A non-associative arithmetic for shapes of channel networks*

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INTRODUCTION

The purpose of this paper is to describe a method for analysis of one type of pictorial information that is abstracted from maps. The picture is a line diagram or graph that, in the language of graph theory, is a planted plane tree in which each vertex has a valency 1 or 3. In hydrology and geomorphology this type of graph is interpreted as a channel network that encompasses the topological properties of the network of rivers and streams comprising a drainage system. A recent survey paper by Dacey identifies a large number of properties of channel networks. Considering that many of these properties are clearly displayed by sketches of channel networks, the mathematical derivations seem unnecessarily complicated. This disparity in level of difficulty may reflect that the pictorial representation of a graph has a structure that is more amenable to analysis than does the conventional linguistic (i.e., mathematical) representation.

This paper describes a formal model that is seemingly more adaptable to the analysis of properties of channel networks and similar graphs than is the combinatorial mathematics that is conventionally used. This model is an “arithmetic of shapes” that incorporates a relatively simple notation to express many of the attributes of graphs. It is an arithmetic in that the rules for operations on shapes and combinations of shapes are in many ways similar to the rules of Etherington’s formulation of non-associative arithmetics. The relation between non-associative arithmetics and channel networks was evidently first noted by Smart.

A formal statement of the arithmetic for the shapes of channel networks is provided in this paper. While space limitations preclude demonstration of the utility of this arithmetic model, it evidently yields all basic properties of channel networks. Though this model is formulated in terms of channel networks, the same type of graph also serves as the representation of the partition of a class by bifurcations and, accordingly, has many interpretations other than as channel networks. One application, illustrated by Cavalli-Sforza and Edwards, is for reconstruction of the evolutionary tree leading to the genetic characteristics of an observed population.

The first part of this paper delimits the domain of the arithmetic model by identifying the basic concepts and structure of channel networks. Then models that encompass this structure are displayed.

CHANNEL NETWORKS

The current study of channel networks largely derives from Shreve’s formulation of topologically random channel networks. A basic concept is that of topologically distinguishable channel networks. The following structure for analysis of properties of topologically random channel networks is adapted from Dacey. The concepts, terminology and results of graph theory are largely used, though some of the terminology is modified to reflect the specialized terms commonly used in the study of channel networks. In the terminology of graph theory a channel network is a special type of graph consisting of a collection of edges and vertices that form a planted plane tree in which each vertex has valency 1 or 3. This graph is formulated in this study as a collection of links and nodes. The length and shape of links is not taken into account.

Definition 1. The two nodes of each link are distinguished as up-node and down-node. A fork is formed by the coincidence of nodes of three distinct links—the down-node of two distinct links and the up-node of a third link—and these three nodes are called members of a fork. The two links whose down-nodes are members of a fork are called the branches of a fork and these branches are oriented with respect to the third link and are distinguished as the left-branch and the right-branch of the fork. A nodal point is an isolated node that does not coincide with any other node. An up-node (down-node) that is a nodal point is called a source (outlet). An exterior link is a link whose up-node is a source, and a terminal (or outlet) link is a link whose down-node is an outlet. A path is a sequence of one or more links in which no link appears more than once.

Definition 2. A channel network of magnitude \( n \geq 1 \) is a collection \( \lambda_n \) of links, along with the resulting forks and nodal points, that have the following properties.

(a) Each node of every link in \( \lambda_n \) is a source, an outlet or a member of one fork.
The term "ambilateral" was suggested by R. L. Shreve and first used in the context of channel networks by Smart.\(^7\)

Figure 1 illustrates the ambilateral classes formed by all distinguishable channel networks of magnitude 1 through 5. Two fundamental properties of channel networks are the number of topologically distinguishable channel networks of magnitude \(n\) and the number of ambilateral classes needed to account for these channel networks. A basic tool for the study of these properties is identified by

Definition 5. Let \(\lambda_n\) and \(\lambda_m\) denote channel networks of magnitudes \(n\) and \(m\). The channel networks \(\lambda_n\) and \(\lambda_m\) and a new link \(l\) are said to form a (terminal) splice when a fork is formed by the up-node of \(l\) and the outlets of \(\lambda_n\) and \(\lambda_m\), such that this fork is the only point common to \(\lambda_n\) and \(\lambda_m\). This splice is denoted by \(L(\lambda_n, \lambda_m)\) when \(\lambda_n\) is the right-branch of this fork.

Figure 2 illustrates this operation.

Property 1. The \(L(\lambda_n, \lambda_{n-m})\) is a channel network of magnitude \(n\).

Property 2. Each channel network \(\lambda_n, n>1\), is formed by exactly one splice.

**PRELIMINARIES**

The arithmetic for shapes is defined in terms of rules for combining shapes. One possible interpretation of this arithmetic is for shapes that are channel networks and for a rule of combination that is the splice formed by pairs of channel networks. Although some of the elements of this arithmetic are illustrated by this interpretation, it is convenient to use a more general formulation. The notation for this formulation has been developed by Etherington.\(^3\)

Upper case letters \(A, B, C, \ldots\) plus \(X\) are used to denote specific shapes. A rule for combining pairs of shapes is also specified and the combinations of shapes \(A\) and \(B\) is a shape and is denoted by the product notation \(AB\). The shape \(AB\) may be combined with a shape \(C\), the same or different from \(A\) and \(B\), in two ways that are denoted by \(C(AB)\) and \((AB)C\). Depending upon the shapes and rule of combination \(C(AB)\) and \((AB)C\) may be the same or different shapes. Any number of shapes may be combined in this pairwise fashion. To avoid the cumbersome use of multiple brackets, frequent use is made of a notation in which groups of dots separate the factors and fewness of dots establishes the precedence in combining shapes. Thus \(A::BC:CD::CE=A[(BC)[[(CD)^P(CE)]]]\), where \(D^P=DD\).

The following terminology reflects this notation.
Definition 1. The rule for combining shapes is called multiplication and the shape $AB$ is called the product of $A$ and $B$. If $A$ is the product of $B$, $C$, ... , then $B$, $C$, ... , are called factors of $A$. A shape defined by factors that are absorbed one at a time is called a primary shape.

An example of a primary shape is $A:BC·D·E$.

The notation simplifies for a system of shapes that are obtained as products of a unique shape. Let $X$ denote this unique shape. The shape $XX^2 = X^3$ is evidently also unique. However, for $n \geq 3$, there are alternative ways of expressing the product of $n$ $X$'s. The five possible products of $4$ $X$'s are $XX·X·X$, $X·XX·X$, $X·XX·X$, $XX·XX$, and $X·X·X·X$. The first four of these products are primitive shapes. If the product rule is specified as commutative, the expressions for primitive shapes are indistinguishable, but they are distinguishable when the product rule is specified as non-commutative. It is this distinction that suggests the terminology of commutative shapes and non-commutative shapes.

A basic distinction between commutative and non-commutative shapes is that the $2^{n-1}$, $n \geq 2$, ways of absorbing $n$ factors one at a time are not distinguishable for commutative shapes and are distinguishable for non-commutative shapes. One consequence is that only one primary shape is the product of $n$ factors of commutative shapes and it is represented by $X^*$. In contrast, a more elaborate notation is required to represent the $2^{n-1}$ primary shapes that are the product of $n$ factors of non-commutative shapes.

The simpler notation used for commutative shapes is illustrated by the following examples. Two primitive shapes are represented by $X^m$ and $X^n$. Additional shapes are represented as products, powers, and iterated powers of these two shapes so that $X^m·X^n$, $(X^m)^n$ and $(X^n)^m$ are also shapes. The following operations are used to express these latter shapes as an index of $X$. The product of two powers of the same shape is indicated as a sum in the index of the shape, the power of a power is indicated as a product in the index of the shape, and an iterated power is indicated as a power in the index.

For example, where $X$ is the shape,

\begin{align*}
X^3X^2 &= X^{3+2} \\
(X^3)^2 &= X^{3·2} \\
((X^3)^2)^2 &= X^{3·2·2} \\
(X^3)^{2+2} &= X^{3·(2+2)}
\end{align*}

This superscript notation is cumbersome and frequent use is made of the convention that $X^n$ is represented by simply $a$. By this convention, the expression $a+b+c=d$ is in the arithmetic of indices and is equivalent to $X^aX^bX^c = X^d$. Similarly, $ab = c$ is equivalent to $X^aX^b = X^c$. In the arithmetic of indices the letters $a$, $b$, $c$, ... may be positive integers or sums, products, and powers of positive integers, but the letters $m$ and $n$ are always positive integers.

The essential features of this notation are reflected in

Definition 2. Let $A_1$ and $A_2$ be any two shapes $a_1$ and $a_2$ that are formed by factors of $X$. Then $a_1+a_2$ is the shape of the product $A_1A_2$, and $a_1a_2$ is the shape of the product formed by substituting $A_1$ for each of the factors of $A_2$.

This notation is adapted to the structure of channel networks by the following interpretations. The basic shape $X$ is a link along with its up-node and down-node. The shape $a$ is a channel network. The shape $a+b$ is a channel network and the rule of combination is the splice of the channel networks $a$ and $b$. The shape $ab$ is a channel network and the rule of combination is that each exterior link of $b$ is replaced by $a$.

Figure 3 illustrates these operations, while Figure 1 illustrates the representation of channel networks as factors of $X$.

**ARITHMETIC OF COMMUTATIVE SHAPES**

The system $C$ of commutative shapes is defined by the following six rules.

C1 (Existence Rule). The shape $X=X^1$ is in $C$.

C2 (Closure Rule). If $X^a$ and $X^b$ are shapes that are in $C$ then $X^{a+b}$ and $X^{ab}$ are shapes in $C$.

The following three rules pertain to shapes that are in $C$ and are expressed in the arithmetic of indices. In this arithmetic, $a$ stands for $X^a$.

C3 (Associative Rule). $(a+b)+c=a+(b+c)$ provided that $a\neq c$, and $ab·c=a·bc$.

C4 (Commutative Rule). $a+b=b+a$, and $ab=ba$ provided that $a\neq b$ and $b\neq 1$. For $b=1$, $a-1=1\cdot a=1$.

C5 (Distributive Rule). $a(b+c)=ab+ac$, and $(b+c)a=ba+ca$ provided that $a\neq 1$.

C6 (Formation Rule). If $X^a$ is in $C$, then for $n \geq 2$, the shape $X^n=XX^{n-1}$.

Property 1. For each positive integer $n$, $X^n$ is in $C$ and is a primitive shape.

Definition 1. Two shapes $X^a$ and $X^b$ in $C$ are said to be isomorphic if, and only if, $a=b$. Two shapes that are not isomorphic are called ambilaterally distinct shapes.

Property 2. The shape $X^a$ is isomorphic with any shape that is a primitive shape formed by $n$ factors of $X$.

Property 3. If $a$, $b$, $c$ and $d$ are shapes in $C$ such that $a=c$ and $b=d$, then $a+b=b+a$, $c+d$ and $d+c$ are all equal and, hence, are isomorphic.

Property 4. If $c$ is a shape in $C$, then $c=1$ or there exist in $C$ shapes $a$ and $b$ such that $c=a+b$.

Definition 2. Let $\delta(a)$ be the value in ordinary arithmetic of the index $a$ of the shape $X^a$, and $\delta(a)$ is called the degree of the shape $X^a$.

By Definition 1.4, the collection of all shapes of degree $n$ that are mutually isomorphic is called an ambilateral class of degree $n$.

Definition 3. Let $Q(n)$ be the minimum number of ambilateral classes necessary to contain all shapes in $C$ that have degree $n$. 

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Property 5. For each of the degrees 1, 2 and 3 there is a single ambilateral class and, hence, a single ambilaterally distinct shape. The two ambilaterally distinct shapes of degree 4 are isomorphic with 4 or 2·2.

The proof of this property is displayed in order to illustrate the verification of statements about shapes in the arithmetic of indices. The statement is obvious for degree 1 and for degree 2 since the only factoring of 2 is as 1+1. The four cases of degree 3 to consider are (1+1)+1, 1+(1+2), 2+1 and 1+2. By Property 2, the first two shapes are isomorphic with 4, while 51 implies (1+1)+1=2+1 and 1+(1+1)=1+2. To obtain the two distinct shapes of degree 4, first observe that

\[ 4 = 1+3, \quad 4 = 3+1, \quad 4 = (1+2)+1 = (2+1)+1 \]

[Note: The index here is specified as \( n \), but in the text, it is always \( i \).]

Since there is only one shape of degrees 1, 2 and 3, put \( 1 = 1 \), \( 2 = 2 \) and \( 3 = 3 \). This ordering of classes of degree 5 and less is as follows.

\[ Q(1) = 1, \quad Q(2) = 2, \quad Q(3) = 3 \]

This structure may be used to verify the following basic, though well-known, result.

Theorem 1. \( Q(1) = Q(2) = 1 \) and for \( n \geq 2 \),

\[ Q(2n-1) = \sum_{i=1}^{n-1} Q(i)Q(2n-1-i) \]

and

\[ Q(2n) = \sum_{i=1}^{n-1} Q(i)Q(2n-i) + \frac{1}{2}Q(n)[Q(n)+1] \]

A simple, non-recursive expression for \( Q(n) \) is not known, but the values form the sequence 1, 1, 1, 2, 3, 6, 11, 23, 46, 98, \ldots .

With the value of \( Q(n) \) established, an algorithm given by Harding was used to identify the composition of \( n \), for specified values of \( n \) and \( i \), without enumerating all shapes that precede \( n \) in \( \sum \).
To solve this equation, first find the maximum value of \( n-s-1=r-1 \) that is compatible with (*) using this value, then find the maximum value of \( k \) that is compatible with (*). Finally, using these two values, find the value of \( h \) so that (*) is satisfied. In identifying all of these values, the constraints on \( r, s, h \), and \( k \) are obeyed.

This completes the display of the notation and ordering of shapes for the arithmetic model for the ambilateral classes of channel networks. The classes of topologically distinguishable shapes are of greater interest to the study of channel networks and an arithmetic model for the non-commutative shapes is displayed next.

**ARITHMETIC OF NON-COMMUTATIVE SHAPES**

The notation suggested by Etherington\(^3\) and used to formulate the system \( C \) of commutative shapes is inadequate to express the system \( A \) of non-commutative shapes. One limitation is that the primary shape formed by \( n \) factors of the basic shape is unique for commutative shapes but for non-commutative shapes has \( 2^{n-2} \) possible interpretations. Accordingly, \( 2^{n-2} \) different symbols are required to represent the primary non-commutative shapes that correspond to \( X^n \) in \( C \). To distinguish these shapes a new notation is devised. Comparison of the systems \( C \) and \( A \) is facilitated by using \( Y \) to represent the basic shape for the system \( A \).

The following notation takes into account the \( 2^{n-2} \) primitive shapes formed by \( n \) factors of \( Y \). Sequences of \( 0 \)'s and \( 1 \)'s are used to differentiate the primitive shapes. There are \( 2^n \) different sequences formed by \( n \) \( 0 \)'s and \( 1 \)'s. Let \( i_0, 1 \leq i \leq 2^{n-2} \), stand for one of the sequences formed by \( (n-2) \) \( 0 \)'s and \( 1 \)'s. If \( i_n \) and \( i_m \) are identical sequences of \( 0 \)'s and \( 1 \)'s, then that \( n=m \), this is denoted by \( i_n=i_m \) and otherwise \( i_n \neq i_m \). The symbols \( i_0, i_1 \) and \( i_2, i_3 \) stand for the sequences of \( n-1 \) \( 0 \)'s and \( 1 \)'s for which the first \( n-2 \) entries are identical to those of \( i_n \) and the \((n-1)-\)th entry is, respectively, \( 0 \) and \( 1 \).

The primitive shapes formed by \( n \) factors of \( Y \) are represented by \( \{Y_{i_0}: 1 \leq i \leq 2^{n-2}\} \). This notation does not make it explicit the particular sequence of \( (n-2) \) \( 0 \)'s and \( 1 \)'s that is represented by an \( i_n \), but this information is not needed. The shape represented by a known sequence of \( 0 \)'s and \( 1 \)'s is, however, determined by the notation. The shapes \( Y \) and \( Y^n = Y_0^n \) are unique, while for \( n \geq 3 \), the \( Y_{i_n}^n \) are defined recursively by

\[
\begin{align*}
Y_{i_0} & = Y Y_{i_0}^{-1} \\
Y_{i_1} & = Y Y_{i_1}^{-1} Y. 
\end{align*}
\]

(\( f \))

This procedure uniquely determines the shape associated with a particular sequence of \( 0 \)'s and \( 1 \)'s. For example, the four possible interpretations of \( Y_4^4 \) are \( Y_0^4, Y_1^4, Y_2^4, Y_3^4 \), but these correspond, respectively, to \( Y: Y Y \), \( Y Y Y: Y \), \( Y Y Y: Y \), \( Y Y Y: Y \).

Figure 1 illustrates this notation.

The system \( A \) of non-commutative shapes is defined by the following six rules.

**A1 (Existence Rule).** The shape \( Y = Y^1 \) is in \( A \).

**A2 (Closure Rule).** If \( Y \) and \( Y^b \) are shapes that are in \( A \) then \( Y^{a+b} \) and \( Y^a \) are shapes in \( A \).

The next three rules pertain to indices of \( Y \).

**A3 (Associative Rule).** \( (a+b)+c \neq a+(b+c) \) provided that \( a \neq c \) or \( b, c \)

\( \neq a \).

**A4 (Commutative Rule).** \( a+b \neq b+a \), and \( ab \neq ba \) provided that \( a \neq 1 \) and \( b \neq 1 \). For \( b = 1 \), \( a \cdot 1 = 1 \cdot a = a \).

**A5 (Distributive Rule).** \( (a+b)=ab+ac \), and \( (b+c)a \neq ba+ca \) provided that \( a \neq 1 \).

These five rules are the same as the rules \( C1-C5 \) of the system \( C \) with the exception of the commutative rule for addition in the arithmetic of indices. Further, the following formation rule differs from \( C6 \) in order to account for the \( 2^{n-2} \) expressions required for primitive shapes.

Clearly, the shapes \( Y \) and \( Y^2 = YY \) are in \( A \).

Definition 1. Put \( Y_i^2 = Y \) and each \( i_n, 1 \leq i \leq 2^{n-2} \), represents a different sequence of \( n-2 \) \( 0 \)'s and \( 1 \)'s.

**A6 (Formation Rule).** For \( n \geq 3 \), if \( \{Y_{i_n}: 1 \leq i \leq 2^{n-2}\} \) are shapes in \( A \), then these shapes are defined by (\( f \)) for each \( i, 1 \leq i \leq 2^{n-2} \).

Property 1. For each integer \( n \geq 3 \) and \( 1 \leq i \leq 2^{n-2} \), \( Y_{i_n} \) is a shape in \( A \).

Definition 2. Two shapes \( Y^a \) and \( Y^b \) in \( A \) are called map-isomorphic, which is written as \( Y^a \sim Y^b \), if, and only if, \( a = b \). Shapes that are not map-isomorphic are called topologically distinguishable.

Property 2. Two shapes \( Y_{i_n} \) and \( Y_{j_m} \) are topologically distinguishable if \( n \neq m \) or if \( n = m \) and \( i_n \neq j_m \).

Definition 3. Suppose \( A_1 \) and \( A_2 \) are shapes in \( A \) such that \( A_2 \) is a factor of \( A_1 \), but no product expression for \( A_2 \) includes \( A_1 \) as a factor. Then \( A_1 \) and \( A_2 \) are topologically distinguishable.

Property 4. If \( A \) is a shape in \( A \), then there exists an exponent \( a \) such that \( A \sim Y^a \). In particular, for any shape \( Y_{i_n} \) in \( A \) there exists an \( a \) such that \( Y_{i_n} \sim Y^a \).

Definition 4. Let \( A \) be a shape in \( A \) for which \( A \sim Y^a \). The \( X \)-transform of \( A \) is \( X^a \) and is denoted by \( X(A) \). The arithmetic for the indices of an \( X \)-transform obeys the rules \( C3 \), \( C4 \) and \( C5 \) of the system \( C \).

Property 5. If \( A \) is a shape in \( A \) and \( X \) is a shape in \( C \), then \( X(A) \) is a shape in \( A \).

Definition 4. Suppose \( Y^a \) and \( Y^b \) are shapes in \( A \) such that \( X(Y^a) = X^a \) and \( X(Y^b) = X^b \). If, by the rules of the system \( C \), \( a = b \), then \( Y^a \) and \( Y^b \) are said to belong to the same ambilateral class. Alternatively, if \( a \neq b \), \( Y^a \) and \( Y^b \) belong to different ambilateral classes and are called ambilaterally distinct.

Notice that two shapes may be classified as "topologically distinguishable" or as "ambilaterally distinct." As in the previous section, a shape \( Y^a \) in \( A \) is frequently denoted by \( a \).

Property 6. If \( a_1 \) and \( a_2 \) are shapes in \( A \) that are ambilaterally distinct, then \( a_1 \) and \( a_2 \) are topologically distinguishable.

Conversely, shapes that are topologically distinguishable may or may not be ambilaterally distinct.
Definition 5. Let \( d(a) \) be the value in ordinary arithmetic of the index \( a \) of the shape \( Y^a \) and \( d(a) \) is called the degree of \( Y^a \).

Property 7. If \( a \) and \( b \) are shapes in \( A \) and \( d(a) \neq d(b) \), then \( a \) and \( b \) are ambilaterally distinct.

Property 8. If \( a \) is a shape in \( A \) and \( a = b_1 + c_1 = b_2 + c_2 \) then \( b_1 = b_2 \) and \( c_1 = c_2 \).

Theorem 1. Let \( Q(n) \) denote the minimum number of ambilateral classes required to contain all shapes in \( A \) that have degree \( n \). Then \( Q(n) \) satisfies Theorem 3.1.

The remainder of this paper simply displays, without comment, the procedure for ordering shapes.

Definition 6. All shapes in \( A \) that are mutually map-isomorphic are said to belong to the same topological class. Let \( N(n) \) denote the minimum number of topological classes required to contain all shapes in \( A \) of degree \( n \).

Definition 7. The collection \( \Gamma_n \) contains \( N(n) \) shapes of degree \( n \),

\[ \Gamma = \bigcup_{n=1}^{\infty} \Gamma_n \]

The \( i \)-th shape of the \( N(n) \) shapes of degree \( n \) in \( \Gamma_n \) is denoted by \( n_i \), \( 1 \leq i \leq N(n) \). If \( r_i, s_i \) and \( n_i \) correspond to the shapes \( a, b \) and \( c \) and if \( c = a + b \), then an alternative notation for this shape is \( n_i = r_i + s_i \). Further, the shape \( I_1 \) is in \( \Gamma \) and the shape \( r_1 + s_1 \) is in \( \Gamma \) if, and only if, \( r_1 \) and \( s_1 \) are both in \( \Gamma \).

Ordering Rule. Suppose \( n_i \) and \( n_j \) are shapes in \( \Gamma \) such that

- \( n_i = r_i + s_i \)
- \( n_j = t_j + u_j \)

If \( n_i \) and \( n_j \) are topologically distinguishable, then \( i < j \) so that \( n_i \) precedes \( n_j \) if, and only if, one of the following conditions is satisfied.

- \( r \leq s, t > u \);
- \( r = s, t = u, r = t, h < p \);
- \( r = s, t = u, r = t, h < p, k < q \);
- \( r < s, t < u, s < u \);
- \( r < s, t < u, s = u \), \( r = t \)
- \( r < s, t < u, s = u, t = f_1, h < p \);
- \( r < s, t < u, s = u, t = f_1, h = p, k < q \).

Further, if \( r > s \) and \( t > u \), then \( u_1 + t_1 \) precedes \( s_1 + r_1 \) if, and only if, \( n_1 \) precedes \( n_1 \). If \( n_1 \) and \( n_1 \) are map-isomorphic, then \( i = j \).

Theorem 2. \( N(n) = (2n-2)!/(n-1)!n! \).

Definition 8. Put \( p_n(0) = 0 \) and

\[ p_n(j) = \sum_{i=1}^{n} N(i)N(n-i), \quad 1 \leq j \leq n-1. \]

If \( n \) is odd and \( i \leq p_n(\frac{1}{2}n - \frac{1}{2}) \) or \( n \) is even and \( i \leq p_n(\frac{1}{2}n) \), then

\[ i = \begin{cases} \frac{p_n(n-s-1) + (k-1)r + h}{r \neq s}, \\ \frac{p_n(n-s+1) + \frac{h}{2} + \frac{k}{2} + h}{s = r} \end{cases} \tag{1} \]

where \( 1 \leq h \leq N(r), 1 \leq k \leq N(s) \) and \( r \leq s \). When \( i \) exceeds the upper bound, \( n_i \), is related to a shape \( n_j \) for which \( j \) does not exceed the upper bound and, hence, \( n_i \) is determined by (1). If \( n \) is odd and \( i \geq p_n(\frac{1}{2}n - \frac{1}{2}) \), then \( n_i = s_i + r_i = n_i \), and \( j = 2p_n(\frac{1}{2}n) - N(n)-i + 1 \).

The equation (1) is solved by the same iterative procedure recommended for (1) of Theorem 3.2.

REFERENCES