{n-1} \neq x_n$

Solution does not exist.

Case 3: $x_1 \geq x_2 \geq ... \geq x_n$ (at least one "$>$")

Then $kx_2x_3...x_n - x_1 < x_1$.

Case 3-1: $2x_1 = kx_2x_3...x_n + \sqrt{k^2x_2^2x_3^2...x_n^2 - 4(x_2^2 + x_3^2 + ... + x_n^2)}$

Case 3-2: $2x_1 = kx_2x_3...x_n - \sqrt{k^2x_2^2x_3^2...x_n^2 - 4(x_2^2 + x_3^2 + ... + x_n^2)}$

Therefore, the sum of the n components of the solution $x_2, x_3, ..., x_n,\ kx_2x_3...x_n - x_1$ is smaller than that of the solution $x_1, x_2, ..., x_{n-1}, x_n$.

Transforming $x_1, x_2, ..., x_{n-1}, x_n$ to $kx_2x_3...x_n - x_1$ finite times, resulting two kinds of solution:

$x_1 = x_2 = ... = x_n$ or $x_1 \neq x_2 = ... = x_{n-1} = x_n$. None of them is a solution.

For the rest of the paper, we shall focus on the cases n ≥ 3 and 1 ≤ k ≤ n for positive integer solutions x_1, x_2,..., x_n.

II. General Properties of Solutions

In this section, we shall define our notations and terminologies and discuss some general properties of solutions to equation (1).

We call a solution x_1, x_2,..., x_n of (1) an ordered solution if $x_1 \geq x_2 \geq ... \geq x_n$.

Let $x_i^{'} = kx_1x_2...x_{i-1}x_{i+1}...x_n - x_i$. By Theorem 3, we know if x_1, x_2, ..., x_i,..., x_n form a solution of (1) then x_1, x_2, ..., $x_i^{'}$,..., x_n also form a solution. We say the new solution comes from the original solution by applying Theorem 3 on index i.

Lemma 1. Let X: x_1, x_2, ..., x_i,..., x_n be an ordered solution and i ≥ 2. Then

$x_i^{'} = kx_1x_2...x_{i-1}x_{i+1}...x_n - x_i > x_1$.

Definition 1. Let X: x_1, x_2, ..., x_i,..., x_n and Y; y_1, y_2, ..., y_i,..., y_n be two ordered solutions of (1). We define X <Y (X comes before Y in lexical order) iff there exists i such that 1 ≤ i ≤ n and x_j= y_j for 1 ≤ j < i and $x_i < y_i$. We say X is the minimum solution if X < Y for any other solution Y of (1).

Theorem 5. If X: x_1, x_2, ..., x_i,..., x_n is an ordered solution of (1), then Xi': $x_i^{'}$,x_1, x_2,..., x_{i-1}, x_{i+1}, ..., x_n, where $x_i^{'}$ is as defined in Lemma 1, is also an ordered solution and X < Xi' for 2 ≤ i ≤ n. In notation, X |-i Xi'.

Proof. Directly follows Theorem 3 and Lemma 1.

Note that for an ordered solution x_1, x_2, ..., x_i,..., x_n, $x_1^{'}$ may be <, >, or = x_1. For examples, for n = 7 & k = 5: 3, 1, 1, 1, 1, 1, 1 is a solution, applying Theorem 3 to index 1, we get a new solution 2, 1, 1, 1, 1, 1, 1 which is smaller. From 2, 1, 1, 1, 1, 1, 1, we get 3, 1, 1, 1, 1, 1, 1, which is larger. For n = 5 & k = 4: 2, 1, 1, 1, 1 is a solution, applying Theorem 3 on index 1, we get 2, 1, 1, 1, 1 itself. So if we apply the rule in Theorem 3 on index 1 to an ordered solution X: x_1, x_2, ..., x_i,..., x_n, the resulting solution X1': $x_1^{'}$, x_2,..., x_n may not be an ordered solution.

Let insert $x_1^{'}$ in proper order into x_2,..., x_n to form on ordered solution X1”. Then we shall use the notation X |-1 X1”.

Definition 2. Let X, Y be ordered solutions of (1).

We define X |- Y iff X |-i Y for some i = 1, 2,..., n, and X |-* Y iff there exists X1, X2, …, Xm such that X = X1, Y = Xm, and Xi |- Xi+1 for i = 1, 2,…, m-1.

Note that x_i and $x_i^{'} = kx_1x_2...x_{i-1}x_{i+1}...x_n - x_i$ are the two solutions of the quadratic equation

$y^2 - (kx_1x_2...x_{i-1}x_{i+1}\ldots x_n)y + (x_1^2 + x_2^2 + ... + x_{i-1}^2 + x_{i+1}^2 + ... + x_n^2) = 0$.

When i> 1, $x_i < x_i^{'}$. When i = 1, x_1 may >, <, or = $x_1^{'}$.

Theorem 6. If equation (1) has one solution then it has infinitely many solutions.

Proof. Let X be an ordered solution of (1). Then

X |-n X1 |-n X2 |-n …

X $<$ X1 $<$ X2 $<$… There are infinitely many solutions.

Definition 3. Let X be an ordered solution of (1) and X |-1 Y. If X $\leq$Y, then we call X a fundamental solution of (1).

It is still an open question whether (1) has a fundamental solution other than the minimum solution for some n & k.

Let X be an ordered solution of (1) and X |-i Yi for i = 1, 2,…, n. If X is not a fundamental solution then Y1 $<$ X and Yi $>$ X for i = 2, 3, …, n. Furthermore, Yi |-1 X for i = 2, 3, …, n and Y1 |- X.

Definition 4. If X |- Y where X != Y are ordered solutions of (1), then we say X and Y are adjacent ordered solutions.

The following theorem is obvious.

Theorem 7. Let X = x_1, x_2, ..., x_i,…, x_n be an ordered solution of (1). X has $|\{x_1, x_2, ..., x_i, \ldots, x_n\}|$ adjacent solutions if X is not a fundamental solution, otherwise it may have one less adjacent solutions than $|\{x_1, x_2, ..., x_i, \ldots, x_n\}|$ (in case X |-1 X).

Theorem 8. There is a solution with all components even for equation (1) if and only if $n = 4$ and $k = 1$.

Proof: When $n = 4$ and $k = 1$, $x_1 = x_2 = x_3 = x_4 = 2$ is a solution of

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_1x_2x_3x_4.$$

Case 1: $n = 3$

We will see in a following theorem that solution does not exist when $n = 3$ and $k = 2$.

Assume $x_1 = 2y_1, x_2 = 2y_2, x_3 = 2y_3$ is a solution of $x_1^2+x_2^2+x_3^2 = kx_1x_2x_3$. Then $y_1^2+y_2^2+y_3^2 = 2ky_1y_2y_3$. There is no solution for $k > 1$. For $k = 1$, $y_1^2 + y_2^2 + y_3^2 = 2y_1y_2y_3$ does not have solution.

Case 2: $n \geq 4$

Let $x_i = 2y_i$ for $i = 1, 2,..., n$, be a solution to equation (1). Then

$4(y_1^2 + y_2^2 + ... + y_n^2) = 2^nky_1y_2...y_n$

$y_1^2 + y_2^2 + ... + y_n^2 = k2^{n-2}y_1y_2...y_n$

We define a function $f(n) = k2^{n-2} - n$.

We can see $f(3) = 2k - 3, f(4) = 4k - 4, f(5) = 8k - 5$.

We want to show that $f(n) > 0$ for $n > 4$.

It is clear that $\frac{d}{dn}(k2^{n-2} - n) = \frac{1}{4}k2^n \ln 2 - 1 > 0$ for $n > 2$.

Thus $f(n)$ is an increasing function when $n > 2$.

When $n \geq 4, \min(f(n)) = 0$ iff $n = 4$ and $k = 1$.

$f(n) > 0$ when $n \geq 4$ and $k \geq 1$ except $n = 4$ and $k = 1$.

Therefore,

$y_1^2 + y_2^2 + ... + y_n^2 = k2^{n-2}y_1y_2...y_n$ has no solution when $n > 4$.

Thus the theorem holds.

Theorem 9. There is a solution with all components has a common divisor c ($c > 2$) for equation (1) if and only if $n = 3$ and $k = 1$.

Proof: When $n = 3$ and $k = 1$, $x_1 = x_2 = x_3 = 3$ is a solution to $x_1^2 + x_2^2 + x_3^2 = x_1x_2x_3$.

Let $x_i = cy_i$ for $i = 1, 2,..., n$, be a solution to equation (1).

$y_1^2 + y_2^2 + ... + y_n^2 = c^{n-2}ky_1y_2...y_n$.

Let $f(n) = c^{n-2}k - n$.

$\frac{d}{dn}(c^{n-2}k - n) = c^{n-2}(\ln c)\,k - 1 > 0$ for $n > 2$.

Thus $f(n)$ is an increasing function when $n > 2$.

When $n \geq 3, \min(f(n)) = 0$ iff $c = 3$, $n = 3$ and $k = 1$.

Thus $f(n) > 0$ when $n \geq 4$ and $k \geq 1$.

$y_1^2 + y_2^2 + ... + y_n^2 = k2^{n-2}y_1y_2...y_n$ has no solution when $n \geq 4$.

The theorem holds.

Theorem 10. $x_i = 2y_i + 1$ ($i = 1, 2,..., n$), is a solution to equation (1), then either n and k are both even, or n and k are both odd.

Theorem 11. $x_i = c$ ($i = 1, 2,..., n$) is a solution to equation (1), then $n = k$ and $c = 1$, or $n = 3$ and $k = 1$, or $n = 4$ and $k = 1$.

Proof: Let $x_i = c$ ($i = 1, 2,..., n$) be a solution to equation (1). Then

$nc^2 = kc^n$, $n = kc^{n-2}$

If $c = 1$, then $n = k$.

If $c = 2$, then $n = 4$ and $k = 1$. (from a previous theorem)

If $c = 3$, then $\frac{n}{k} = 3^{n-2}$.

For $c \geq 3$, we define $f(n) = kc^{n-2} - n$.

$\frac{d}{dn}(kc^{n-2} - n) = kc^{n-2}\ln c - 1$

$kc^{n-2}\ln c - 1 > 0$.

$f(n) = kc^{n-2} - n$ is an increasing function.

If $f(3) = kc^{3-2} - 3 = 0$, then $k = 1$ and $c = 3$, since $c \geq 3$.

$f(n) > 0$ for $n > 3$.

Therefore there is no solution of the form $x_i = c$ ($i = 1, 2,..., n$) when $n > 3$ and $c \geq 3$.

Theorem 12. Solution does not exist when $n = 3$ and $k = 2$.

Proof: Suppose that x_1, x_2, x_3 is a solution. Then $x_1^2+x_2^2+x_3^2 = 2x_1x_2x_3$.

x_1, x_2, x_3 can not be all odd.

Case 1: x_1, x_2, x_3: one even, two odd

Say $x_1 = 2a + 1$, $x_2 = 2b + 1$, $x_3 = 2c$.

$(2a + 1)^2 + (2b + 1)^2 + (2c)^2 = 2(2a + 1)(2b + 1)2c$

$4a^2 + 4a + 4b^2 + 4b + 4c^2 + 2 = 4(2a + 1)(2b + 1)c$

left side= $2 \bmod 4$, right side = $0 \bmod 4$.

Solution does not exist for this case.

Case 2: x_1, x_2, x_3 : two even, one odd

Say $x_1 = 2a$, $x_2 = 2b$, $x_3 = 2c + 1$.

Left side= $(2a)^2 + (2b)^2 + (2c + 1)^2 = 1 \bmod 4$

Right side= $0 \bmod 4$

Solution does not exist for this case

Case 3: x_1, x_2, x_3: all even

Say $x_1 = 2a$, $x_2 = 2b$, $x_3 = 2c$.

$(2a)^2 + (2b)^2 + (2c)^2 = 2(2a)(2b)(2c)$

$a^2 + b^2 + c^2 = 4abc$

Solution does not exist for this case.

Therefore, solution does not exist when $n = 3$ and $k = 2$.

Theorem 13. If x_1, x_2, x_3 is a solution to the equation (1) for $n = 3$ and $k = 3$ if and only if $3x_1, 3x_2, 3x_3$ is a solution to the equation (1) for $n = 3$ and $k = 1$.

Proof: Let x_1, x_2, x_3 be a solution for $n = 3$ and $k = 3$, then $x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3$.

Then $9x_1^2 + 9x_2^2 + 9x_3^2 = 9 \times 3x_1x_2x_3$.

$(3x_1)^2 + (3x_2)^2 + (3x_3)^2 = (3x_1)(3x_2)(3x_3)$.

Therefore, $3x_1, 3x_2, 3x_3$ is a solution for the case $n = 3$ and $k = 1$.

Let $3x_1, 3x_2, 3x_3$ is a solution for the case $n = 3$ and $k = 1$.

Then $(3x_1)^2 + (3x_2)^2 + (3x_3)^2 = (3x_1)(3x_2)(3x_3)$.

Then $x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3$.

Therefore, x_1, x_2, x_3 is a solution for the case $n = 3$ and $k = 3$.

Theorem 14. If $x_1^2 + x_2^2 + x_3^2 = x_1x_2x_3$, then $x_i = 0 \bmod 3$ for $i = 1, 2, 3$.

Proof: We can easily prove this theorem by using the previous theorem. We give another proof in the following.

We want to show that $x_1, x_2,\ x_3$ is a solution, then $x_i = 0 \bmod 3$ for $i = 1, 2, 3$.

If $x_1 = 3a+x$, $x_2 = 3b+y$, $x_3 = 3c+z$ with $0 \le x, y, z \le 2$ is a solution, then $(3a+x)^2 + (3b+y)^2 + (3c+z)^2 = (3a+x)(3b+y)(3c+z)$.

$9a^2 + 6ax + 9b^2 + 6by + 9c^2 + 6cz + x^2 + y^2 + z^2$

$= 27abc + 9abz + 9yac + 3yaz + 9xbc + 3xbz + 3yxc + xyz$

left side $= x^2 + y^2 + z^2 \bmod 3$

right side$= xyz \bmod 3$

Case 1: all of $x, y, z \neq 0$.

Case 1-1: all of $x, y, z = 1$, then left side$= 0 \bmod 3$, right side $= 1 \bmod 3$.

Case 1-2: all of $x, y, z = 2$, then left side$= 0 \bmod 3$,right side $= 2 \bmod 3$.

Case 1-3: $x = 1, y = 1, z = 2$, then left side$= 0 \bmod 3$, right side $= 2 \bmod 3$.

Case 1-4: $x = 1, y = 2, z = 2$, then left side$= 0 \bmod 3$,right side $= 1 \bmod 3$.

Case 2: only $x = 0$

Case 2-1: $y = 1, z = 1$

left side$= 2 \bmod 3$

right side$= 0 \bmod 3$

Case 2-2: $y = 1, z = 2$

left side$= 2 \bmod 3$

right side$=0 \bmod 3$

Case 2-3: $y = 2, z = 2$

left side$= 2 \bmod 3$

right side$= 0 \bmod 3$

Case 3: $x = 0,\ y = 0,\ z \neq 0$

Case 3-1: $z = 1$

left side$= 1 \bmod 3$

right side$= 0 \bmod 3$

Case 3-2: $z = 2$

left side$= 1 \bmod 3$

right side $= 0 \bmod 3$

Therefore $x_1 = 0 \bmod 3, x_2 = 0 \bmod 3$ and $x_3 = 0 \bmod 3$.

K.Guy and R.Nowakowski asked:"For what pairs of integers a, b does ab exactly divide $a^2 + b^2 + 1$?" [20]. We answer this question in the following theorem.

Theorem 15. If $a^2+b^2+1 = kab$ has solutions, then $k = 3$. And there are infinite solutions.

Proof: We already proved that solution does not exist when $n = 3$ and $k = 2$ in the equation (1).

When $k = 3$, $a = b = 1$ is a solution.

If $k = 1$, then $a^2 + b^2 + 1 = ab$. It is not true since $a^2 + b^2 + 1 > ab$.

Therefore, solution dose not exist when $k = 1$ or 2. Solutions do exist when $k = 3$.

Let a and b with $a \ge b \ge 1$ be a solution to $a^2 + b^2 + 1 = 3ab$.

We will show that $3a - b$, a is another solution for $a^2 + b^2 + 1 = kab$, i.e., $(3a - b)^2 + a^2 + 1 = 3(3a - b)a$, i.e.,$(3a - b)^2 + a^2 + 1 = 3(3a - b)a$.

$(3a - b)^2 + a^2 + 1 - 3(3a - b)a = a^2 + b^2 + 1 - 3ab = 0$.

It is clear that $3a - b > a$ and $a > b$.

Therefore, there are infinite solutions.

III. Search for Solutions

We shall present an algorithm for checking all potential ordered solutions in lexical order (< in Definition 1) with each component xi ≤ Limit where Limit is an input constant.

The algorithm will skip any range where the non-existence of solutions can be inferred by the following proposition.

Proposition 1. If $kx_1x_2...x_i > x_1^2$+x_2^2+...+x_i^2+(n-i) with i < n then $kx_1x_2...x_n > x_1^2$+x_2^2+...+x_n^2 for any x_{i+1}, x_{i+2},. . . , x_n such that $x_i \ge x_{i+1} \ge x_{i+2} \ge \ldots \ge x_n > 0$.

We can consider one extra x_j at a time until j = n or x_j = 1. Hence the product will always be bigger than the sum of squares under the condition.

Our systematic search/checking algorithm with cut-offs based on Proposition 1:

Algorithm 1 Systematic Search for Solutions [8]

Check all possible ordered solutions with each

component 1 ≤ x[i] ≤ Limit

s[i]=x[1]*x[1] + x[2]*x[2] +...+ x[i]*x[i]

p[i]=k*x[1]*x[2]*...*x[i]

The average time complexity of Algorithm 1 is about O(Limit2), since the cut-offs usually occur at early stages when i = 2 or 3.

There are no solutions with all x_i ≤1,000,000 for (n, k) cases: (3, 2), (4, 2), (4, 3), (5, 2), (5, 3), (6, 1), (6, 2), (6, 4), (6, 5), (7, 4), (7, 6), (8, 2), (8, 3), (8, 4), (8, 5), (8, 6), (8, 7), (9, 1), (9, 2), (9, 3), (9, 4), (9, 5), (9, 7), (9, 8). There is a strong possibility that no solutions at all on these cases.

The minimum solutions for 3 ≤n ≤9 and 1 ≤ k ≤ n other than the cases listed above contain no components > 4.

For a fix n, the processing time decreases as k increases since lager k induces more cut-offs (Proposition 1). For a fix k, the processing time increases as n increases since the equation (1) gets more complex on larger n.

Algorithm 2 Generating Solutions [8]

We use Algorithm 1 to find the minimum solution M (within the limit). If the minimum solution is found, proceeds as follows:

Keep a lexically ordered linked list L of all ordered-solutions found so far. Initially L consists of the minimum solution M alone. Let M be $m_1, m_2, \ldots, m_n$.

```
for (i = 1; i <= n; i++) {
if (i==1 or m_i != m_{i-1}) {
generate a new ordered solution Y using
the rule in Theorem 3 on index i;
if (Y > M) and all yi within limit
then insert Y to L in lexical order;
}
}
```

Traverse through L until the end. Let X: x_1, x_2, ..., x_n be initialized to the 2nd ordered solution in the linked list L.

```
while (X != null) {
for (i = 2; i <= n; i++) {
        if (x_i != x_{i-1}) {
    generate a new solution Y using the
rule in Theorem 3 on index i.
if Y > X and all y_j with in limit
then insert Y to L in lexical order
//if Y ≤ X, Y is already in L
}
}
X = the next ordered solution in L.
}
```

The time complexity of Algorithm 2 is O(s2) where s is the number of ordered solutions within the limit when case (n, k) has solutions.

IV. Integer Sequences

In this section, we shall present some interesting sequences arise from Markoff-Hurwitz Equations. Some of those sequences are in [21].

Given n & k, the leading term of the ordered solutions of equation (1) in non-decreasing order form an integer sequence if the solutions exist. So far we have not found same leading term from two different ordered solutions for a given (n, k).

Sequence for n=3 and k=3:[22]

1, 2, 5, 13, 29, 34, 89, 169, 194, ...

This sequence is called Markoff numbers.

The number of distinct prime divisors of any of the first 93 terms is less than 6. The second, third, 4th, 5th, 6th, 10th, 11th, 15th, 16th, 18th, 20th, 24th, 25th, 27th, 30th, 36th, 38th, 45th, 48th, 49th, 69th, 79th, 81th, 86th, 91th terms are primes.

The odd numbers in this sequence are of the form 4k+1.

The even numbers in this sequence are of the form 4k+2.

Assuming that each solution (x,y,z) is ordered $x \geq y \geq z$, the open problem is to prove that each x value occurs only once. There are no counterexamples in the first 1046858 terms. According to Sarnak on Apr 30 2015, all claims to have proved the unicity conjecture have turned out to be false.

The numeric value of C = lim (number of Markoff numbers < x) / log^2(3x) given in Zagier's paper and quoted above suffers from an accidentally omitted digit and rounding errors. The correct value is C = 0.180717104711806.

Sequence for n=3 and k=1:[23]

3, 6, 15, 39, 87, 102, 267, 507, ...

This sequence is the previous sequence multiplied by three.

A list of x's in nondecreasing order over all solutions of $x^2 + y^2 + z^2 = xyz$, with $x \geq y \geq z$.

x,y,z is a solution of $x^2 + y^2 + z^2 = 3xyz$ if and only if 3x,3y,3z is a solution of $x^2 + y^2 + z^2 = xyz$.

Sequence for n=4 and k=1:[24]

2, 6, 22, 82, 262, 306, 1142, ...

Sequence for n=4 and k=4:[25]

1, 3, 11, 41, 131, 153, 571, ...

a,b,c,d is a solution to $a^2 + b^2 + c^2 + d^2 = 4abcd$ if and only if 2a, 2b, 2c, 2d is a solution to $a^2 + b^2 + c^2 + d^2 = abcd$.

Sequence for n=5 and k=1:[26]

4, 5, 9, 12, 23, 31, 33, 35, ...

Sequence for n=5 and k=4:[27]

2, 7, 26, 55, 97, 362, 433, ...

Sequence for n=5 and k=5:[28]

1, 4, 19, 91, 379, 436, 2089, ...

Sequence for n=6 and k=3:[29]

2, 4, 10,11, 23, 26, 64, 68, ...

Sequence for n=6 and k=6:[30]

1, 5, 29, 169, 869, 985, 5741, ...

Sequence for n=7 and k=1:[31]

3, 5, 10, 18, 23, 37, 39, 58, ...

Sequence for n=7 and k=2:[32]

2, 6, 15, 22, 47, 82, 118, ...

Sequence for n=7 and k=3:[33]

3, 7, 17, 18, 47, 62, 99, 123, ...

Sequence for n=7 and k=5:[34]

2, 3, 9, 14, 43, 67, 89, 206, ...

Sequence for n=7 and k=7:[35]

1, 6, 41, 281, 1721, 1926, ...

Sequence for n=8 and k=1:

4, 14, 31, 52, 110, 194, ...

Sequence for n=8 and k=8:

1, 7, 55, 433, 3079, 3409, ...

Sequence for n=9 and k=6:

2, 4, 11,23, 64, 131, 134, ...

Sequence for n=9 and k=9:

1, 8, 71, 631, 5111, 5608, ...

Sequence for n=10 and k=1:

4, 8, 13, 20, 29, 47, 48, ...

V. Solutions For Some Given n and k

We will list some solutions to Markoff-Hurwitz Equations for given n and k.

n=3 and k=1:

(3, 3, 3), (6, 3, 3), (15, 6, 3), (39, 15, 3), ...

There are 89 solutions within the Limit = 1215752192.

n=3 and k=3:
(1, 1, 1), (2, 1, 1), (5, 2, 1), (13, 5, 1), ...
There are 89 solutions within the Limit = 1215752192.
n=4 and k=1:
(2, 2, 2, 2), (6, 2, 2, 2), (22, 6, 2, 2), ...
There are 59 solutions within the Limit = 1215752192.
n=4 and k=4:
(1, 1, 1, 1), (3, 1, 1, 1), (11, 3, 1, 1), ...
There are 59 solutions within the Limit = 1215752192.
n=5 and k=1:
(4, 3, 3, 1, 1), (5, 3, 3, 1, 1), (9, 4, 3, 1, 1), ...
There are 508 solutions within the Limit = 1215752192.
n=5 and k=4:
(2, 1, 1, 1, 1), (7, 2, 1, 1, 1), (26, 7, 1, 1, 1), ...
There are 76 solutions within the Limit = 1215752192.
n=5 and k=5:
(1, 1, 1, 1, 1), (4, 1, 1, 1, 1), (19, 4, 1, 1, 1), ...
There are 39 solutions within the Limit = 1215752192.
n=6 and k=3:
(2, 2, 1, 1, 1, 1), (4, 2, 1, 1, 1, 1), (10, 4, 1, 1, 1, 1), ...
There are 203 solutions within the Limit = 1215752192.
n=6 and k=6:
(1, 1, 1, 1, 1, 1), (5, 1, 1, 1, 1, 1), (29, 5, 1, 1, 1, 1), ...
There are 33 solutions within the Limit = 1215752192.
n=7 and k=1:
(3, 2, 2, 2, 1, 1, 1), (5, 2, 2, 2, 1, 1, 1), ...
There are 388 solutions within the Limit = 1215752192.
n=7 and k=2:
(2, 2, 2, 1, 1, 1, 1), (6, 2, 2, 1, 1, 1, 1), ...
There are 168 solutions within the Limit = 1316134912.
n=7 and k=3:
(3, 2, 1, 1, 1, 1, 1), (7, 3, 1, 1, 1, 1, 1), ...
There are 173 solutions within the Limit = 1215752192.
n=7 and k=5:
(2, 1, 1, 1, 1, 1, 1), (3, 1, 1, 1, 1, 1, 1), ...
There are 99 solutions within the Limit = 1316134912.
n=7 and k=7:
(1, 1, 1, 1, 1, 1, 1), (6, 1, 1, 1, 1, 1, 1), ...
There are 24 solutions within the Limit = 1215752192.
n=8 and k=1:
(4, 2, 2, 2, 1, 1, 1, 1), (14, 4, 2, 2, 1, 1, 1, 1), ...
There are 180 solutions within the Limit = 1215752192.
n=8 and k=8:
(1, 1, 1, 1, 1, 1, 1, 1), (7, 1, 1, 1, 1, 1, 1, 1), ...
There are 23 solutions within the Limit = 1215752192.
n=9 and k=6:
(2, 1, 1, 1, 1, 1, 1, 1, 1), (4, 1, 1, 1, 1, 1, 1, 1, 1), ...
There are 80 solutions within the Limit = 1215752192.
n=9 and k=9:
(1, 1, 1, 1, 1, 1, 1, 1, 1), (8, 1, 1, 1, 1, 1, 1, 1, 1), ...
There are 18 solutions within the Limit = 1215752192.
n=10 and k=1:
(4, 4, 3, 1, 1, 1, 1, 1, 1, 1), (8, 4, 3, 1, 1, 1, 1, 1, 1, 1), ...
There are 371 solutions within the Limit = 1215752192.

VI. Conjectures

There are no solutions with all x_i $\leq$1,000,000 for (n, k) cases: (3, 2), (4, 2), (4, 3), (5, 2), (5, 3), (6, 1), (6, 2), (6, 4), (6, 5), (7, 4), (7, 6), (8, 2), (8, 3), (8, 4), (8, 5), (8, 6), (8, 7), (9, 1), (9, 2), (9, 3), (9, 4), (9, 5), (9, 7), (9, 8). There is a strong possibility that no solutions at all on these cases.

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