APPLICATION OF CONTINUED FRACTIONS FOR FAST EVALUATION
OF CERTAIN FUNCTIONS ON A DIGITAL COMPUTER

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Abstract

The purpose of this paper is to develop a method for evaluation
of certain elementary functions on a digital computer by the use of
continued fractions. The time required for this evaluation is drastically
reduced by using "short" operations like shift and add, instead of
multiplications. Consistency is the most important factor that allows
the expansion of a function into a continued fraction. Several cases
are discussed and in particular the solution of the quadratic equation
is discussed in more detail to demonstrate the convergence of the method.

1. INTRODUCTION

The idea of using continued fraction representations for generating
a solution to a limited class of quadratics was first introduced by
Robertson [3].

Consider the finite continued fraction with k partial numerators
p_i and k partial denominators q_i = 1, 2, ..., k, whose value is
\( \frac{A_k}{B_k} \), i.e.,

\[
\frac{A_k}{B_k} = \frac{p_1}{q_1 + \frac{p_2}{q_2 + \frac{p_3}{q_3 + \cdots + \frac{p_k}{q_k}}}}
\]

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Science Foundation under Grant No. US NSF GJ813.
A convenient way of writing (1.1) is
\[ \frac{A_k}{B_k} = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \frac{p_3}{q_3} + \cdots + \frac{p_k}{q_k}. \]

A<sub>k</sub> and B<sub>k</sub> are determined from the recursions:
\[
\begin{align*}
A_i &= q_1 A_{i-1} + p_i A_{i-2} \\
B_i &= q_1 B_{i-1} + p_i B_{i-2} \quad i = 2, 3, \ldots, k,
\end{align*}
\]
with initial values,
\[
\begin{align*}
A_0 &= 0 & A_1 &= p_1 \\
B_0 &= 1 & B_1 &= q_1.
\end{align*}
\]

It is clear that A<sub>k</sub> and B<sub>k</sub> can be separately and simultaneously determined in two binary arithmetic units in k-1 addition times if the p<sub>i</sub> and q<sub>i</sub> are chosen to be simple in the binary sense.

The digit set for p<sub>i</sub> and q<sub>i</sub> for the purposes of this paper is [1/2,1]; it will be stated later that the continued fraction \( A_k/B_k \) assume all values in the limit, over the interval
\[
(1.3) \quad \frac{\sqrt{2}-1}{2} \leq \lim_{k \to \infty} \frac{A_k}{B_k} \leq \sqrt{2}. \]

The range defined in (1.3) includes the range of normalized floating point binary fractional parts, [1/2,1]. This property indicates that a suitable continued fraction representation exists, such that conversion to conventional binary can be achieved by repetitive use of two binary adders in parallel, followed by a division to determine the quotient \( A_k/B_k \).

The main reason for selecting p<sub>i</sub>, q<sub>i</sub>\( \in \{1, \frac{1}{2}\}, i=1, 2, \ldots \) is that the four multiplicative operations required for each iteration in (1.2) are
reduced to "shift" and "add" operations. These operations will be called "short" operations throughout this paper, mainly because the time required to perform these operations is shorter than the time required to perform "long" operations, e.g., Multiplication, Division.

The purpose of this paper is to develop algorithms for fast evaluation of certain elementary functions by using "short" operations in several registers simultaneously. In order to be able to do so we make use of consistent functions. These functions will be defined at the end of section 2.

Selection rules for p and q in each iteration is an important step for the development of the algorithm. Selection rules were extensively studied by Trivedi [4], where a complete set of such rules were developed for the quadratic equation.
2. BILINEAR TRANSFORMATIONS AND THE RICCATI EQUATION.

In this section we develop a special case of the analysis of Wynn [7].

The general continued fraction will be regarded as a sequence of bilinear transformations of the form:

\[ f_k = \frac{p_k}{q_k + f_{k+1}}, \quad k = 1, 2, \ldots, \]

where \( f_k(x) \) is a function of \( x \). Therefore

\[ f_1 = \frac{p_1}{q_1} \frac{p_2}{q_2} \frac{p_3}{q_3} \cdots \frac{p_n}{q_n + f_{n+1}} \]

\[ = \frac{A_n + f_{n+1}A_{n-1}}{B_n + f_{n+1}B_{n-1}}, \quad n = 1, 2, \ldots, \]

where the functions \( A_n \) and \( B_n \) satisfy the recursion

\[ A_n = q_n A_{n-1} + p_n A_{n-2} \]

\[ B_n = q_n B_{n-1} + p_n B_{n-2} \quad n = 1, 2, \ldots, \]

with the initial values

\[ A_0 = 0 \quad A_1 = p_1 \quad B_0 = 1 \quad B_1 = q_1. \]

For the purposes of this paper \( p_i, q_i, i = 1, 2, \ldots \), will be selected from the digit set \( \{1/2, 1\} \) so that the recursion (2.4) can be performed by using only "short" operations.
The main purpose of this section is to show that there exist functions for which bilinear transformations of the form (2.1) can be used, and such that the functions $f_k$, $k = 1, 2, \ldots$, are consistent, i.e. equal.

Consider the Riccati equation

\begin{equation}
(2.5) \quad y_1' + a_1 y_1^2 + b_1 y_1 + c_1 = 0
\end{equation}

where $y_1$ is a function of the variable $x$, and $a_1, b_1$ and $c_1$ are functions of $x$ or constants. The property of this equation as noted by P. Wynn [6] is that if the dependent variable $y_1$ is replaced by the bilinear transformation (2.1) then the functions $f_k$, $k = 1, 2, \ldots$ also satisfy the Riccati equation

\begin{equation}
(2.6) \quad y_k' + a_k y_k^2 + b_k y_k + c_k = 0
\end{equation}

We develop below the recursion for the coefficients of the $(k+1)^{st}$ equation by means of the coefficients of the $k^{th}$ equation.

Let

\begin{equation}
(2.1) \quad y_k' + a_k y_k^2 + b_k y_k + c_k = 0
\end{equation}

be the $k$-th Riccati equation.

From (2.1) we have

$$y_k = \frac{p_k}{q_k + y_{k+1}} \quad , \quad p_k, q_k \in \{1, 1/2\}$$

then since

$$y_k' = \frac{-p_k y_{k+1}'}{(q_k + y_{k+1})^2}$$

we have

$$\frac{-p_k y_{k+1}'}{(q_k + y_{k+1})^2} + a_k \frac{p_k^2}{(q_k + y_{k+1})^2} + b_k \frac{p_k}{q_k + y_{k+1}} + c_k = 0$$
If we multiply by \(-\left(\frac{q_k + y_{k+1}}{p_k}\right)^2\) to normalize the coefficient of \(y_{k+1}',\) we get

\[
y_{k+1}' - \frac{c_k}{p_k} y_{k+1}^2 - \left(b_k + \frac{2c_k q_k}{p_k}\right) y_{k+1}' - \left(a_k p_k + b_k q_k + \frac{c_k q_k^2}{p_k}\right) = 0
\]

and the recursion that follows is

\[
\begin{align*}
a_{k+1} &= -\frac{c_k}{p_k} \\
b_{k+1} &= -b_k - \frac{2c_k q_k}{p_k} \\
c_{k+1} &= -a_k p_k - b_k q_k - \frac{c_k q_k^2}{p_k}
\end{align*}
\tag{2.7}
\]

We note that all the operations involved in (2.7) require only "short" operations, since both \(p_k\) and \(q_k\) are simple binary constants.

**Lemma 1:**

\[\Delta = b_k^2 - 4a_k c_k = \text{constant} \quad k = 1, 2, \ldots\]

**Proof:** We use the recursion (2.7) and get

\[
\begin{align*}
b_{k+1}^2 - 4a_{k+1} c_{k+1} &= b_k^2 + \frac{4b_k c_k q_k}{p_k} + \frac{4c_k q_k^2}{p_k} - 4a_k c_k \\
&\quad - \frac{4b_k c_k q_k}{p_k} - \frac{4c_k q_k^2}{p_k} = b_k^2 - 4a_k c_k
\end{align*}
\]
We define below the term consistency:

**Definition:** If

\[ f_{k+1}(x) = f_k(x) \quad \text{for all } x \quad k = 1, 2, \ldots \]

then the function \( f(x) \) is consistent.

The purpose of this paper is to develop consistent functions, and then to use them along with the recursions (2.4) and (2.7) to find continued fraction representations for the value of \( f(x) \), for a given \( x \).
3. **SOLUTION OF** \( ax^2 + bx - c = 0 \)

We show now how the solution of a quadratic equation with two distinct roots of opposite sign, and in particular the square root problem, can be found by the technique of section 2.

Let

\[
(3.1) \quad a_1 x_1^2 + b_1 x_1 - c_1 = 0
\]

be a given quadratic equation.

The substitution we use is of the form

\[
(3.2) \quad x_i = \frac{p_i}{q_i + x_{i+1}}
\]

where \( p_i, q_i \in \{1, \frac{1}{2}\} \quad i = 1, 2, \ldots \).

For the \(k\)th step we have

\[
a_k \frac{p_k^2}{(q_k + x_{k+1})^2} + b_k \frac{p_k}{q_k + x_{k+1}} - c_k = 0 \quad k = 1, 2, \ldots
\]

or

\[
c_k x_{k+1}^2 + (2c_k q_k - b_k p_k) x_{k+1} + c_k q_k^2 - a_k p_k^2 - b_k p_k q_k = 0.
\]

The recursion that follows is:

\[
(3.3) \quad a_{k+1} = c_k
\]

\[
b_{k+1} = 2c_k q_k - b_k p_k
\]

\[
c_{k+1} = -c_k q_k^2 + (a_k p_k + b_k q_k) p_k \quad k = 1, 2, \ldots
\]
and the resulting quadratic equation is

\[(3.4) \quad a_{k+1} x_{k+1}^2 + b_{k+1} x_{k+1} + c_{k+1} = 0\]

This method of approximating the solution of \((3.1)\) can be used if we develop a technique for selecting \(p_k\) and \(q_k\) \(k = 1, 2, \ldots\) from the coefficients of the \(k\)-th quadratic equation, i.e., \(a_k, b_k\) and \(c_k\).

The following lemma will be stated without a proof.

**Lemma 2:**

Let the \(k\)\(^{th}\) approximation to the solution of \((3.1)\) be

\[x_1 = \frac{p_1}{q_1} + \frac{p_2}{q_2} + \ldots + \frac{p_k}{q_k}\]

Then for the digit set \(p_k, q_k \in \{1, 1/2\}\) we have in the limit

\[M = \max x_1 = \sqrt{2}\]
\[m = \min x_1 = \frac{\sqrt{2}-1}{2}.\]

Using the result of the Lemma it can be seen that the consistency of the procedure can be achieved in each step if

\[(3.5) \quad m \leq x_k \leq M \quad k = 1, 2, \ldots\]

By imposing condition \((3.5)\) we need only one set of selection rules for \(p_k\) and \(q_k\), \(k = 1, 2, \ldots\) for the range \([m, M]\).

Note that for a binary computer with floating point arithmetic we only need a range between \(1/4\) and 1 because of normalization. Also note that for \(x_1\) outside our range a simple mapping can be used.
Let

\[(4.1) \quad y' + ay^2 + by + c = 0\]

be a Riccati equation, with \(a\), \(b\), and \(c\) constants.

In order to find the solution of (5.1) we integrate by parts:

\[(4.2) \quad \frac{dy}{ay^2 + by + c} = -dx\]

and the solutions are:

\[(4.3) \quad \int \frac{dy}{ay^2 + by + c} = \frac{1}{\sqrt{b^2 - 4ac}} \ln \frac{2ay + b - \sqrt{b^2 - 4ac}}{2ay + b + \sqrt{b^2 - 4ac}} \]

\[= \frac{1}{\sqrt{b^2 - 4ac}} \arctanh \frac{b + 2ay}{\sqrt{b^2 - 4ac}}, \text{ when } b^2 - 4ac > 0;\]

\[= \frac{-2}{b + 2ay}, \quad \text{ when } b^2 - 4ac = 0;\]

\[= \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ay + b}{\sqrt{4ac - b^2}}, \text{ when } b^2 - 4ac < 0.\]

The above solutions can be used now for the continued fraction expansion of the inverse functions which appear explicitly in the solution.

We start with the case

\[(4.4) \quad y = \tan x\]
The Riccati equation for (4.4) is
\[ y' = y^2 + 1, \quad y(0) = 0. \]

We note that \(-\Delta = \sqrt{4ac - b^2} = 2\).

Now we use a bilinear transformation of the form (2.1). The result is a differential equation of the type (2.6) with the recursion (2.7). By Lemma 1 it follows that the solution for each equation \(k\) is of the type (4.3) with \(-\Delta > 0\), and therefore we get for the \(k^{th}\) step:
\[ -x_k - d_k = \arctg \frac{2a_k y_k + b_k}{2}, \]

where \(d_k\) is a constant of integration.

The solution is therefore
\[ y_k = -\frac{b_k}{2a_k} - \frac{1}{a_k} \tan (x_k + d_k) \quad k = 1, 2, \ldots \]

Except for the first part of the solution which is a linear transformation, we see the consistency of the method, because if a set of selection rules are developed for \(\tan x\) it can be used for each step and therefore evaluation of this function will be possible.

Another important function which can be included is \(e^x\).

We have \(y' = y, \quad \Delta = 1\)

The \(k^{th}\) step solution is
\[ -x_k - d_k = \ln \frac{2a_k y_k + b_k - 1}{2a_k y_k + b_k + 1}, \]

or
\[ y_k = -\frac{b_k + 1}{2a_k} - \frac{1}{a_k (e^{x_k} - 1)} \quad k = 1, 2, \ldots \]
Again we note that if a set of selection rules can be developed for $e^x$ then it is possible to carry the process for each step and therefore to find the continued fraction expansion for the exponential function.

For the case where $\Delta = 0$ we have several possibilities:

(a) $b = a = 0$, $y' + c = 0$ with the solution $y = -cx + d$

(b) $b = c = 0$, $y' + ay^2 = 0$ with the solution

$$y_k = \frac{1}{ax+b}$$
REFERENCES


