Exact solution for the initialization time of packet radio networks with two station buffers

by DANIEL MINOLI
Bell Telephone Laboratories
Holmdel, New Jersey

INTRODUCTION

The Packet Radio Network (PRNET), a store-and-forward packet-switching system sharing a single radio channel via multi-access techniques and spread spectrum, is an effective communication medium for data and voice transmission in situations requiring fast deployment, non-fixed hardware locations, encryption and anti-jamming in hostile military environments.1-3,14,15

Many investigators have developed analytical solutions for the performance of channel access schemes typically employed (pure ALOHA, ALOHA, CSMA, etc.).1-13 These studies have shown, among other things, that PRNET's can be easily saturated and become unstable unless efficient routing and flow control algorithms are used. To enable point-to-point packet transportation the network station assigns a code (label) to each repeater; the process of assigning such labels is referred to as "network initialization." The initialization procedure assumes that the network topology is not known a priori and is changing with time. Thus, the initialization procedure involves mapping of network topology, determining network structure (labels for repeaters) and transmitting labels to the repeaters.

Notwithstanding its importance,10 the problem of initialization has not been studied extensively except in References 5, 16, 17 and 18.

In this paper we present a Markov chain model which enables one to obtain in closed form the optimal rates at which repeaters and station must transmit initialization packets and labels to minimize the network initialization time in a one-hop network, two buffer station. This is an extension of the work reported in References 17 and 18.

In Reference 17 a Markov Chain model for initialization of 1-hop packet radio networks was discussed; with this model the total initialization time must be obtained numerically by solving a set of linear simultaneous equations. For an \( m \) repeater network the number of such equations turns out to be \( O(m^3) \). Hence, this model is not applicable for analyzing large networks.

The complexity of such models can be reduced by modifying the label queue management from a random selection discipline to a first-in, first-out (FIFO) discipline; the number of linear simultaneous equations that need to be solved can be reduced from \( O(m^3) \) to \( O(m^2) \). In this case we can actually go a step further, and obtain closed-form solutions. In Reference 18 the exact solution for the initialization time with one station buffer for the label queue was derived.

In this paper we formulate a new Markov Chain initialization model based on FIFO queue management and we derive exact solutions for this new model when the station has two buffers for the complete interference case. Our most important result is that the network initialization time is relatively insensitive to the station transmission rate, but the repeater transmission rate must be carefully chosen to achieve rapid initialization.

INITIALIZATION AND INITIALIZATION PROTOCOL

Network initialization must be performed whenever the network resumes operation from cold, or whenever the network topology changes. Such topology changes may occur quite frequently; this may be due to decreases in repeater transmission ranges due to battery power drainage or equipment failure; or due to the severe variations in received signal strength caused by the topology of the terrain, man-made structures, foliage, multipath distortion and fading. In addition to this problem of monitoring RF connectivity, the potential mobility of the network must be taken into consideration; the initialization algorithm and its efficiency are particularly significant in such cases.4

The initialization procedure considered in this paper—and which is typical for this operation—consists of the following steps: a non-initialized repeater transmits Repeater-On-Packets (ROPs) informing the station of its existence and unique identification; a station program determines a label for the specific repeater and the station places a Label Packet (LLF) in its Label queue for transmission to the repeater. After the repeater has received the LLF and acknowledged its receipt, the repeater is considered labeled. The time required to initialize all repeaters is the network initialization time.

We consider a PRNET with a single station and \( m \) repeaters in which all devices can communicate directly; a slotted ALOHA access scheme is assumed. In such an environment, the available channel is time slotted into segments of
duration equal to the packet transmission time, and devices are required to start transmission at the beginning of a time slot; if two or more devices transmit a packet in the same slot, none of the packets will be correctly received. When a packet is successfully received by a destination device, it is acknowledged with some acknowledgment protocol. An originating device retransmits its packet after a time-out if it does not receive the ACK. Thus if a collision occurs, after a preset time-out retransmission of the packets is scheduled at some randomly selected future slot.

The initialization process begins with all \( m \) repeaters broadcasting Repeater-On-Packets (ROP) to request a label from the station; we assume that the repeaters will transmit the ROPs into a given slot with probability \( p \). After the station receives a ROP, the station will prepare a label for the repeater and insert it into its label queue; we assume that this requires zero processing time. Duplicate ROPs received by the station are ignored.

When the station label queue is non-empty, the station will transmit into a given slot with probability \( q \). The label selected for transmission is selected with FIFO discipline from the basic queue. After the repeater receives a label packet from the station, it forwards one End-To-End Acknowledgment (ETE ACK) immediately, and halts all transmission. The station will not delete labels from the label queue until it correctly receives the corresponding ETE ACK. Upon receipt of the ETE ACK, the repeater is considered to be labeled. We assume that repeaters will not forward ROPs or labels to other repeaters.

It can be noted that at any time during the initialization process a repeater will have one of the following statuses:

1. Transmitting ROPs, but has not successfully sent one to the station.
2. Transmitting ROPs, but has already successfully sent one to the station.
3. Having received label from station in previous slot, will forward ETE ACK to station in next slot.
4. Awaiting another label from station.
5. Completed initialization—Repeater has received label from station and forwarded a successful acknowledgment.

The initialization process is begun with all repeaters having Status 1 and is completed when all repeaters have Status 5. Figure 1 illustrates the relationship between the aforementioned statuses.

Given the status of all \( m \) repeaters in the network, we define the state of a Markov chain at a particular time as the quintuple \( s = (m_1, m_2, m_3, m_4, m_5) \) where \( m_i \) denotes the number of repeaters with status \( i \) at that time. From the assumptions it is immediately clear that:

\[
\sum_{i=1}^{5} m_i = m \quad m_2 + m_3 + m_4 \leq 2 \quad m_3 + m_4 \leq 1
\]  

(1)

Transition probabilities for this model with complete interference are derived in Appendix A.

**MAIN RESULTS—STUDY OF INITIALIZATION TIME FOR TWO-STATION BUFFERS FOR FIFO LABEL QUEUE SERVICE DISCIPLINE**

In this section we present the closed form solutions for the initialization time when the station label queue has two positions, i.e., \( b=2 \). Furthermore, we study by numerical methods the optimum values of transmission rates by repeaters and station which result in minimum initialization time as a function of number of repeaters and the interference between repeaters. Finally, in the fifth section, we compare the initialization time for one- and two-station buffers.

**Two-station buffer case with FIFO service management**

The total single hop \( m \) repeater initialization time (derived in Appendix B) is given by:

\[
I = \frac{1}{mp(1-p)^{m-1}} + \frac{1}{q(1-p)} + \frac{2}{1-q} + \sum_{i=0}^{\infty} E[T_i] \quad (2)
\]

where

\[
E[T_i] = \frac{1}{p(m-i-1)(1-q)(1-p)^{m-i-1} + q(1-p)^i + h_0} + \frac{1}{q(1-p)^i + h_0}
\]

\[
+ \frac{1}{(m-i-1)(1-q)(1-p)^{m-i-1} + q(1-p)^i + h_0} + \frac{1}{(m-i-1)(1-q)(1-p)^{m-i-1} + q(1-p)^i + h_0}
\]

(3)

\[
\left\{1 + \frac{1}{q(1-p)^i + h_0}\right\}^{-1}
\]

(4)

(5)

(6)

(7)
and where:

\[ p = \text{Repeater transmission rate} \]
\[ q = \text{Station transmission rate} \]
\[ J(i) = \text{Average number of interfering repeaters, given } i \text{ repeaters are initialized.} \]

The expression is exact for complete interference, i.e.,

\[ I(i) = m - 1 - i. \]

**NUMERICAL RESULTS FOR THE TWO-STATION BUFFER CASE**

Figures 2, 3, and 4 depict the optimal parameter values and initialization time for complete interference, for \( J(i) = 2 \), and no interference, respectively, for an \( m \)-repeater network. The optimal values were obtained by point-by-point search and are accurate to two significant digits.

Figure 5 shows the initialization time as a function of \( m \), parameterized on \( J(i) \). The initialization time for a one-buffer system is shown for comparison.

The following can be observed:

1. The optimal station transmission is nearly constant at \( q^* = 0.40 \), showing a slight tendency to decrease as \( m \) increases.
2. There is only a very small difference in the optimal station rate at low or high interference.
3. For complete interference $p^* < 1/m$, and for large $m$, $p^* = 0.7/m$ (empirically determined).
4. As the interference decreases, $p^*$ increases (as expected); $p < 1/m$ and $p^* = 0.8/m$ for large $m$.
5. As the interference decreases, the initialization time decreases. There is a reduction of about 15 percent in initialization time as we go from complete interference to zero interference.

COMPARISON OF ONE-BUFFER AND TWO-BUFFER CASES

Single station buffer

It was shown in Reference 18 that the initialization time for a single hop $m$-repeater packet radio network is given by:

$$I = \sum_{i=0}^{m-1} \left\{ \frac{1}{(m-i)p(1-p)^{m-i-1}} + \frac{1}{(1-q)(1-p)^{m-i-1}} \left( 1 + \frac{1 - (1-q)(1-p)^{m-i-1}}{q(1-p)^{m-1}} \right) \right\}$$

A comparison between the previous cases shows:

1. The station's optimal transmission rate decreased 10 percent as we went from one buffer to two buffers. This is explained by the fact that in a two-buffer situation there is no pressing urgency to clear the one occupied buffer since another is available for accepting ROPs.
2. The repeater's transmission rates are unchanged on the average, indicating that the repeaters can be ignorant of the station's buffer number.
3. The initialization time decreased as we went from one buffer to two buffers. Approximately 15 percent less time is required with two buffers than with one buffer.

CONCLUSIONS

Closed-form solutions for the initialization time of a single-hop packet radio network were obtained. The solutions are for the case in which the station has one or two buffers for storing and sending labels and when it uses a first-in-first-out queue management strategy. The slotted ALOHA access scheme was assumed. The optimum values of repeater and station transmission rates as a function of the interference pattern of repeaters were experimentally obtained. These optimum rates result in minimum initialization times.

The following conclusions emerge from the studies:

1. The initialization time with two buffers at the station is approximately 15 percent smaller than with one buffer, for the same interference pattern and network size.

2. The optimal station transmission rate, $q^*$, is nearly independent of $m$ (number of repeaters), for both values of the buffer size; however, $q^*$ decreased 10 percent as we went from one buffer to two buffers. Thus, $q$ is a function of the station's architecture only.

3. The optimal repeater transmission rates were independent of the buffer size on the average, indicating that the repeaters need not be aware of the station's buffer structure.

4. The optimal repeater transmission rate increases about 20 percent as interference goes from complete to zero; the initialization time decreases about 15 percent as we go from complete interference to zero interference.

5. In both cases for large $m$ optimal repeater transmission rates were proportional to $1/m$ where $m$ is number of repeaters in the network.

Figure 4—Zero interference, two buffers.
APPENDIX A—INITIALIZATION MODEL

Transition probabilities

Because of the assumptions of complete interference and FIFO it is easy to compute the state transition matrix. For the complete interference case, a packet is successfully received by the station, if all other repeaters and the station are silent. Similarly, for a packet to be received successfully by a repeater it is required that all repeaters be silent.

We associate a status with each repeater during the initialization process, as in the second section. The state of the system is defined by the number of repeaters in each status. Given the present state of the chain \((m_1, m_2, m_3, m_4, m_5)\), where \(m_i\) is the number of repeaters in the status \(i\), the following list enumerates all possible transitions. We consider \(m_3=0\) and \(m_5=1\) separately.

1. No repeater received a label in the previous slot \((m_3=0)\)
   a. Successful ROP when station queue is not empty. This requires \(m_1>0, 0<m_2+m_4<b\). Transition to state \((m_1-1, m_2+1, 0, m_4, m_5)\) occurs with probability \((1-q)m_1p(1-p)^{m_2+m_4+1}=Z_{11}'\).
   b. Successful ROP when station queue is empty. This requires \(m_1>0, m_2+m_4=0\). Transition to state \((m_1-1, m_2+1, 0, m_4, m_5)\) with probability \(m_1p(1-p)^{m_2+m_4}=Z_{11}\). Since \(Z_{11}'\) and \(Z_{11}\) represent probabilities of mutually exclusive events, we will use one notation \(Z_1\), which will denote \(Z_1'\) or \(Z_1\), depending on the event.
APPENDIX B—DERIVATION OF SOLUTION FOR A TWO-BUFFER MODEL WITH FIFO SERVICE DISCIPLINE

By a cycle of a Markov chain we mean a section of the state transition diagram. We consider cycles $C_i$ such that

1. The $C_i$s are disjoint.
2. Any state of the chain belongs to one cycle.
3. $C_i$ and $C_{i+1}$, when considered as graphs, are isomorphic for all but possibly the first and last case.

The solution strategy involves computing the expected time to traverse one cycle, and then summing these over cycles $0, 1, \ldots, m-1$.

**General strategy**

It can be observed that closed-form computations are relatively easy when the number of states by which one can enter a cycle is one. To achieve this, we define the following cycles:

- **Cycle $-1$**: Start with $(m, 0, 0, 0, 0)$
  - End with $(m-1, 1, 0, 0, 0)$

- **Cycle $i$**
  - Start with $(m-i-1, 1, 0, 0, i)$
  - End with $(m-i-2, 1, 0, 0, i+1)$, $i = 0, 1, 2, \ldots, m-2$

- **Cycle $m-1$**
  - Start with $(0, 1, 0, 0, m-1)$
  - End with $(0, 0, 0, 0, m)$

Note that Cycle $-1$ involves only the successful sending of a ROP to the station while Cycle $m-1$ involves only the labeling of the last repeater after all others have been labeled, after this repeater has successfully sent a ROP.

Thus at the beginning of Cycle $i$, $i = 0, 1, \ldots, m-2$, there are $m-i-1$ repeaters with status 1, one repeater with status 2, and $i$ repeaters with status 5.

Figure 6 illustrates the states for a typical cycle $0 \leq i \leq m-2$. SR, SL, SE, UE represent successful ROP, successful label, successful ETE ACK, and unsuccessful ETE ACK, respectively; notice that now there are three separate classes of routes which the chain can take to traverse the cycle. Each cycle starts with 1 ROP in the buffer. Either one of the following classes of routes can be taken:

1. Another ROP is received by the station before a label is received by a repeater (follow the lower route passing through states $4_i, 5_i$, and possibly $6_i$).
2. The label is received and acknowledged by the repeater before another ROP is received (upper route without cross over passing through states $2_i, 0_i$, and possibly $3_i$).
3. The label is dispatched several times but before it is successfully acknowledged another ROP is received (upper route with cross over passing through states $2_i, 3_i, 6_i$, and $5_i$).

Our calculations involve examining the expected time to traverse each of these classes of routes, and deciding with what probability each is chosen. Let $E[T_i]$ be the expected time to go through cycle $i$; then the total expected initialization time is the sum of the expected times required to traverse each cycle:

$$ E \left[ T \right] = \sum_{i=1}^{m-1} E[T_i] \quad \text{(B.1)} $$

Since Cycle $-1$ and $m-1$ are different, we address these first. Throughout the analysis we use the fact that the geometrical distribution is memoryless; that is, if $\tau$ is a geometrically-distributed random variable; then

$$ \text{Prob}(\tau > i + j | \tau > i) = \text{Prob}(\tau > j) $$

for $i, j$ integers.

In our case $\tau$ is the time spent in a specific state of the Markov chain; the time to leave any state is geometrically distributed.

Consider the following particular scenario. The chain is in state $A$. If event $E$ occurs transition to state $B$ occurs; else if $\bar{E}$ (its complement) occurs go to state $C$. Remain in state $C$ until some other event $F$ occurs then return to state $A$. See Figure 7. Let $E[T_{AB}]$ and $E[T_{CA}]$ be the expected time to go from $A$ to $B$ and $C$ to $A$, respectively. Because of the abovementioned memoryless property we obtain

$$ E[T_{AB}] = \frac{1 + (1-p(E))E[T_{CA}]}{p(E)} $$

From the collection of the Computer History Museum (www.computerhistory.org)
Cycles \(-1\) and \(m-1\)

Figure 8 depicts that section of the state transition diagram for cycle \(-1\) and \(m-1\).

Evaluation of \(E[T_{-1}]\)

This involves a SR, given that all \(m\) repeaters are uninitialized; this requires one repeater to transmit and all others to remain silent. Thus the expected time of Cycle \(-1\) is:

\[
E[T_{-1}] = \frac{1}{P_{E-1}} = \frac{1}{mp(1-p)^{m-1}} \quad (B.2)
\]

Evaluation of \(E[T_{m-1}]\)

Observe that, except for the number of unlabeled repeaters, this section of the state diagram is the same as a cycle for a one-buffer problem. As in Reference 18 we obtain

\[
E[T_{m-1}] = E[T_{m-1}] + E[T_{m-1}] = \frac{1}{q(1-p)} + \frac{1}{P_{E-1}} \quad (B.3)
\]

where \(P_{E-1} = \text{Prob}(\text{ETE Ack is successful})\).

Since every repeater is silent at this stage, we only require that the station be silent; thus, \(P_{E-1} = (1-q)\), giving the expected time for Cycle \(m-1\):

\[
E[T_{m-1}] = \frac{1}{q(1-p)} + \frac{2}{1-q} \quad (B.4)
\]
Evaluation of $E[T_i]$, $i=0, 1, \ldots, m-2$

For a typical Cycle $i$, we are interested in quantifying the time needed to traverse it. Such time is made up of various components, indicated as follows:

**Computation of $E[T_{1;i\rightarrow i+1}]$**

To compute this expression, assume that at the current slot the state of the chain is $I_i$, then condition on the outcome of the next slot. It is easy to derive

$$P_{1;i}=\text{Prob (go from } 1, \text{ to } 2_i)$$

$$=q(1-p)^{i+1}$$

(B.5)

$$P_{1;i}=\text{Prob (go from } 1, \text{ to } 4_i)$$

$$=(1-q)(1-p)^{m-i-1}(m-i-1)$$

(B.6)

Then, referring to Figure 6, we see that

$$E\left[T_{1;i\rightarrow i+1}\right]=$$

$$1+E[T_{2;i\rightarrow i+1}]$$

if a label is delivered

$$1+E[T_{4;i\rightarrow i+1}]$$

if a ROP is received

$$1+E[T_{1;i\rightarrow i+1}]$$

if neither of the above

(B.7)

where we have made use of the memoryless property previously described. Unconditioning, we obtain:

$$E[T_{1;i\rightarrow i+1}]=$$

$$(1+E[T_{2;i\rightarrow i+1}])P_{1;i}+(1+E[T_{4;i\rightarrow i+1}])P_{1;i}$$

$$+(1+E[T_{1;i\rightarrow i+1}])((1-P_{1;i})(1-P_{2;i}))$$

(B.8)

Solving for $E[T_{1;i\rightarrow i+1}]$, we obtain:

$$E[T_{1;i\rightarrow i+1}]=\frac{1+P_{1;i}E[T_{2;i\rightarrow i+1}]+P_{2;i}E[T_{4;i\rightarrow i+1}]}{P_{1;i}+P_{1;i}}$$

(B.9)

We notice that to proceed we need $E[T_{2;i\rightarrow i+1}]$ and $E[T_{4;i\rightarrow i+1}]$; we address $E[T_{4;i\rightarrow i+1}]$ next.

**Computation of $E[T_{4;i\rightarrow i+1}]$**

Referring to Figure 6, we see that

$$E[T_{4;i\rightarrow i+1}]=E[T_{3;i\rightarrow i+1}]+E[T_{5;i\rightarrow i+1}]$$

(B.10)

We now evaluate the expressions on the right-hand side of Equation B.10. It is quite simple to show that:

$$E[T_{3;i\rightarrow i+1}]=\frac{1}{q(1-p)^{m-i-1}}$$

(B.11)

since we require a successful label delivery from the state $(m-i-2, 2, 0, 0, i)$. To obtain $E[T_{5;i\rightarrow i+1}]$ we must condition on the outcome of the next slot, namely whether the ETE Ack is successful or not. Using the analysis applied to the one buffer case we find:

$$E[T_{5;i\rightarrow i+1}]=\frac{1}{P_{4;i}}\left(1+E[T_{6;i}](1-P_{5;i})\right)$$

(B.12)

Note that:

$$P_{5;i}=(1-p)^{m-i-1}(1-q)$$

(B.13)

since we are in state $(m-i-2, 1, 1, 0, i)$, and the station

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*Figure 7—Transition time under memoryless distributions.*

*Figure 8—Cycles $-1$ and $m-1$.*
and all other unlabeled repeaters must be silent for the ETE to be successful. Also

\[ E[T_{6i+1}] = \frac{1}{P_{6i+1}} = \frac{1}{q(1-p)^{m-i}} \]  

(B.14)

since one less repeater is now active after state 6\textsubscript{i} is reached.

Finally:

\[ E[T_{4i+1}] = q(1-p)^{m-i-1}(1-q) \]  

(B.15)

\[ E[T_{2i+1}] = \left\{ \begin{array}{ll} 1 & \text{if ETE Ack is successful} \\ 1+E[T_{3i+1}] & \text{if ETE Ack is unsuccessful} \end{array} \right. \]  

(B.16)

Evaluation of \( E[T_{5i+1}] \)

Assuming the current state of the chain to be 2\textsubscript{i} and conditioning on the event of the next slot, namely whether the ETE Ack is successful, we immediately see

\[ E[T_{5i+1}] \]

\[ = \left\{ \begin{array}{ll} 1 & \text{if ETE Ack is successful} \\ 1+E[T_{3i+1}] & \text{if ETE Ack is unsuccessful} \end{array} \right. \]  

(B.16)

Note that (see Figure 6):

\[ P_{3i+1} = \text{Prob (ETE Ack is successful from state 2)} \]

\[ = (1-q)(1-p)^{m-i-1} \]  

(B.17)

Unconditioning, we get:

\[ E[T_{5i+1}] = P_{3i+1}(1+E[T_{3i+1}]) \]

\[ + (1+E[T_{3i+1}]) (1-P_{3i+1}) \]  

(B.18)

Since

\[ E[T_{6i+1}] = \frac{1}{(m-i-1)p(1-p)^{m-i-2}} \]  

(B.19)

(due to the fact that the station is automatically silent when the label queue is empty) we obtain:

\[ E[T_{2i+1}] = 1+P_{2i+1} \]

\[ + \frac{1}{(m-i-1)p(1-p)^{m-i-2}} \]

\[ + (1-P_{2i+1})E[T_{3i+1}] \]  

(B.20)

Thus, we need to find \( E[T_{3i+1}] \).

Evaluation of \( E[T_{3i+1}] \)

Note this calculation is more complicated than the single-buffer case because it is possible to transfer from 3\textsubscript{i} to 6\textsubscript{i}, i.e.: another ROP is received before label delivery. Again, we condition on the next slot.

\[ E[T_{3i+1}] \]

\[ = \left\{ \begin{array}{ll} 1+E[T_{3i+1}] & \text{if label successful} \\ 1+E[T_{6i+1}] & \text{if ROP successful} \\ 1+E[T_{3i+1}] & \text{if neither} \end{array} \right. \]  

(B.21)

Unconditioning:

\[ E[T_{3i+1}] = (1+E[T_{3i+1}])P_{3i+1} + (1+E[T_{6i+1}])P_{6i+1} + (1+E[T_{3i+1}]) (1-P_{3i+1} - P_{6i+1}) \]  

(B.22)

where

\[ P_{3i+1} = \text{Prob (ROP is successful from state 3)} \]

\[ = (m-i-1)p(1-p)^{m-i-2}(1-q) \]  

(B.23)

\[ P_{6i+1} = \text{Prob (label is successful from state 3)} \]

\[ = q(1-p)^{m-i-1} \]  

(B.24)

Solving, we obtain:

\[ E[T_{3i+1}] \]

\[ = \frac{1+P_{3i+1}E[T_{3i+1}]+P_{6i+1}E[T_{6i+1}]}{P_{3i+1}+P_{6i+1}} \]  

(B.25)

\[ E[T_{6i+1}] \]

\[ = \frac{E[T_{6i+1}]}{q(1-p)^{m-i-1}} \]  

(B.27)

and \( E[T_{5i+1}] \) was derived two sections ago. Hence, we obtained an equation for \( E[T_{3i+1}] \) in terms of \( E[T_{3i+1}] \). Together with the equation of the third subsection, we have a system which must be solved simultaneously.

Evaluation of \( E[T_i] \)—The final result

We now pull things together; from the beginning of the third subsection we have:

\[ E[T_{i+1}] = \frac{1}{P_{i+1}+P_{i+1}} + \frac{P_{i+1}}{P_{i+1}+P_{i+1}}E[T_{i+1}] \]

\[ + \frac{P_{i+1}}{P_{i+1}+P_{i+1}}E[T_{i+1}] \]  

(B.28)
From the next subsection we have:

$$E[T_{i+1}^{4}] = E[T_{i}^{4}] + \frac{1}{P_{i+1}} \{1 + E[T_{i}^{5}](1 - P_{i+1})\} \quad (B.29)$$

We only need $E[T_{i+1}^{2}]$. From the next two subsections we have:

$$E[T_{i+1}^{2}] = 1 + P_{i} E[T_{i}^{2}] + (1 - P_{i}) E[T_{i+1}^{4}] \quad (B.30)$$

and

$$E[T_{i+1}^{5}] = \frac{1}{P_{i} + P_{i+1}} E[T_{i}^{5}] + \frac{P_{i}^{2}}{P_{i} + P_{i+1}} E[T_{i+1}^{4}] + E[T_{i+1}^{6}] \quad (B.31)$$

The following procedure is employed:

1. Substitute $E[T_{i+1}^{5}]$ (which has just been computed) into Equation B.31.
2. Solve the system of Equations B.30 and B.31 for $E[T_{i+1}^{2}]$.
3. The answer to 2 is in terms of $E[T_{i}^{5}]$, which was just derived: substitute this expression.

We thus have $E[T_{i+1}^{2}]$, namely:

$$E[T_{i+1}^{2}] = \left[1 + (1 - q)(1 - p)^{m-i-1} \frac{1}{(m-i-1)p(1-p)^{m-i-2}} \right. \left. + [1 - (1 - q)(1 - p)^{m-i-1}] \frac{1}{(m-i-1)p(1-p)^{m-i-2}(1-q) + q(1-p)\mu_{0}} \right. \left. + \frac{1}{(1-p)^{m-i-1}(1-q)} \frac{1}{(m-i-1)p(1-p)^{m-i-2}(1-q) + q(1-p)\mu_{0}} \right] \times \left[1 + \frac{1}{q(1-p)\mu_{0}}\right]$$

$$\left. \left\{1 - \frac{[1 - (1 - q)(1 - p)^{m-i-1}]q(1-p)\mu_{0}}{(m-i-1)p(1-p)^{m-i-2}(1-q) + q(1-p)\mu_{0}}\right\}^{-1}\right. \right] \right\}$$

Substitute Equations B.29 and B.32 into Equation B.28; we obtain Equation 3, that is,

$$E[T_{i+1}^{2}] = E(T_{i}) = \text{Equation (3)}$$

Thus, we have accomplished our goal of determining the expected time to traverse a typical cycle. Summing over all cycles gives the total expected initialization time (Equation 2).