Optimal three-dimensional flight control of a supersonic fighter

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ABSTRACT

This paper discusses optimal three-dimensional flight control of a supersonic fighter through an onboard automatic guidance and control system. This system is simulated to check the speed requirements of the algorithms to be solved before being implemented in real hardware. A very-high-speed digital computer is used for this time-critical simulation. In the simulation optimal trajectories are generated in real time, on line. Results obtained from the particular problem of a real-time, online minimum-time supersonic chandelle with prescribed final point are displayed.
INTRODUCTION

This paper discusses optimal three-dimensional flight control of a supersonic fighter through an onboard automatic guidance and control system. The essential portion of this system consists of a fast computer system called the mission computer and another fast computer system called the flight control computer. The function of the mission computer is to generate real-time, online optimal trajectories; the function of the flight control computer is to track the trajectories generated by the mission computer. The automatic guidance and control system is simulated to check the speed requirements of the algorithms to be solved before implementing it in real hardware. Naturally the simulation is very time-critical and therefore requires a very-high-speed computer to perform the simulation. In this paper, the computer used for this simulation is a very-high-speed special-purpose digital computer designed specifically for time-critical, continuous system simulation tasks.

Simulation of real-time, online optimal trajectories of supersonic flight has been discussed by Lin. In these references, for each maneuver a family of trajectories is precalculated off line and stored in the mission computer. These trajectories are then generated in real time, on line, by the mission computer by table lookup. They are then checked by computer simulation to see if they can be adopted in real-time, online computation. If so, the flight control computer will track these optimal trajectories generated by the mission computer. However, a substantial computer memory space is required to store all these precalculated trajectories; therefore this approach creates a problem, since the mission computer has a limited memory space. To avoid this problem, optimal trajectories in this paper are generated entirely in real time, on line, without table lookup. The particular problem of a real-time, online minimum-time supersonic chandelle with prescribed final point is used as an example in this paper. Several techniques are used to obtain real-time, online optimal trajectories of this maneuver. They include (1) modeling of the aerodynamic and engine characteristics of a typical lightweight, high-thrust-to-weight ratio supersonic fighter; (2) introduction of a set of dimensionless variables, which leads to general results for a whole class of vehicles having similar physical characteristics; (3) general properties of optimal trajectories; and (4) use of the switching theory.

THEORETICAL ANALYSIS

If the thrust is considered as nearly aligned with the velocity vector \( V \), then the motion of a point mass lifting vehicle over a flat nonrotating earth, with the assumption of symmetrical flight, is governed by the following set of nonlinear ordinary differential equations.

\[
\begin{align*}
\dot{X} &= V \cos \psi \cos \phi \\
\dot{Y} &= V \cos \phi \sin \phi \\
\dot{Z} &= V \sin \phi \\
\dot{\phi} &= g \frac{L \cos \phi}{W} - \cos \gamma \\
\dot{\psi} &= g \frac{L \sin \phi}{VW} \\
\end{align*}
\]

In these equations, the position vector is composed of the longitudinal range \( X \), the lateral range \( Y \), and the altitude \( Z \); and the velocity vector is composed of the speed \( V \), the flight path angle \( \gamma \), and the heading \( \psi \). For turning flight, during a relatively short interval, we can neglect the mass flow equation and consider the weight as practically constant. The acceleration of the gravity \( g \) is assumed to be constant. The aerodynamic and propulsive forces are determined by the following relations

\[
\begin{align*}
L &= V^2 \rho (Z) V^2 C_{\alpha} \\
D &= V^2 \rho (Z) V^2 C_D \\
T &= \zeta T_{max}(Z, M) \\
\end{align*}
\]

where \( \rho (Z) \) is the atmospheric density given as a tabular function, and

\[
0 \leq \zeta \leq 1
\]

represents the thrust control parameter. For a parabolic drag polar, as function of the Mach number, we have

\[
C_D = C_{D0}(M) + K(M) C_L^2
\]

where the zero-lift drag coefficient, \( C_{D0} \), and the induced drag coefficient, \( K \), are functions of the Mach number. Because of the lift-drag relation, the flight is controlled by the lift coefficient \( C_L \) which is equivalent to the angle of attack \( \alpha \), the bank angle \( \phi \), and the thrust magnitude \( T \).

The Hamiltonian to the variation problem is

\[
H = P_x V \cos \psi \cos \phi + P_y V \cos \psi \sin \phi + P_z V \sin \phi + P_\phi g \left( \frac{T - D}{W} - \sin \gamma \right) + P_\psi g \frac{L \cos \phi}{W} - \cos \gamma + P_\phi g \frac{L \sin \phi}{VW \cos \gamma}
\]

It is known that the problem has the integrals

\[
H = C_0, \quad P_x = C_1, \quad P_y = C_2, \quad P_z = C_3 \quad \text{with } C_0 > 0.
\]

The problems considered here are minimum time problems. Hence, for maximization of the Hamiltonian, \( C_0 > 0 \). By using the Hamiltonian integral only two of the three remaining adjoint variables \( P_x, P_y \) and \( P_z \) need to be found. In vertical flight all three variables are involved, hence their presence imposes a difficulty in solving the optimization problem, the
same difficulty encountered in three-dimensional flight problems. As compared to the vertical flight, three-dimensional flight has two more variables, i.e., the lateral range $Y$ and the heading $\psi$, which may pose a problem in obtaining the optimal solution; however, the adjoint variables $P_v$ and $P_\psi$ associated with these variables have been found in Eqs. (8). In the case of horizontal flight, the adjoint variables $P_\gamma$ and $P_\delta$ are not present, and the remaining adjoint variable $P_v$ is given by the Hamiltonian integral. The problem is completely solved in the book by Lin. 

Since the aerodynamic and the engine characteristics are functions of the Mach number, it is convenient to use the following dimensionless variables:

$$M = V/a(Z), \quad \omega = 2W/kp(Z)S$$

where $a(Z)$ is the speed of sound and $p(Z)$ is the ambient pressure. Both $a(Z)$ and $p(Z)$ are given as tabular functions; and $\omega$ is the dimensionless wing loading, which is a function of the ambient pressure $p$, and is also a variable representing the altitude. For convenience of notation, we define

$$
\begin{align*}
\lambda &= C_l/C_l^*, \\
\lambda &= \sqrt{C_d/C_l^*}, \\
\tau &= C_l^*(M, w) - T_{max}W
\end{align*}
$$

where $C_l^*$ is the maximum lift-to-drag ratio, which is a performance characteristic; and $\lambda$ is the normalized lift coefficient, which can be considered a control variable for the lift control. The normalized lift coefficient is rescaled so that when $\lambda = 1$, the lift coefficient is equal to the lift coefficient $C_l^*$ for maximum lift-to-drag ratio. Note both $E^*(M)$ and $C_l^*(M)$ are functions of the Mach number and that the maximum thrust-to-weight ratio $T_{max}(M, \omega)$ is a function of the Mach number as well as $\omega$. The lift control is bounded by an upper limit $\lambda_{max}(M)$ that corresponds to $C_l_{max}(M)$. It is assumed that $C_l_{max}(M)$, $K(M)$, $\tau_{max}(M, \omega)$ and $C_l_{max}(M)$ are known functions of the Mach number. For numerical computation, we use data of a supersonic fighter assembled in Lin, but the same procedure applies to any other set of data.

The three-dimensional turning flight is a difficult maneuver; hence we use the load factor $n$ as a lift control variable to represent the angle of attack. The load factor $n$ is equal to the dimensionless life force $\ell$ defined as

$$n = \ell = L/W = M^2 C_{l}/\omega = \Delta \lambda$$

where

$$\Delta = M^2 C_{l}^*/\omega$$

The load factor $n$ can also be used as a control variable to replace the normalized lift coefficient $\lambda$, and it is an important parameter that can limit the flight domain. From Eq. (11), since $\lambda \leq \lambda_{max}(M)$, the flight domain is bounded by the curve

$$n = \Delta \lambda_{max}(M)$$

Hence, the load factor $n$ is subject to the constraints

$$|n| \leq n_{max} = \inf \left[ n_{n}, \frac{M^2 C_{l_{max}(M)}}{\omega} \right]$$

where the constant value $n_n$ is a physiological/structural constraint and $n_{max}$ is the maximum permissible load factor, either for $C_l = C_{l_{max}}$ or $n = n_n$. This domain of flight may be further restricted by the line of maximum dynamic pressure

$$M^2 \leq \frac{2 \bar{q}_{max}}{k_p} = \frac{S}{W} \bar{q}_{max}$$

and the line of maximum Mach number obtained by solving the equation $dV/dt = 0$ with $\tau = \tau_{max}$ such that

$$\tau_{max} = \frac{1}{2E^*} \left( \Delta + \frac{n^2}{\Delta} \right) + \sin \gamma$$

By using the notations introduced thus far, the Hamiltonian (7) becomes

$$H = C_l V \cos \gamma \cos \psi + C_l V \cos \gamma \sin \psi + P_\gamma V \sin \gamma$$

$$+ P_\delta \left[ \tau_{max}(M, \omega) - \frac{1}{2E^*} \left( \Delta + \frac{n^2}{\Delta} \right) - \sin \gamma \right]$$

$$+ P_\gamma \frac{E^*}{V} (\cos \phi - \cos \gamma) + P_\delta \frac{n \sin \phi}{V \cos \gamma}$$

In this formulation, the control variables are the thrust parameter $\xi$, the bank angle $\phi$, and the load factor $n$. The thrust parameter $\xi$ and the load factor $n$ are subject to constraints (5) and (14) respectively, and the bank angle $\phi$ is subject to the constraint $|\phi| \leq \phi_{max}$. The $\phi_{max}$ can be either a constant or a function of the state variables, depending on the problem considered.

Regarding the thrust control, we consider the adjoint $P_v$, called the switching function. Then to maximize the Hamiltonian, if

$$P_v > 0, \quad \xi = 1 \quad \text{• Boost arc (B arc)}$$

$$P_v < 0, \quad \xi = 0 \quad \text{• Coast arc (C arc)}$$

$$P_v = 0 \quad \text{for } t \in [t_1, t_2] \quad \text{• Sustained arc (S arc)}$$

$v = \text{variable}$

The optimum trajectory is a combination of boost arc (B arc), coast arc (C arc), and sustained arc (S arc). At the junction of the different thrust control arcs, $P_v = 0$. For a junction between arcs, a C-B sequence is optimum if at the junction $dP_v/dt > 0$; for a reverse condition, a B-C sequence is optimum.

The aerodynamic control consists of the bank angle $\phi$ and the load factor $n$. The optimal aerodynamic control can be obtained by using the technique of the domain of maneuverability presented by Lin. First, whenever interior bank angle and interior load factor are used, we have

$$\tan \phi = \frac{P_\gamma}{P_v \cos \gamma}$$

$$n^2 = \frac{\Delta E^*}{V^2 P_\gamma^2} \left( P_\gamma^2 + \frac{P_\delta^2}{\cos \gamma} \right)$$

By applying Eq. (19) to Eq. (20), we have

$$n = \Delta E^* \frac{P_v}{V^2 P_\gamma \cos \phi}$$

If all $P_v$, $P_\gamma$, and $P_\phi$ approach zero simultaneously, the in-
determination of $\phi$ and $n$ can be resolved by applying L'Hôpital's rule, which leads to

$$\tan \phi = \frac{A}{B}$$

and $n$ is given in Eq. (19).

The adjoint equations of $P_v$ and $P'Y'$ are coupled with the equation of $P_z$. With the existing integrals (8), one of the three adjoint equations is given in Eqs. (27), with $\xi = 1$ and $\phi = \phi_{\text{max}}$ and $n$ is given in Eq. (25). For boundary bank angle and boundary load factor, if $n = n_s$, then for a constant $\phi_{\text{max}}$ the adjoint equations are given in Eqs. (27) with $\xi = 1$ and $\phi$ is given in Eq. (19). If $n = M^2C_{l_{\text{max}}}^2/\omega$

$$\frac{dP_v}{dt} = -C_v + P_v \left[ \xi \tau(2 - \tau_{\text{max}}) + 2 \sin \gamma \right]$$

$$+ \frac{1}{2E^*} \left[ \left( \Delta - \frac{n^2}{\Delta} \right) C_{l_{\text{max}}} \left( \Delta + \frac{n^2}{\Delta} \right) E^* - 4 \pi \right]$$

$$+ P_v \frac{2n}{n \cos \gamma} \left( \cos \gamma - \cos \phi \right) + P_v \frac{2n \sin \gamma}{n \cos \gamma}$$

In the above equations, the Hamiltonian integral has been used for simplification, and the subscript $M$ is defined as

$$(28)$$

In Eqs. (27), $\xi = 1$, $\phi$ is given in Eq. (19), and $n$ is given in Eq. (21). For interior bank angle and boundary load factor, if $n = n_s$, the adjoint equations are given in Eqs. (27) with $\xi = 1$, and $\phi$ is given in Eq. (19). If $n = M^2C_{l_{\text{max}}}^2/\omega$

then for a constant $\phi_{\text{max}}$ the adjoint equations are given in Eqs. (29), with $\phi = \phi_{\text{max}}$ and $n = n_s$. In the most general case, the solution requires the estimate of five parameters $C_1, C_2, C_3$ and the initial adjoint variables $P_v$, $P'Y'$. This, coupled with the optimal switching from one control regime to another, constitutes the main difficulty of the problem. Success in obtaining the solution depends on the knowledge of the particular flight program considered. A very-high-speed digital computer is used for the computation. The total computing time for a single pass through the entire equations for optimal flight in three dimensions is approximately 457.5 microseconds. This shows that integration frame rate of up to 2186 per second can be accomplished.
\[
\frac{dP_y}{dt} = \frac{1}{V} \left[ -C_0 + P_0 \left( \frac{2g}{V} \left( n \cos \phi - \cos \gamma \right)\right) - g \cos \phi \frac{dn}{dV} \right]
\]

\[
+ P_0 \left( \frac{2g n \sin \phi}{V \cos \gamma} - \frac{g \sin \phi}{\cos \gamma} \frac{dn}{dV} \right) = 0
\]

(30)

Thus from this equation it is clear that \( P_y \neq 0 \) and \( P_0 \neq 0 \).

If

\[
\frac{\partial n}{\partial V} = 0
\]

then Eq. (30) becomes

\[
\frac{dP_y}{dt} = \frac{1}{V} \left[ -C_0 + P_0 \frac{2g}{V} \left( n_{\text{max}} \cos \phi - \cos \gamma \right) \right]
\]

(31)

The variable thrust along a sustained arc is obtained by taking the derivative of this equation using the available singular conditions

\[
\frac{d^2P_y}{dt^2} = -C_0 \frac{g n_{\text{max}}}{V^2} < 0
\]

(32)

where the term in parentheses is a function of the state variables and constants of integration. Since the order of the singular arc is \( q = 1 \), then according to the generalized Lengendre-Clebsch condition, a necessary condition for the optimality condition of the singular arc is that

\[
C_0 \frac{g n_{\text{max}}}{V^2} \leq 0
\]

(33)

This condition is not satisfied with \( C_0 > 0 \). If \( n = M^2 C_{\text{max}} / \omega \), Eq. (30) becomes

\[
\frac{dP_y}{dt} = -\frac{1}{V} \left[ C_0 + P_0 \frac{2g}{V} \left( 2 \cos \gamma + n \cos \phi C_{\text{max}} \right) \right]
\]

\[
+ P_0 \frac{g n \sin \phi}{V \cos \gamma} C_{\text{max}} = 0
\]

(34)

By taking the derivative of Eq. (34) using the available singular conditions, the equation for the intermediate thrust control is obtained in the form

\[
\frac{d^2P_y}{dt^2} = \frac{A \xi}{V^2} + \left( \ldots \right) = 0
\]

(35)

where

\[
A = -\frac{g}{V^2} \left[ C_0 + C_{\text{max}} \left( P_0 \frac{g n \cos \phi}{V} + P_0 \frac{g n \sin \phi}{V \cos \gamma} \right) \right]
\]

\[
+ \left[ 2 + C_{\text{max}} + C_{\text{max}} \frac{M^2}{n_{\text{max}}^2} \right] \right]
\]

(36)

and the term in parentheses in Eq. (35) is a function of the state variables and constants of integration. According to the generalized Lengendre-Clebsch condition for the optimality of the singular arc, \( A > 0 \). If \( C_{\text{max}} \) is independent of the Mach number, the condition \( A > 0 \) is not satisfied. This is particularly true for the case of maneuver at low Mach number. For any prescribed function \( C_{\text{max}}(M) \), the condition \( A > 0 \) defines a small region in the state variable and adjoint variable space where singular arc can be optimal.

With only \( B \) arc and \( C \) arc involved, the optimal thrust control is a combination of \( B \) arc and \( C \) arc. At the junction of a \( B \) arc and a \( C \) arc, \( P_y = 0 \). For continuity of the load factor this occurs either when \( n = n_s \) or \( n = M^2 C_{\text{max}} / \omega \), or \( P_y = 0 \). If all \( P_y \), \( P_0 \), and \( P_0 \) approach zero simultaneously, and the interior bank angle and interior load factor are used, then the indeterminations of \( \phi \) and \( n \) are given by Eq. (22) and Eq. (23), respectively. At this point, from the equation for \( P_y \) in Eq. (30) with \( P_y = 0 \) and \( P_0 = 0 \),

\[
\frac{dP_y}{dt} = -\frac{C_0}{V} < 0
\]

(37)

Hence the connection is from a \( B \) arc to a \( C \) arc. If all \( P_y \), \( P_0 \), and \( P_0 \) approach zero simultaneously, and the boundary bank angle and interior load factor are used, the indetermination of \( n \) is given by Eq. (26). At this point, from the equation for \( P_y \) in Eq. (30) with \( P_y = 0 \) and \( P_0 = 0 \), Eq. (37) is true. Hence the connection is from a \( B \) arc to a \( C \) arc. If the discontinuity of the angle of attack is neglected, the switching is always at \( n = n_s \) or \( n = M^2 C_{\text{max}} / \omega \). If it is at \( n = n_s \), then a switching from a \( C \) arc to a \( B \) arc is optimal if at the switching point \( dP_y/dt > 0 \), i.e., from Eq. (31), we have

\[
P_y \frac{2g}{V} \left( n_{\text{max}} \cos \phi - \cos \gamma \right) + P_0 \frac{2g n \sin \phi}{V \cos \gamma} > C_0
\]

(38)

For a switching from a \( B \) arc to a \( C \) arc the above inequality is reversed. If the switching is at \( n = M^2 C_{\text{max}} / \omega \), then a switching from a \( C \) arc to a \( B \) arc is optimal if at the switching point \( dP_y/dt > 0 \), i.e., from Eq. (34) we have

\[
C_0 + P_0 \frac{g}{V} \left( 2 \cos \gamma + n \cos \phi C_{\text{max}} \right)
\]

\[
+ P_0 \frac{g n \sin \phi}{V \cos \gamma} C_{\text{max}} < 0
\]

(39)

This inequality is reversed for a switching from a \( B \) arc to a \( C \) arc.

COMPUTATIONAL RESULTS

The problem of minimum-time supersonic chandelle with free final longitudinal range \( X_f \) and free final lateral range \( Y_f \) is completely solved in real time, on line, by Lin. Instead of free final longitudinal and lateral ranges, as in Lin, this paper focuses on prescribed final longitudinal range \( X_f \) and prescribed final lateral range \( Y_f \). Furthermore, the final altitude is prescribed. Thus this problem is minimum-time supersonic chandelle with prescribed final point. This particular three-dimensional turn can be analyzed in comparison with the trajectories on a horizontal and a vertical plane, as shown in Figure 1.

Figure 1 gives a comparison of three 180° turning maneuvers, i.e., the horizontal or level turn, the vertical turn or Immelman, and the chandelle. These three turnings are useful maneuvers in combat. In the figure, I, II, and III represent three different positions of the intruder’s inbound. Consider the offset point of our aircraft. The goal of all three turning maneuvers is to reach the attack cone, defined as any position from which we can maneuver and overtake the target in the position of the weapon-firing...
ranges. In general, the attack cone lies in the 5 to 7 o'clock position behind the target and at coaltitude or slightly below the target altitude. Depending on the position of the intruder’s inbound, an appropriate turning maneuver is chosen for interception. For example, if the position of the intruder’s inbound is Location I, then the horizontal turning maneuver is used. This maneuver as shown in the figure is a turn to a heading. If the position of the intruder’s inbound is Location II, the vertical turning maneuver needs to be performed to get to the attack cone. The vertical turning maneuver as shown in the figure is a particular vertical turn called the Immelman. If the position of the intruder’s inbound lies between Locations I and II, e.g., in Location III in the figure, the appropriate tactical maneuver to choose is the chandelle, which is a three-dimensional 180° climbing turn. For simplification, in this paper we define the offset point as the position when the fighter arrives at Mach two and is ready to immediately initiate the minimum-time supersonic chandelle to reach the attack cone. The altitude of this position is referred to as the initial altitude of the offset point.

In this problem we have the terminal conditions

\[ t = 0, \ X = 0, \ Y = 0, \ Z = Z_0, \ \dot{V} = V_0, \ \gamma = \gamma_0, \ \psi = 0^\circ \]  

(40)

\[ t_f = \text{min.}, \ \ X = X_f, \ \ Y = Y_f, \ \ Z = Z_f, \ \ V_f = \text{free}, \ \ \gamma_f = \text{free}, \ \ \psi_f = 180^\circ \]  

(41)

Hence, we have the transversality conditions

\[ C_0 = 1, \ P_{\psi_i} = 0, \ P_{\gamma_i} = 0 \]  

(42)

Since \( P_{\psi_i} = P_{\gamma_i} = 0 \), by continuity of the bank angle and load factor the last portion of the trajectory must be flown with boundary bank angle and boundary load factor. For \( \eta_t = \eta_s \), if the terminal point is considered a switching point, then condition (38), with \( P_{\gamma_i} = 0 \) at the final time, dictates a final C arc. If this resulting inequality reverses, the final arc is a B arc. For \( \eta_t = \frac{M^2C_{\text{max}}}{\omega_5} \), if the terminal point is considered a switching point, condition (39) with \( P_{\gamma_i} = 0 \) at the final time dictates a final C arc. If this resulting inequality reverses, the final arc is a B arc. The problem in terms of \( C_1, C_2, C_3, P_{\psi_i}, \) and \( P_{\gamma_i} \) is a five-parameter problem. The five parameters \( C_1, C_2, C_3, P_{\psi_i} \) and \( P_{\gamma_i} \) are to be selected to satisfy the final
conditions (41) and the transversality conditions (42). For the solution we guess $C_1$, $C_2$, $C_3$, $P_{\psi_0}$, and $P_{n_0}$ and start the integration of the state equations (1) and adjoint Eqs. (27) and (29), along with the use of the optimal thrust and aerodynamic control law. At the final heading $\psi_f = 180^\circ$, the conditions on $X = X_f$, $Y = Y_f$, $Z = Z_f$, $P_{\psi_f} = 0$, and $P_{n_f} = 0$ are used to adjust the five unknown parameters $C_1$, $C_2$, $C_3$, $P_{\psi_0}$, and $P_{n_0}$.

Figures 2a–2h show the results of the computation obtained by running a very-high-speed digital computer in real time, on line, using the supersonic fighter as the model. These results are obtained by using the example problem of minimum-time supersonic chandelle, turning from an initial point of $X_0 = 0$ km, $Y_0 = 0$ km, $Z_0 = 8$ km, with an initial Mach two, to a prescribed final point of $X_f = 10.5$ km, $Y_f = 9$ km, $Z_f = 19$ km for Trajectory 1, and $X_f = 10.5$ km, $Y_f = 8$ km, $Z_f = 19$ km for Trajectory 2. The constraints $\phi_{\text{max}} = 1.5$ radians and $n_s = 4.5$ are imposed. The indetermination in evaluating the initial value of $P_z$ when $\gamma_0 = 0^\circ$ is avoided by using initially a slightly positive value of $\gamma_0$, since the trajectory has the tendency to start with a climb for a high initial Mach two. For the same prescribed final altitude $Z_f = 19$ km and the same prescribed final longitudinal range $X_f = 10.5$ km, Trajectory 1 has a longer prescribed final lateral range $Y_f = 9$ km and hence requires longer time to complete ($t_f = 40.1$ seconds), whereas Trajectory 2 has a shorter prescribed final lateral range $Y_f = 8$ km and hence takes a shorter time to complete ($t_f = 38.8$ seconds).

REFERENCES


