Reasoning about Global Behavior of Ordinary Differential Equations by Combining Qualitative and Quantitative Analysis

Toyoaki Nishida
Shuji Doshita

Department of Information Science
Kyoto University
Sakyo-ku, Kyoto 606, Japan
nishida@kuis.kyoto-u.ac.jp
doshita@kuis.kyoto-u.ac.jp

Abstract

Understanding a system of differential equations begins with drawing a rough, qualitative picture of overall behaviors of the system. This paper presents an attempt at integrating numerical methods and knowledge-based methods with qualitative reasoning as a kernel. Essence of our approach is threefold: (a) representing geometric and topological aspects of solution curves relevant to qualitative analysis as mappings between hyperplanes in the phase space, (b) computing mappings that characterize the behavior by local analysis of solution curves, and (c) deriving global behaviors by analyzing structural information of the composite mappings representing solution curve. Preliminary results obtained from this approach are demonstrated for two-dimensional ordinary differential equations.

1 Introduction

Understanding a system of differential equations begins with drawing a rough, qualitative picture of overall behaviors of the system. In particular, identifying and classifying asymptotic behaviors which a given system exhibits after a long run towards plus or minus infinity enable to set up a plan for more detailed analysis. Five issues should be addressed in order to automate such early stages of analysis:

- what features to focus on
- what conclusion are drawn from such features
- how to represent such features
- how to build representation for such features from given description of differential equations
- how to derive global behaviors from the representation.

For the first two questions, dynamical systems theories [3, 2], mathematical theories for reasoning about qualitative nature of differential equations, suggest that studying geometric and topological properties of solution curves in phase space provides a powerful means for understanding the behaviors of differential equations qualitatively. Dynamical systems theories described above provide a conceptual and theoretical framework for describing what we see in the phase space. For example, consider Van der Pol’s equation:

\[
\begin{align*}
\frac{dx}{dt} &= -2x^3 + 2x + 2y \\
\frac{dy}{dt} &= -x.
\end{align*}
\]

Figure 1 shows several solution curves for (1) in the phase space spanned by state variables \(x\) and \(y\). Solution curves indicate how the value of \(x\) and \(y\) change over time. It is easy for us to grasp the rough image of the behavior, as “there seems to be a cyclic solution curve around \((0, 0)\) to which nearby solution curves are attracted. Oh, in fact all solution curves except the one at the origin seem to tend towards the cycle. And, since a cyclic orbit means a periodic behavior the above means that the system (1) will almost always exhibit the same periodic oscillation after a good amount of time has elapsed.” However, it is not trivial to have computer programs follow a similar intelligent path of reasoning. Thus, for the remaining three questions raised above, dynamical systems theories do not provide a direct answer.

The purpose of this paper is to propose answers to the remaining three questions raised above. Roughly, our answer is the following:
Figure 1: Solution Curves in the Phase Space for Van der Pol's Equation (1)

- representing geometric and topological aspects of solution curves as mappings between hyperplanes in the phase space
- deriving mappings that characterize the behavior by local analysis of solution curves
- reasoning about global behaviors by analyzing structural information of the composite mappings representing solution curves.

These features are embodied in a couple of programs PSX2PWL and PSX2NL which can autonomously explore the phase space for global behavior by combining qualitative and quantitative analysis. Qualitative analysis in this case involves predicting or interpreting the behavior, while quantitative analysis means using numerical simulations to track particular solutions or solving nonlinear simultaneous equations. In general, the more complex the class of differential equations becomes, the more the analyzer has to rely on numerical and approximate methods and the less rigorous answer can be expected as a result.

2 Preliminaries

Consider an ordinary differential equation

\[ \dot{x} = f(x), \]  

where \( x \) is a vector of state variables \((x_1, \ldots, x_n) \in \mathbb{R}^n\). Each state variable \( x_i \) gives some value in \( \mathbb{R} \) as a function of \( t \in \mathbb{R} \). An \( n \)-dimensional space spanned by \( \{x_i\} \) is called the phase space.\(^2\) For each point \( c = (c_1, \ldots, c_n) \) in the phase space, formula (2) specifies the rate and the orientation of state change: \( \dot{x}|_{x=c} = (\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt})|_{x=c} \). In other words, formula (2) defines a vector field in the phase space. A specific solution corresponding to initial state \( a = (a_1, \ldots, a_n) \) is a curve such that it passes on the point corresponding to \( a \) in the phase space and it is tangent to the vectors specified by the vector field at each point. Such a curve is called a solution curve, a trajectory, or an orbit. Open intervals of orbits are called orbit intervals.

Theoretically, the vector field specified by (2) is said to generate a flow \( \psi(x, t) : U \times I \rightarrow \mathbb{R}^n \) for all \( x \) in \( U \subset \mathbb{R}^n \) and \( t \) in some open interval \( I \subset \mathbb{R} \). Thus, an orbit for initial condition \( x = c \) is identified with a function \( \lambda_t \psi(c, t) : I \rightarrow \mathbb{R}^n \). If uniqueness of solution to differential equation holds, orbits never intersect with others nor with themselves (called the non-intersection constraint). The collection of all orbits in the phase space is called the phase portrait.

There is an obvious correspondence between geometric properties of orbits and aspects of dynamical behavior. For example, points in the phase space at which the right-hand side of the formula (2) is zero are called fixed points, and they correspond to equilibrium states which will not evolve forever. Fixed points are classified into hyperbolic and non-hyperbolic fixed points, and hyperbolic fixed points are in turn classified into sinks, sources, saddles, and some other peculiar subcategories. Orbits near a sink arbitrarily approach the sink as \( t \rightarrow +\infty \), while orbits near a source do so when \( t \rightarrow -\infty \). Most orbits near a saddle first approach it and then go away from it.\(^3\) Closed orbits correspond to periodic behaviors.

Generally, it is important to capture asymptotic behaviors of solutions as \( t \rightarrow \pm \infty \). Informally, an orbit which attracts nearby orbits as \( t \rightarrow \infty \) is called an attractor. The set of all points such that orbits passing on them tend towards an attractor \( A \) as \( t \rightarrow \infty \) is called the domain of attraction (of \( A \)). Similarly, we define repellors and the domain of repelling by replacing \( t \) by \(-t\) in the above definition.

As for two-dimensional Euclidean flows, it is proved

\(^{2}\)Although we study two-dimensional phase spaces (i.e., \( n = 2 \)), we attempt to present ideas as general as possible whenever they are applicable to general \( n \)-dimensional phase spaces.

\(^{3}\)These are not definitions but properties. For definition see [3] for example.
that solution curves in either (1) diverge for place at infinity, (2) approach a fixed point, or (3) approach a closed orbit (called a limit cycle), as $t \to \pm \infty$.

3 Reasoning about Phase Portraits with Bundle of Orbits

Our concern in this paper is engineering issues for making dynamical systems theories computational. In particular, it is crucial to find an adequate representation so that reasoning about phase portrait can be made effectively.

As a general strategy, we focus on bundle of orbits rather than individual orbits, thereby we can derive useful conclusions which cannot be made about single orbits. We first introduce flow mappings as representation of flow and then we describe how properties about flow are derived by reasoning with flow mappings.

3.1 Representing Flow

First, let us introduce several $(n - 1)$-dimensional hyperplanes, called sampling hyperplanes, in the $n$-dimensional phase space to “sample” data about bundles of orbits. When $n = 2$, sampling hyperplanes are straight lines, which we call sampling lines.

Consider a continuous set of orbits $\phi$ such that $\phi$ intersects sampling hyperplanes $p_1$ and $p_2$ at $r_1$ and $r_2$, respectively, as shown in figure 2, and $\phi$ introduces a homomorphic mapping from $r_1$ to $r_2$. This means that mapping $\phi_0$ preserves the order of landmarks in two-dimensional phase spaces. We call the interval $\phi_0$ of $\phi$ delimited by $r_1$ and $r_2$ a bundle of orbit intervals. We also refer to $\phi$ as a bundle of orbits. We represent this as $\phi : r_1 \rightarrow r_2$. $r_1$ or $r_2$ may also be $(n - k)$-dimensional region ($2 \leq k \leq n$) when they are (part of) invariant manifolds. $r_1$ is called the origin of $r_2$, and $r_2$ the destination of $r_1$.

Although it is hard to find an effective representation for $r_1$ and $r_2$ in $n$-dimensional space, we can currently step aside from this problem if we limit our concern to two-dimensional phase space. In two-dimensional phase space, $r_1$ and $r_2$ are points or continuous line segments which can be completely specified by the location of at most two end points.

Orbits delimiting a bundle of orbits are called landmark orbits, and points on sampling lines at which landmark orbits meet the sampling line are called landmarks. It should be noted that the exact location of landmarks is not needed in qualitative analysis. Instead, only the total ordering of landmarks on a sampling line is needed.

In order to represent flow in a region, we first divide the flow into a minimal collection of bundles of orbit intervals each of which maps a boundary segment or a repelling orbit in the region into another boundary segment or an attracting orbit in the region. The collection of mappings corresponding to bundles of orbits introduced above is called flow mappings.

For example, the flow of Van der Pol’s equation in a rectangular region $ABCD$ in figure 3 is represented as a sum of flow mappings:

$$\phi_1 = \phi_{1,1} : A \phi_1^{-1}(R) \rightarrow RA$$
$$\phi_{1,2} : A \phi_1^{-1}(R) S \rightarrow \phi_1(S) \phi_1(R)$$
$$\phi_{1,3} : B \phi_1^{-1}(S) \rightarrow SB$$
$$\phi_{1,4} : \phi_1^{-1}(S) F \rightarrow \phi_1(F) \phi_1(S)$$
$$\phi_{1,5} : E \phi_1^{-1}(P) \rightarrow FE$$
$$\phi_{1,6} : \phi_1^{-1}(P) Q \rightarrow \phi_1(Q) \phi_1(P)$$
$$\phi_{1,7} : X \rightarrow \phi_1(Q)$$
$$\phi_{1,8} : FR \rightarrow \phi_1(R) F$$

for region $ABEF$ and

$$\phi_2 : CDFQ \rightarrow QEC$$

for region $ECDF$. 
3.2 Reasoning about Global Behavior

Now suppose we have found in the given phase portrait a pattern as shown in figure 4, in which bundle of orbits $\phi$ is transverse to regions $I$ and $J$ on the same hypersurface such that $J \subset I$. This pattern is called a contracting recursive bundle of orbit intervals, and it entails that all orbits transverse to $I$ are also transverse to $J$, and never leave the region occupied by $\phi$ as far as uniqueness of solution holds. If the phase space is two-dimensional, region $J$ is finitely bounded, and region $I$ and $J$ do not share boundary, then $\phi$ contains one or more limit cycle $\{a_1, \ldots, a_k\}$ such that all orbits transverse to $I - J$ approach one of these limit cycles as $t \to \infty$. A bundle of orbit intervals delimited by the innermost and outermost limit cycles in $\{a_1, \ldots, a_k\}$ (including these two limit cycles) are called an attracting bundle of orbit intervals.

Consider again the Van der Pol's equation (1) and its phase portrait shown in figure 1. Notice that the observation we have made in section 1 is only a conjecture and hence is not a proved one. Based on the theoretical framework introduced above, we can give a more concrete justification as follows:

1. let us introduce a straight line $AB$ as shown in figure 5;
2. track a bundle of orbits transverse to $AB$ at $PQ$ by numerical integration until it is about to cross $AB$ again to the same direction; let us denote the images of $P$ and $Q$ by that section of bundle of orbits as $\phi(P)$ and $\phi(Q)$, respectively;
3. since $\phi(P)\phi(Q) \subset \phi(P)\phi(Q)$ holds, we can conclude that there exists an attracting bundle of orbit intervals which is transverse to $\phi(P)\phi(Q)$.

Note that the above is not a mathematical proof, but can be called a numerical justification. Numerical proof is not rigid in a mathematical sense but it makes a lot of sense in practical applications. For example, numerical justifications provide a lot of insights and a good motivation of seeking for a rigorous mathematical proof.

4 Outline of the Algorithm

Roughly, PSX2NLM performs the following steps to reason about the long-term behavior:

(step 1) divide the given region of analysis into smaller convex cells;
(step 2) characterize the flow in each cell as a set of flow mappings;
(step 3) reason about global flow by chaining flow mappings for each cell.

\footnote{Actually, only two landmark orbits passing on $P$ and $Q$ are tracked in a physical sense.}
does not much depend on the class of ODEs in hand, while the other two steps do. In the rest of this paper, we survey algorithms for local and global analysis.

5 Algorithms for Local Analysis

The purpose of local analysis is to generate a set of flow mappings for each cell in the phase space. A general strategy we take for local analysis is to analyze the flow at the boundary of a given cell to infer the flow inside the cell.

Let us call a maximal continuous segment of the boundary at which orbits enter (leave) the cell an entrance (exit) segment. Orbits are tangent to the boundary at other sections of the boundary. We call each maximally continuous section of such boundary segments a singular node. A singular node usually consists of a single point. Singular nodes are further classified into three: if the orbit passing through a singular node lies outside the region immediately before and after the visit, the singular node is called a convex node; otherwise if the orbit lies inside the region immediately before and after the visit, it is called a concave node; otherwise it is called a degenerate singular node and does not play an important role in local analysis.

Identifying singular nodes, their type, and their approximate location provide a useful clue of capturing local flow. So far, we have studied two-dimensional flows and obtained the results described in the following two subsections.

5.1 Two-Dimensional Piecewise Linear Differential Equations

Piecewise linear differential equations are specified as a collection of locally-defined linear flows. As for linear flow, it is easy to identify all fixed points and their type, and to divide the phase space into convex regions so that the interior of each region may not contain fixed points or intersect with real eigenspaces or boundaries between linear regions. In addition, it is possible to identify all convex and concave nodes on the boundary of a given convex region by simple computation, and generate all possible mappings between boundary segments by analyzing distribution of convex nodes and concave nodes.

Notice that we have to take into account several special cases. First, the boundary of a cell may lie on an invariant manifold of a fixed point. We consider such boundary segment as a special case of convex nodes, for the orbit lies outside immediately before and after it passes on the boundary segment. Second, a fixed point may lie on the boundary. Such a fixed point is taken as a special case of an entrance segment if it is a source; an exit segment if it is a sink; a part of a convex node if it is a saddle.

Orbits in closed cells originates from entrance segments and tend to exit segments. The following properties are important in identifying the flow pattern in a closed cell.

Property 1 Let the number of convex nodes and concave nodes be \( n_c \) and \( n_e \), respectively. Then,

\[
\frac{1}{2} = n_c + n_e.
\]

Property 2 If at least one concave node is involved on the boundary of a cell, there exists a sequence of three consecutive convex nodes that have no concave node in between. The second of the three convex nodes is called the center node.

Figure 6 illustrates a typical distribution of singular nodes on the boundary of a closed cell.

As for two-dimensional linear flows, every invariant manifold...
Property 3 If a concave node $c$ is involved in the boundary, then the orbit passing on $c$ intersects each of the two sequences of the boundary segments between $c$ and the center node.

Property 4 The orbit passing on a concave node $c$ does not intersect boundary segments adjacent to $c$.

A sequence of all boundary segments ordered clockwise from the center node is called a left-cyclic boundary segment list. Boundary segment $s$ of a cell is said to be to the right (left) of boundary segment $p$ of the same cell if $s$ appears after (before) $p$ in the left-cyclic boundary segment list. An orbit is said to pass through a cell from left to right if the boundary segment through which the orbit leaves the cell is to the right of the boundary segment through which the orbit enters the cell. Orbits that pass through a cell from right to left are similarly defined. The following property holds:

Property 5 For all cells, one of the following conditions holds:

- all orbits that pass through the cell from left to right (left-right flow)
- all orbits that pass through the cell from right to left (right-left flow).

It is easy to see whether a local flow is left-right or vice versa. For example, one might see the orientation of the flow at the first boundary segment of the left-cyclic boundary segment list. If it is an entrance segment, then the flow is left-right; otherwise the flow is right-left.

Property 6 It follows from the non-intersection constraint of orbits and property 3 that all orbits inside a cell are nested around the center node.

Example

If the number of concave nodes on the boundary is at most one (as shown in figure 7(a)), we can uniquely determine the pattern of flow in the cell from the flow at the boundary, as shown in figure 7(b). Note that the absolute location of singular nodes is not necessary in global analysis; only their location relative to other significant location is needed. This is an example in which qualitative analysis has brought about useful conclusion.

In contrast, if the cell has more than one concave node, the number of possibilities increases quickly. For example, if there are three concave nodes on the boundary as shown in figure 8, there are eleven possible ways of qualitatively different patterns of local flow in this cell. Figure 9 shows a few of them.

Local flow in an open cell is more complex, for equation (5) does not hold any more. However, we can reduce the problem of analyzing local flow in open cells to that of closed cells, by extending the set of boundary segments with additional virtual boundary entities such as points at infinity or an edge at infinity and giving them appropriate attributes.

For example, consider a local flow in an open cell shown in figure 10, which has a concave node $A$ on the...
boundary. Although both entrance segments and exit segments exist on the boundary, the number of convex node segments is one less than the required. Hence, we regard the edge at infinity a convex node segment. In this case, local flow in this cell at the inter-boundary segment level is uniquely characterized as the following set of mappings:

\[ l_{1,\infty} A \rightarrow \phi(A) l_{2,\infty} \oplus C \rightarrow AB \oplus C \rightarrow D \phi(A), \]

where \( l_{a,\infty} \) stands for a point at infinity. Figure 11 illustrates this interpretation.

We have made an extensive study of local flow defined by linear differential equations and we have implemented a program called PSX2PWL. The details are reported in [4].

### 5.2 General Two-Dimensional Nonlinear Differential Equations

Lots of difficulties arise when we are to understand the behavior of two-dimensional nonlinear differential equations: no general theory is known, behavior is complex, and so on. In this section, we address the incom-

**Figure 8:** a Closed Cell

**Figure 9:** Several Qualitatively Different Patterns of Local Flow for the Cell shown in Figure 8

**Figure 10:** Example of Local Flow in an Open Cell

**Figure 11:** Interpretation of the Flow in figure 10
pleteness of information, which means that complete information may not be available due to the complexity of mathematical problems encountered. This implies, for example, that the procedure may fail when it is to divide the phase space into uniform cells or to annotate flow at the boundary of cells.

PSX2NL, a successor of PSX2PWL, is designed to overcome some of these difficulties. First, PSX2NL can switch to multiple analysis procedures, ranging from those producing rigorous results through requiring to solve complex equations to those producing less reliable result without requiring much. When everything goes well, PSX2NL analyzes the flow using the most rigorous method it knows; otherwise it gradually resorts to numerical methods which can only provide plausible information. The kernel of integrating these heterogeneous analysis procedures is a qualitative analysis procedure which determines such issues as when to run a numerical simulator and how to interpret the result.

Second, we introduce a grammatical inference to cope with the case when no analytical information is available. The basic procedure for handling the case is to enumerate all possible patterns of flow in turn, compare each pattern with a given flow, and pick out one which achieves the best match. To represent a pattern of flow, we use a \textit{flow pattern} consisting of a set of flow mappings and description of geometric objects referred to by flow mappings. \textit{Flow grammar} specifies the set of all possible structurally stable flow patterns one may encounter.

We use a tuple \((n_v, n_c, n_s, n_{ss})\) as an index to a finite set of flow patterns which has \(n_v\) convex nodes, \(n_c\) concave nodes, \(n_s\) saddles, and \(n_{ss}\) sinks or sources. \(fp(n_v, n_c, n_s, n_{ss})\) denotes the set of all flow patterns with a given index. Although it appears a four-dimensional addressing system, \(fp(n_v, n_c, n_s, n_{ss})\) in fact is three-dimensional, due to the following theorem:

\textbf{Theorem 1} For any \(C^1\) flow and a cell \(R\), let the number of saddle nodes and sinks or sources in \(R\) be \(n_v\) and \(n_{ss}\), respectively. Let the number of convex nodes and concave nodes at the boundary \(\partial R\) of \(R\) be \(n_c\) and \(n_s\), respectively. Then we have the following relation:

\[ n_v - n_c = 2 \times (n_s - n_{ss} + 1). \quad (6) \]

For more detail the reader is referred to [5].

We have finished a preliminary implementation of a program called PSX2NL for analyzing behavior of general two-dimensional ordinary differential equations [6]. We will now turn to several examples.

\section*{Example}

Consider the flow in region \(ABEF\) in figure 3.\(^6\) PSX2NL recognizes concave nodes \(P, Q, R,\) and \(S\) and use the Runge-Kutta algorithm to numerically trace the orbit passing on these concave nodes.

Consider a flow:

\[ \begin{align*}
\frac{dx}{dt} &= x^2 - xy \\
\frac{dy}{dt} &= -y + x^2.
\end{align*} \quad (7) \]

The result of local analysis obtained by PSX2NL is shown in figure 12, where qualitative knowledge suggests that orbits passing through concave nodes \(A, B, C,\) and \(D\) are crucial and should be tracked. Interesting use of qualitative reasoning is to cope with error or slow convergence of a numerical simulation. For example, if a tracked orbit enters some neighborhood of a known fixed point, say \(X\) in figure 12, one rule suggests that it would be beneficial to tentatively conclude that the asymptotic destination of the orbit is the fixed point and shift the focus to other interesting phenomenon. Alternatively, PSX2NL suspects the possibility of the existence of a limit cycle if the speed of convergence is slow, as is for the orbit passing on \(C\) and tending to the fixed point \(Y\) in figure 12.

\section*{6 Algorithm for Global Analysis}

The purpose of global analysis is to examine the structure of compositions of local mappings obtained in local analysis. The global characteristics of phase portraits by constructing in turn compositions of mappings representing a bundle of orbit intervals.

Formally, given a couple of bundles of orbit intervals \(\phi_1 : I \to J\) and \(\phi_2 : J \to K,\) composition \(\phi_2 \circ \phi_1\) of \(\phi_1\) and \(\phi_2\) is defined as follows:

\[ \phi_2 \circ \phi_1(x) = \{ y | \exists z \in J : \phi_1(z) = x, \phi_2(z) = y \}. \]

\(\phi_2 \circ \phi_1\) is homomorphic if \(\phi_1\) and \(\phi_2\) are homomorphic. Mapping \(\phi_m \circ \ldots \circ \phi_1\) is called a \textit{contracting recursive}.

\(^6\)The principle for dividing the phase space into cells is different from that employed for PSX2PWL. PSX2NL divides the phase space so that no fixed point may be located on the boundary. This is generally easy for systems which have no dense cluster of fixed points, as usually is the case. In addition, whenever possible PSX2NL tries to decompose the cell so that at most one fixed point is contained in each cell. This would not be very smart if the flow was linear. However, it is less affected by the incompleteness of information.
mapping, if its range is a subset of its domain, namely,
\[ \phi_m \circ \ldots \circ \phi_1(I) \subseteq I. \]

Similarly, \( \phi_m \circ \ldots \circ \phi_1 \) is called an extending recursive mapping, if
\[ \phi_m \circ \ldots \circ \phi_1(I) \supseteq I. \]

The existence of a contracting (extending) recursive mapping entails the existence of an attracting (repelling) bundle of orbit intervals.

The complexity of global analysis significantly differs, depending on whether uniqueness of solution is the case. If uniqueness of the solution holds, it is relatively easy to implement an effective algorithm for global analysis.

When uniqueness of solution does not hold, the analysis of phase portrait becomes harder, for orbits can tangle in an arbitrarily complex manner. See orbits in figure 13 for illustration of such situation. Here, bundle of orbit intervals \( \phi_1 \) and \( \phi_2 \) merge together and flow into \( \phi_3 \), part of which in turn \( \phi_3 \) merges with \( \phi_8 \) and flows into \( \phi_4 \). There is a loop structure consisting of \( \phi_4 \), \( \phi_6 \), and part of \( \phi_8 \). This is not a contracting recursive bundle of orbits in a strict sense, for it contains a nondeterministic branch to \( \phi_7 \). But it is useful to recognize these loops as a kind of recursive mappings and use them as a clue of finding recursive mappings in a strict sense. The notion of coincidence is used to

Figure 12: A Snapshot of Local Analysis by PSX2NL for (7)

Figure 13: Collection of Bundle of Orbit Intervals with Complex Connectivity
Figure 14 illustrates how PSX2NL extends landmark orbits. For example, the orbit passing at a vertex $F$ is tracked and its image $\phi(F)$ by the flow in cell $ECDF$ is obtained. The orbit is further extended in cell $ABEF$ and $\phi_1 \circ \phi_2(F)$ is obtained, and so forth. The process continues until the global characteristic of the behavior is captured.

7 Concluding Remarks

This work was inspired by early efforts for combining qualitative and quantitative analysis to explore phase portraits [7, 1, 8]. Unlike our predecessors, we have put more stress on inventing representation so that qualitative and quantitative methods interact in a more effective fashion. In this paper, we have presented a method of representing orbits as mapping and pointed out that it is computationally feasible for two-dimensional flows: completely for piecewise linear differential equations and partly for other nonlinear ordinary differential equations. Although the algorithm for producing initial set of mappings depends on the class of given differential equations, the algorithm for global analysis is common and can be shared.

Two issues remain: extending the method for the remaining cases of two dimensional ordinary differential equations and for higher dimensional systems. To achieve these goals, we have to rely more on heuristics, requiring more tight coupling of qualitative and quantitative analysis.

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References


