Realization of Boolean Polynomials Based on Incidence Matrices

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THIS PAPER describes a general algebraic method of finding minimum contact networks for any given Boolean polynomial. Solutions obtained by this method may in general be any kind of connection with any number of contacts for each variable. Furthermore, any practical requirements such as series-parallel cases usually found in most electronic devices, and single contact for specified variables, can all be considered in the calculation, if necessary. A routine algorithm on incidence matrices will automatically yield any ingenious connections. The node-branch and branch loop incidence matrices which were revealed by G. R. Kirchhoff[1] 1847, are adopted as unknown, especially those of modulo 2 which were elaborated by O. Veblen[2] in 1916 are mostly used. However, simultaneous use of modulo zero (or infinity), 2 and other integers was found useful for combinatorial consideration in the multi-contact case. General Galois fields are already used in switching theory by Moisil[3] and his Rumanian group.

General non-series-parallel synthesis of switching 2-terminals[4] by incidence matrices modulo 2 began in 1939 and has been developed further.[5,6] Especially R. Gould had established a systematic method for the general multicontact case. His significant improvements of solutions of the four variable problem[6,7] prove the usefulness of incidence matrices.

The main novelties of this article are as follows:

(1) A rigorous use of incidence matrices gives an essentially new approach to find all possible complicated connections. The method can also be used for finding a single solution or giving proofs of various minimalities.

(2) Topological enumeration of nodes, branches and degrees of freedom gives a criterion of realizability.

(3) Realization[8] of individual connections from each incidence matrix was reduced to a routine process by the new graphical or algebraic "*ambit-method*". It can also be used for network synthesis of other kinds. Fortunately an algebraic-topological review[9,10] of network equations from the geometric standpoint of H. Weyl[11] and G. Kron[12] clarifies the possibility because topological properties relating to connections of networks belong to affine geometry[10] which has no "metric"; that is, these cases are perfectly independent of any branch causality such as Ohm's law in d.c. or a.c.[10] Such *affine network theory* can be called the "*pre-Ohmic*" or "*Ohm-free*" *network theory* to which the *Boolean network* theory belongs.[13] This shows that incidence matrices can be used in the synthesis of networks with rectifiers, hysteresis or any other nonlinearity.

(4) Loops corresponding to ties are expressed by vectors, and cut-sets for barriers are expressed by covectors which mean pairs of initial and terminal hyperplanes in a branch-number-dimensional *affine* space. Hermann Weyl's "*method of orthogonal projection*"[11,12] is generalized to affine projection for vectors and to intersection with a subspace for covectors, and forms the general foundation of the whole topological network theory.[10]

(5) The invariant transformations of variables for given Boolean functions form a non-commutative group[15] which plays an important role in this method.

(6) Semi-ordered vector sets. All vectors of only zero and unity components (modulo 2) of A-dimensional space form an additive group of order 2^A. All loop vectors (modulo 2) of a network form its subgroup. However, its subset of vectors of a single loop passing the relay branch, or more shortly expressed as "*single relay-loop vectors*" does not generate a group because only a sum of an odd number of these vectors generates a relay-loop vector, and further, a sum can be either a single relay-loop or a multiple loop which consists of a single relay-loop and unseparated or separated loops of contact branches. Therefore, if the numbers of independent single relay-loop vectors and all dependent relay-loop vectors of this subset are respectively denoted with C and K, the total number W of the relay-loop vectors is given by

$$W = C + K = {}_CC_1 + {}_CC_3 + \ldots + {}_CC_n = 2^{C-1} \quad (2)$$

where

$$n = C - 1 \text{ if } C \text{ is even,}$$
$$n = C \quad \text{if } C \text{ is odd.}$$

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Dually, the total number V of cut-set covectors cutting the relay branch is given by

$$V = B + H = 2^{B-1} \qquad (1)$$

where B and H are numbers of independent single relay-cut-set covectors and all dependent covectors.

An order relation of vectors and covectors will be defined, taking an example on the following three vectors U^κ, V^κ and W^κ.

$$\begin{aligned} U^\kappa &= [U^1\ U^2\ U^3\ U^4\ U^5\ U^6] \\ &= [1\ \ 1\ \ .\ \ 1\ \ 1\ \ 1], \\ V^\kappa &= [1\ \ .\ \ .\ \ 1\ \ .\ \ 1], \\ W^\kappa &= [.\ \ 1\ \ 1\ \ .\ \ 1\ \ 1], \end{aligned}$$

there holds

$$U^\kappa \geqq V^\kappa \text{ for all } \kappa \qquad (i)$$

and an inequality holds

$$U^\kappa > V^\kappa \text{ at least for one } \kappa\ (\kappa = 2 \text{ and } 5). \qquad (ii)$$

If any pair of vectors or covectors is in relationships i and ii, the vector U is called *"larger than"* V, or V is called *"smaller than"* U and is expressed as

$$\vec{U} > \vec{V} \quad \text{or} \quad U > V. \qquad (iii)$$

The relation is called a *"vector order relation"* (covering, or inclusion). W has no order relation to U and V. In general, a vector set forms a *"partially ordered"* (*semi-ordered*) set.

If all possible K relay-loop vectors C_k^κ are determined by odd number sums of C linearly independent relay-loop vectors C_q^κ, all W relay-loop vectors generally form a semi-ordered vector set. Because W vectors include all possible relay-loop vectors, if there exists a multiple loop U^κ in these vectors, its single loop part V^κ exists in the remaining vectors, and this multiple loop vector U^κ is larger than its single loop part V^κ. Thus a multiple relay-loop vector U^κ always possess a smaller relay-loop vector V^κ. This is an algebraic criterion that a relay-loop vector is multiple. Therefore, the necessity of singleness of given short circuit conditions C_g^κ algebraically means the nonexistence of a smaller relay-loop vector in all K dependent relay-loop vector C_k^κ. Dually, a multiple relay-cut-set covector singleness of given open circuit conditions $B^f{}_\kappa$ algebraically means the nonexistence of a smaller relay-cut-set covector in all H dependent relay-cut-set covectors $B^h{}_\kappa$.

Statement of Problems

If the problem is given by short circuit conditions, one can start either from the standard sum (canonical form) S_0 or product R_0. Either is easily obtained from the other. Generally there exist certain transformations t_i of variables which keep the given sum S_0 invariant, and these transformations form a non-commutative group[15] concerning successive substitutions. Then, there is a set of transformation elements of the group called *"generators"* from which all other transformations can be generated. Also as is well known, only permutations of pairs of variables such as (xx'), (xy), (xy') or (wx) (zz') are sufficient to be considered as generators. From the total number of each literal in the standard sum and from its configuration, the group generators are determined from R_0 or S_0 by a routine process. If necessary, the multiplication table of the group can be easily made.

Change of Boolean Expressions by a General Process

From the standard sum S_0, the prime implicant S_1 and all other possible Boolean expressions S_i, including their dual product expressions $R_0, R_1, \ldots R_i$, can be algebraically obtained. Though a prime implicant happens to be a monotone function, that is, of purely unprimed literals or all primed literals, a minimum network is often obtainable from its non-monotone expression.

From the Boolean standpoint any S_i is equivalent to R_j for all values of i and j. However in topological design, the simultaneous consideration of S_i and R_j is more convenient, especially for topological enumerations. In this case for each S_i, if it has a corresponding network, the choice of R_j of the same network is desirable. For each S_i, the corresponding R_j is generally determined in this manner by examining whether the factors of R_j form a minimum necessary set of variables which should be zero in order that the S_i under consideration becomes zero. In Example 1,

$$R_i \rightleftarrows S_i \text{ for all } i.$$

However, in general the correspondence is not one to one.

A necessary condition of realization of a standard sum S_0 or G terms by single contacts is that all linearly dependent relay-loops are included in the original sum. If a set of generating loop-vectors, C in number, is realizable as a network, the above is also the sufficient condition and there holds

$$G = 2^{C-1} \qquad (4)$$

Its proof is based on the exclusiveness of make and break contact literals in all terms of S_0, on the odd number in addition and on the non-existence of multiple loops. Dually for the standard product, there holds

$$F = 2^{B-1} \qquad (3)$$

for a realizable case.

Topologization

A set of short circuit conditions of each of S_i is topologically a set of vectors $\overrightarrow{C_g}(i)$ in $A(i)$—dimensional affine space expressing single relay-loops and forms a branch-loop incidence matrix $C_g{}^\kappa(i)$ of $G(i)$ rows and $A(i)$ columns. Dually, a set of open circuits of each of R_j is topologically represented by a set of covectors $\overrightarrow{B^f}(j)$ in $A(j)$ — dimensional affine space expressing single relay-cut-sets and forms a cut-set matrix $B^f{}_\kappa(j)$ of F(j) rows and $A(j)$ columns.

Realizability by the Numbers of Nodes, Branches and Degree of Freedom

Each row of $C_g{}^\kappa$ must be a single relay-loop. Then, the number $\alpha^0(i)$ of nodes must be equal to the "*topological length*" of this loop, that is, the number of $\alpha^1(i)$ of branches. Thus, the network must have at least α^0 nodes which is equal to the maximum number $E(i)$ of unities in a loop vector:

$$\alpha^0(i) \geqq E(i); \tag{6}$$

especially for the prime implicant S_1,

$$\alpha^0(i) \geqq E(1) \tag{8}$$

This is expressed also as

$$\alpha^0(i) - 1 \geqq D(i) = E(i) - 1 \tag{10}$$

where $D(i)$ is a *topological distance* of the terminals of the relay branch, that is, the maximum number of variables (contacts) in a tie.

The rank $C(i)$ of $C_g{}^\kappa$ gives the degree of freedom $P^1(i)$:

$$P^1(i) = C(i) = \text{rank }(C^\kappa{}_g(i)) . \tag{12}$$

Euler's relation

$$P^1 = \alpha^1 - \alpha^0 + 1 \tag{14}$$

gives

$$C \leqq A - E + 1 \tag{16}$$

If this is not satisfied, there does not exist a circuit for this $C_g{}^\kappa(i)$.

$$N = A - E + 1 - C \tag{18}$$

gives the maximum permissible number of additional linearly independent "*pseudoties*", (which mean topological single loops but not Boolean ties by) including make and break contacts of one relay or more in series. However, a relay-loop vector with unities in pair contact components is a single relay-loop pseudotie only when there are no smaller relay-loop vectors. It is a multiple loop vector if there is a smaller relay-loop vector. Dually, each row of $B^f{}_\kappa$ must be a single relay-cut-set covector. If the maximum number of contacts in each row of $B^f{}_\kappa$ (which can be regarded as a *topological thickness* of this barrier between two terminals of the relay) is denoted by $P(i)$, and that of branches (which may be called a *topological width* of the cut-set) is denoted by $Q(i)$, then there holds

$$P^1(i) \geqq P(i) = Q(i) - 1 \tag{9}$$

$$P^1(i) + 1 \geqq Q(i) \tag{5}$$

Especially, for a dual prime implicant R_1,

$$P^1(i) + 1 \geqq Q(1) \tag{7}$$

The proof is based on the singleness of cut-sets, and that P branches can form chords (links) of a cotree.

$$\alpha^0(i) - 1 = B(i) = \text{rank }(B^f{}_\kappa(i)) . \tag{11}$$

Euler's relation

$$\alpha^0 - 1 = \alpha^1 - P^1 \tag{13}$$

gives

$$B \leqq A - Q + 1 \tag{15}$$

If this is not satisfied, there is no circuit for this $B^f{}_\kappa(i)$.

$$M = A - Q + 1 - B \tag{17}$$

gives the maximum permissible number of additional linearly independent "*pseudocuts*", which are topologically single cut-sets but not Boolean cuts, by including make and break contacts of one relay or more in parallel. However, a general cut-set covector including such pair-contacts is either a single relay-pseudocut or a *multiple cut-set*, of which the included single loop is not necessarily a pseudocut.

Base Vectors and Covectors by Semi-Diagonalization

The vector set $C_g{}^\kappa$ defines an affine subspace of C-dimension. Its base vectors $C_q{}^\kappa$ can be determined by transforming $C_g{}^\kappa$ into such a form that each row has at least one unity which is the only one unity in its column. If all rows acquire such unities, such $C_q{}^\kappa$ will be called a "*semi-diagonalized*" form. Further transformation of the first square part of $C_q{}^\kappa$ to a unit matrix by adequate exchange of its rows and columns is dispensable. The semi-diagonalization is more easily done by a routine addition modulo 2 than by multiplication of an inverse of a square part. Then, all K linearly dependent vectors $C_k{}^\kappa$ are obtained from the base vectors $C_q{}^\kappa$ by all odd number sums. $C_k{}^\kappa$ will be called "*subties*" in short. Dually, base covectors $B^a{}_\kappa$ and all H dependent cut-sets $B^h{}_\kappa$ can be determined. $B^h{}_\kappa$ will be called "*subcuts*."

Determination of Connections

If the given vector set $C_g{}^\kappa$ exactly coincides with its base $C_q{}^\kappa$ and all subties $C_k{}^\kappa$, that is, with W $C_w{}^\kappa$,

$$G = C + K = W, \tag{20}$$

then the remaining part is to realize the base $C_q{}^\kappa$ as a connection by its semi-diagonalization. If $C_q{}^\kappa$ is not realizable and N of Eq. 18 is a positive integer, there is a possibility to realize it by the addition of pseudoties $C_n{}^\kappa$ up to N. $C_n{}^\kappa$ should not be smaller than any of $C_g{}^\kappa$, and all subties generated by $C_n{}^\kappa$ and the $C_g{}^\kappa$ should be either multiple loops or pseudoties. If it is still nonrealizable, other Boolean expressions of the same number of contacts should be calculated. If all of them are unrealizable, only an increase of contacts enables realization. If some of the subties $C_k{}^\kappa$ are pseudoties and the rest are all included in $C_g{}^\kappa$, the process is the same. Dually, if $B^f{}_\kappa$ coincides with $B^v{}_\kappa$, that is, $B^a{}_\kappa$ and $B^h{}_\kappa$ and

$$F = B + H = V \tag{19}$$

or $B^h{}_\kappa$ include some pseudocuts, then the realizability of $B^a{}_\kappa$ is examined and if M of Eq. 17 is positive and all subcuts by $B^m{}_\kappa$ and $B^a{}_\kappa$ are either multiple-cuts or pseudoties, the addition of $B^m{}_\kappa$ may change $B^a{}_\kappa$ to a realizable one.

The above case is the simplest. In general, not all of the subties $C_k{}^\kappa$ are included in the original $C_g{}^\kappa$, and the rest of the subties $C_k{}^\kappa$ form multiple loops and Boolean ties which are not included in $C_g{}^\kappa$. These can give a new algebraic definition of the familiar "sneak paths". Subties $C_k{}^\kappa$ may include a "smaller vector" than any vector of the given $C_g{}^\kappa$. Sneak paths and such order relations should be eliminated either by a change of i of S_i, or by an increase of contacts. Dually, the algebraically defined "sneak barriers" and smaller covectors in $B^k{}_\kappa$ should be eliminated either by a change of j of R_j or by an increase of contacts.

Example 1[16]

Decimal expression of short circuit combinations: 4, 2, 1, 0.

				primes	name
1. $S_0 =$	x	y'	z'	2	1
	$\smile x'$	y	z'	2	2
	$\smile x'$	y'	z	2	3
	$\smile x'$	y'	z'	3	4
	1.3	1.3	1.3		

a	b	c	name	
		$y'z'$	$1'$	$R_0 = (x \smile y' \smile z')$
	$x'z'$		$2'$	$(x' \smile y \smile z')$
$x'y'$			$3'$	$(x' \smile y' \smile z)$
$x'y'$	$x'z'$	$y'z'$	$4'$	$(x' \smile y' \smile z')$

The combinations in R_0 are identical with those in S_0 in this example.

2. The generators of the invariant transformations are

$$t_1 = (xy) = \begin{pmatrix} xx' & yy' & zz' \\ yy' & xx' & zz' \end{pmatrix}, \quad t_2 = (xz)\,.$$

All other elements t_i are obtained from these two:

$$t_1t_1 = t_0\text{: unit element,}$$
$$(yz) = t_3 = t_1t_2t_1\,,$$

$$t_1t_2 = \begin{pmatrix} xy \\ yx \end{pmatrix}\begin{pmatrix} xz \\ zx \end{pmatrix} = \begin{pmatrix} xyz \\ yzx \end{pmatrix} = t_4\,,$$
$$t_2t_1 = \begin{pmatrix} xz \\ zx \end{pmatrix}\begin{pmatrix} xy \\ yx \end{pmatrix} = \begin{pmatrix} xyz \\ zxy \end{pmatrix} = t_5\,.$$

The group table is as follows.

TABLE I
THE GROUP TABLE

	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	2	3
2	2	5	0	4	3	1
3	3	4	5	0	1	2
4	4	3	1	2	5	0
5	5	2	3	1	0	4

3. $F = (a \smile 3)(b \smile 2)(c \smile 1)(a \smile b \smile c \smile 4)$

$$\begin{aligned} &= abc && = S_1 \\ &\smile ab1 \smile ac2 \smile bc3 && \smile S_2 \smile S'_2 \smile S''_2 \\ &\smile a12 \smile b13 \smile c23 && \smile S_3 \smile S'_3 \smile S''_3 \\ &\smile 1234 && \smile S_0\,. \end{aligned}$$

All primed expressions are reduced to unprimed S_i by t_j:

$$t_1S'_2 = t_2S''_2 = S_2, \qquad t_2S'_3 = t_3S''_3 = S_3$$

Therefore it is sufficient to consider only $S_i (i = 1, 2, 3, 0)$ in further calculation.

TABLE II

		D			D			D			D
	$S_1 = x'y'$	2		$S_2 = x'y'$	2		$S_3 = x'y'$	2		$S_0 = x\ y'z'$	3
	$\smile x'\ \ z'$	2		$\smile x'\ \ z'$	2		$\smile x\ y'z'$	3		$\smile x'y\ z'$	3
	$\smile\ \ y'z'$	2		$\smile x\ y'z'$	3		$\smile x'y\ z'$	3		$\smile x'y'z$	3
										$\smile x'y'z'$	3
o 6	2 2 2	S_{10}	7	1 2 2 2	S_{20}	8	1 2 1 2 2	S_{30}	12	1 3 1 3 1 3	S_0
x 3	1 1 1	S_{11}									
x 4	2 1 1	S_{12}	x 4	1 1 1 1	S_{21}						
x 4	1 2 1	$t_1 \rightarrow S_{12}$									
x 4	1 1 2	$t_2 \rightarrow S_{12}$									
o 5	2 2 1	S_{13}	o 5	1 2 1 1	S_{22}	x 5	1 1 1 1 1	S_{31}			
o 5	2 1 2	$t_3 \rightarrow S_{13}$	x 5	1 1 2 1	S_{23}						
o 5	1 2 2	$t_2 \rightarrow S_{13}$	x 5	1 1 1 2	$t_3 \rightarrow S_{23}$						

o: realizable x: unrealizable

4. R_{11} and S_{11}

	Terms P		x'	y'	z'	R_{11}	Q
$R_{11} = (x' \smile y'\ \ \ )$	2		1	1	.	1	3
$(x'\ \ \ \smile z')$	2	$B^f_\kappa =$	1	.	1	1	3
$(\ \ \ y' \smile z')$	2		.	1	1	1	3
2 2 2		$B^h_\kappa =$	.	.	.	1	
						(subcut)	

F = factors = 3, B = rank of $B^f_\kappa = 3$.
P = maximum no. of terms = 2, Q = max. no. of branches $= P + 1 = 3$.
nodes: $\alpha^0(R_{11}) = B + 1 = 4$.

If pseudocuts are added

$$\alpha^0(R_{11}) > B + 1 = 4.$$

degree of freedom $P^1(R_{11}) \geqq P = Q - 1 = 2 = 3 - 1$.
$\therefore$ Branches: $\alpha^1(R_{11}) = P^1 + \alpha^0 - 1 \geqq 2 + 4 - 1 = 5$.

	factors D		x'	y'	z'	S_1	E
$S_{11} = x'\ y'$	2		1	1	.	1	3
$\smile x'\ \ \ z'$	2	$C_g{}^\kappa =$	1	.	1	1	3
$\smile\ \ \ y'\ z'$	2		.	1	1	1	3
2 2 2		$C_k{}^\kappa =$	.	.	.	1	

D = max. no. of factors = 2,
E = max. no. of branches in loops $= D + 1 = 3$.
G = no. of terms = 3, C = rank of $C_g{}^\kappa = 3$.

$\alpha^0(S_{11}) - 1 \geqq D = E - 1 = 2$,
$\alpha^0(S_{11}) \geqq E = 3$
$P^1(S_{11}) \geqq C = 3$
$\alpha^1(S_{11}) = P^1 + \alpha^0 - 1 \geqq 3 + 3 - 1 = 5$.

Furthermore, though R_1 corresponds to S_1, their incidence matrices do not form null-factors:

$$B^f{}_\kappa C^\kappa{}_q = 1^f{}_q \neq 0\,, \quad C^\kappa{}_q = C_q{}^\kappa\,.$$

This shows that there is no network which simultaneously satisfies $B^f{}_\kappa$ and $C_g{}^\kappa$. Also both the subcut $B^h{}_\kappa$ and subtie $C_\kappa{}^h$ have the order relation:

$$B^f > B^h\,, \quad C_g > C_k\,.$$

For S_{12}, the corresponding $C_g{}^\kappa$ is given by

	$1x'$	$2x'$	y'	z'	S_{12}
$C_g{}^\kappa =$	1	a	1	.	1
	.	1	.	1	1
	.	.	1	1	1
$C_k{}^\kappa =$	1	a'	.	.	1 , $a' = a + 1$.

The above $C_g{}^\kappa$ is the most general 2 contact case of x' if a is regarded as an unknown (O or 1), as is easily proved.

$$\alpha^1 = 5,\ \alpha^0 \geqq 4,\ P^1 \leqq 2\,.$$

But $G = C = 3$.

Also the subtie C_4 forms a sneak path. Hence, there is no 4 contact realization in S_1. In S_2 obtained from the $C_g{}^\kappa$ and $C_k{}^\kappa$ of S_{21} one gets $\alpha^1 = 5$, $\alpha^0 \geqq 4$ $P^1 \leqq 2$s. But $G = C = 3$. Furthermore, the subtie C_4 forms a sneakpath. Hence, a single contact realization for S_2 does not exist. Thus, there is no 4 contact realization.

Cases of 5 Contacts

	$1x'$	$2x'$	$1y'$	$2y'$	z'	S_{13}
$C_g{}^\kappa(S_{13}) =$	1	b	1	d	.	1
	a	1	.	.	1	1
	.	.	c	1	1	1
$C_k{}^\kappa =$	a'	b'	c'	d'	.	1

$$\alpha^1 = 6, \quad \alpha^0 \geqq 3. \quad \therefore P^1 \leqq 4. \quad G = C = 3.$$

From R_i, $a^0 \geqq B + 1 = 4$.

$\therefore P^1 = a^1 - a^0 + 1 \leqq 6 - 4 + 1 = 3.$

$\therefore N = P^1 - C = 3 - 3 = 0.$

$\therefore a^0 = a^1 - P^1 + 1 = 6 - 3 + 1 = 4.$

Concerning the subtie C_4, if $a' = b' = c' = d' = 0$ or $a = b = c = d = 1$, then $C_4^{\kappa} = \ldots\ldots 1$ is smaller than C_g. Hence, this is unrealizable. The only other possibility is that C_4 expresses $x'y'$. Then

$$a' \smile b' = 1, \quad c' \smile d' = 1, \quad \text{or } ab = cd = 0$$

TABLE III

	a	b	c	d
Case 1	0	0	0	0
2	0	1	0	0
3	1	0	0	0
2′	0	0	0	1 → Case 2 by t_1,
3′	0	0	1	0 → Case 3 by t_1.
1′	0	1	0	1 → Case 1, e.g. $C_1 = C_4$ of case 1.
3″	0	1	1	0 → Case 3 by($1y'$, $2y'$) and t_1,
3‴	1	0	0	1 → Case 3 by ($1x'$, $2x'$).
4	1	0	1	0

Case 1

	$1x'$	$2x'$	$1y'$	$2y'$	z'	S_{13}	
	1	.	1	.	.	1	$1x'$
$C_g^{\kappa} =$	.	1	.	.	1	1	$2x'$ (Fig. 1)
	.	.	.	1	1	1	$2y'$
			.		.	.	
			2		3	1	
$C_4^{\kappa} =$	1	1	1	1	.	1	

$C_4 > C_1$.

$\therefore C_4$ is a multiple loop.

The connection is obtained from C_g^{κ} by the ambit method as shown in Fig. 1, because C_g^{κ} is already semidiagonalized.

Case 2

	$1x'$	$2x'$	$1y'$	$2y'$	z'	S_{13}
	1	1	1	.	.	1
$C_g^{\kappa} =$	.	1	.	.	1	1
	.	.	.	1	1	1
$C_k^{\kappa} =$	1	.	1	1	.	1

	$1x'$	$2x'$	$1y'$	$2y'$	z'	S_{13}	
	1	.	1	.	1	.	$1x'$
$C_g^{\kappa} =$	.	1	.	.	1	1	$2x'$
	.	.	.	1	1	1	$2y'$
			.		.	.	
			3		2	1	

The circuit is obtained from semi-diagonalized C_g^{κ}, (see Fig. 2)

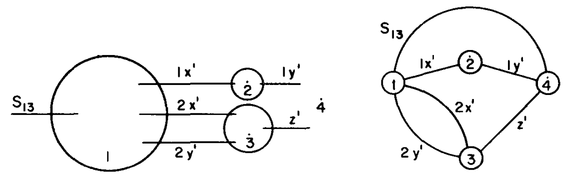


Fig. 1.

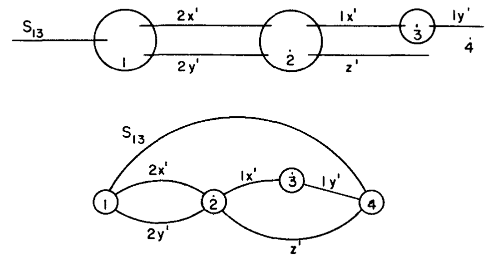


Fig. 2.

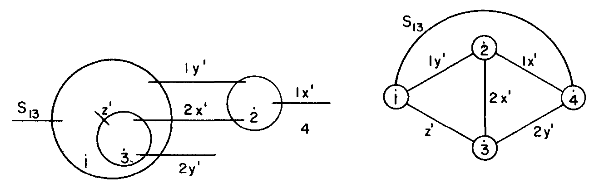


Fig. 3.

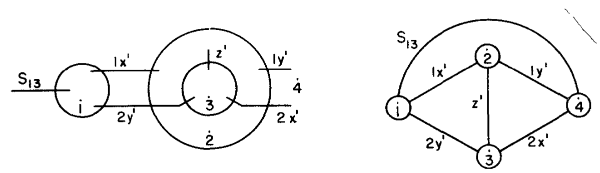


Fig. 4.

Case 3

	$1x'$	$2x'$	$1y'$	$2y'$	z'	S_{13}	
	1	.	1	.	.	1	$1y'$
$C_g^{\kappa} =$	1	1	.	.	1	1	$2x'$
	.	.	.	1	1	1	$2y'$
	.				.	.	
	2				3	1	
$C_k^{\kappa} =$	.	1	1	1	.	1	

This gives a bridge of Fig. 3.

Case 4

									$1y'$		z'	S_{13}	
	1	.	1	.	.	1	1	.	1	.	.	1	$1x'$
$C_g^{\kappa} =$	1	1	.	.	1	1	.	1	1	.	1	.	$2x'$
	.	.	1	1	1	1	.	.	1	1	1	1	$2y'$
									.		.	.	
									2		3	1	

Add C_1 to C_2 in order to obtain semi-diagonalized C_g^{κ}.

$$C_k^{\kappa} = .\ 1\ .\ 1\ .\ 1\ .$$

This yields another bridge of Fig. 4.

For S_{22}, the following case is the most general form of $C_g{}^K$ based on t_3:

$$\begin{array}{lcccccc} & x & 1x' & 2x' & y' & z' & S_{22} \\ & . & 1 & a & 1 & . & 1 \\ C_g{}^K = & . & . & 1 & . & 1 & 1 \\ & 1 & . & . & 1 & 1 & 1 \\ C_k{}^K = & 1 & 1 & a' & . & . & 1 \end{array}$$

$$\alpha^1 = 6,\ \alpha^0 \geqq 4, \quad \therefore P^1 \leqq 3. \quad G = C = 3.$$

$C_4{}^K$ forms a pseudotie and it cannot be a multiple loop.

$$N = P^1 - C = 3 - 3 = 0.$$

Hence, there is no room of adding pseudoties.
This is semi-diagonalized to $C_g{}^K$ by adding C_2 to C_3.

$$\begin{array}{lccccccc} & & & 2x' & y' & & S_{22} & \\ & . & 1 & a & 1 & . & 1 & 1x' \\ C_{q'}{}^K = & . & . & 1 & . & 1 & 1 & z' \\ & 1 & . & 1 & 1 & . & . & x \\ & & & \dot{3} & \dot{2} & & \dot{1} & \end{array}$$

This generates 2 kinds of bridges as shown in Figs. 5 and 6. For S_{23}, the subtie C_4 forms a sneak path xy' or x. Hence, it is not realizable.

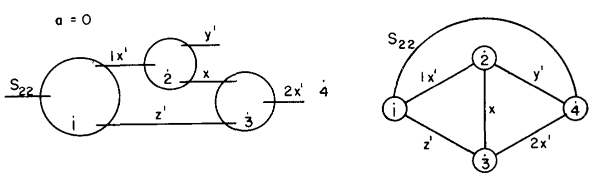


Fig 5.

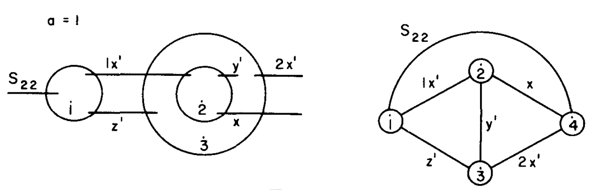


Fig. 6.

Application of t_i yields more networks and the total number is given by Table IV.[16,18]

TABLE IV

Fig.	1	2	3	4	5	6	Sum
i	1–5	1, 2=4, 3=5	1–5 1,	2=4, 3=5	1=4, 2=5, 3	1–5	
	5	3	5	3	3	5	26

Example 2[17]

1. The decimal expression of short circuit combinations is

11, 6, 1, 0 .

That of open circuits is

13, 12, 11, 10, 8, 7, 6, 5, 3, 2, 1, 0 .

$$\begin{array}{ll} S_0 = w\ x'\ y\ z & S_1 = w\ x'\ y\ z \\ \smile w'\ x\ y\ z' & \smile w'\ x\ y\ z' \\ \smile w'\ x'\ y'\ z & \smile w'\ x'\ y' \\ \smile w'\ x'\ y'\ z' & \end{array}$$

16 1·3 1·3 2·2 2·2 11 1·2 1·2 2·1 1·1

$$R_0 = (w \smile x \smile y' \smile z) \quad \cdots \qquad R_1 = (w \smile x \smile y')\ (w' \smile y)\ (w' \smile z)\ (x' \smile y)\ (x' \smile z')$$

48 5·7 5·7 6·6 6·6 11 1·2 1·2 2·1 1·1

2. The invariant transformation is only one:

$$t_1 = (wx)(zz') = \begin{pmatrix} w\ w'\ x\ x'\ y\ y'\ z\ z' \\ x\ x'\ w\ w'\ y\ y'\ z'\ z \end{pmatrix}$$

3. $F = S_1 \smile S_0$.

f for R_1 is long, only the dual prime implicant R_1 is given above.

4. The incidence matrix for S_1 is

$$\begin{array}{lccccccccccc} & w & w' & x & x' & y & y' & z & z' & S_1 & E & \\ & 1 & . & . & 1 & 1 & . & 1 & . & 1 & 5 & w \\ C_g{}^K = & . & 1 & 1 & . & 1 & . & . & 1 & 1 & 5 & x \\ & . & 1 & . & 1 & . & 1 & . & . & 1 & 4 & y' \\ C_k{}^K = & 1 & . & 1 & . & . & 1 & 1 & 1 & 1 & & \\ & & \dot{2} & & \dot{3} & \dot{4} & & \dot{5} & \dot{6} & \dot{1} & & \end{array}$$

The only subtie C_4 is a single loop pseudotie. Furthermore $C_g{}^K$ itself forms $C_q{}^K$. Hence, if it is realizable, it forms an answer. This $C_g{}^K$ is already semidiagonalized as seen in columns w, x, and y'. However, realization of ambits in the order of dotted number proves that this is not realizable as seen in Fig. 7. On the other hand, the standard sum S_0 is not realizable by single contacts as easily seen from its subties or for the reason explained in *3* (see eq. 4).

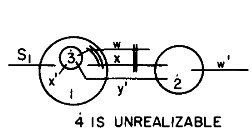


Fig. 7.

5.
$$\begin{aligned} E(S_1) &= 5\,. \quad \alpha^0 \geqq 5\,. \\ P^1 &= \alpha^1 - \alpha^0 + 1 \leqq 9 - 5 + 1 = 5. \\ G &= C = 3\,. \\ N &= P^1 - C \leqq 5 - 3 = 2. \end{aligned}$$

That is, an addition of pseudotie base vectors is possible, the maximum addition being 2.

$$Q(R_1) = 4\,, \quad P^1 \geqq Q - 1 = 3\,.$$

6. At first an addition of one pseudotie will be considered. In this case,

$$P^1 = G + 1 = 4\,, \quad a^1 = 9\,,$$
$$a^0 = a^1 - P^1 + 1 = 9 - 4 + 1 = 6\,.$$

The 8 components of the adding pseudotie form an unknown quantity:

$$C_g{}^K = \begin{array}{ccccccccccc} 1 & . & . & 1 & 1 & . & 1 & . & 1 & & 1 \\ . & 1 & 1 & . & 1 & . & . & 1 & 1 & & 2 \\ . & 1 & . & 1 & . & 1 & . & . & 1 & & 3 \end{array}$$

$$C_n{}^K = \begin{array}{ccccccccccc} a & b & c & d & e & f & g & h & 1 & & 4 \end{array}$$

$$\begin{array}{lccccccccccccccc} & & & & & & & & & & & 1 & 2 & 3 & 4 \\ & 1 & . & 1 & . & . & 1 & 1 & 1 & 1 & 5 & x & x & x & \\ & a' & b' & c' & d' & e & f & g' & h' & 1 & 6 & x & x & & x \\ C_k{}^K = & a' & b' & c & d & e' & f' & g' & h & 1 & 7 & x & & x & x \\ & a & b & c' & d' & e' & f' & g & h' & 1 & 8 & & x & x & x \end{array} \quad 8 = 2^{C-1} = 2^3.$$

$t^1C_2 = C_1,\ t_1C_8 = C_7\,.$

(a) Pseudotie conditions of the subties $C_k{}^K$.

TABLE V

Case										
1	.	.	.	.					1	
2	.	.			.	.			1	$t_1 \rightarrow 9$
3	.	.					1	.	1	$t_1 \rightarrow 10$
4	1	1	.	.	.	.			1	$\rightarrow 9$
5	1	1	.	.			.	1	1	$\rightarrow 10$
6	1	1	1	1	.	.			1	
7	1	1	1	1			.	.	1	
8	1	1	1	1	1	1			1	
9			.	.	.	.			1	
10			.	.			.	1	1	
11			1	1	.	.	.	.	1	$\rightarrow 13$
12			1	1	1	1	1	.	1	
13					.	.	.	.	1	
14	1	1			1	1	.	1	1	$t_1 \rightarrow 12$
15			.	.	1	1	.	1	1	$\rightarrow 10$
16	.	.			1	1	1	.	1	$t_1 \rightarrow 15$
17			1	1	1	1	1	.	1	$\rightarrow 12$

The blank part is perfectly arbitrary.

(b) The pseudotie condition of $C_4{}^K$ algebraically means that at least one pair of make and break contacts of one relay must both be unity. If this is considered in the above combinations of values of $C_k{}^K$, all possible values are given as follows.

TABLE VI

Case	No.										
1a		.	.	.	.	1	1	*g*	*h*	1	
	1	.	.			1	1	.		1	unr.
	2	.		.	.	1	1		1	1	unr.
	3	.	.	.		1	1	1	.	1	unr.
	4	.	.	.	.	1	1	1	1	1	unr.
1b		.	.	.	.	*e*	*f*	1	1	1	
	5	.	.	.	.		.	1	1	1	No. 1
	6		.	.	.		1	1	1	1	unr.
	7	.	.	.	.	1	.	1	1	1	No. 2
	8	.	.	.	.	1	1	1	1	1	= 4
6		1	1	1	1	.	.	*g*	*h*	1	
	9	1	1	1	1	.	.	.	.	1	No. 1 C_6
	10	1	1	1	1	.			1	1	x $C_6 < C_1$
	11	1	1	1	1	.		1	.	1	x $C_6 < C_2$
	12	1	1	1	1		.	1	1	1	x $C_6 < C_g$
7		1	1	1	1	*e*	*f*	.	.	1	
	13	1	1	1	1	.	.	.	.	1	= 9
	14	1	1	1	1		1	.	.	1	6 C_6
	15	1	1	1	1	1	.	.	.	1	No. 2 C_6
	16	1	1	1	1	1	1	.	.	1	4 C_6
8		1	1	1	1	1	1	*g*	*h*	1	
	17	1	1	1	1	1	1	.		1	= 16
	18	1	1	1	1	1	1		1	1	3 C_6
	19	1	1	1	1	1	1	1		1	2 C_6
	20	1	1	1	1	1	1	1	1	1	1 C_6
9a		1	1	.	.	.		*g*	*h*	1	
	21	1	1	.	.	.	.	.		1	3 C_7
	22	1	1	.	.	.	.	.	1	1	4 C_7
	23	1	1	.	.	.	.	1	.	1	1 C_7
	24	1	1	.	.	.	.	1	1	1	2 C_7
9b		*a*	*b*	.	.	.	.	1	1	1	
	25	.	.	.	.	.	.	1	1	1	= 5
	26	.	1	.	.		.	1	1	1	unr.
	27	1	.	.	.	.	.	1	1	1	No. 3
	28	1	1	.	.	.	.	1	1	1	= 24
10a		1	1			*e*	*f*	.	1	1	
	29	1	1	.	.	.		.	1	1	= 22
	30	1	1	.	.	.	1	.	1	1	No. 2 C_7
	31	1	1	.	.	1		.	1	1	6 C_7
	32	1	1			1	1	.	1	1	No. 1 C_7
10b		*a*	*b*		.	1	1	.	1	1	
	33				.	1	1		1	1	= 2
	34		1		.	1	1	.	1	1	No. 3 C_7
	35	1				1	1	.	1	1	26 C_7
	36	1	1			1	1	.	1	1	= 32
12		*a*	*b*	1	1	1	1	1		1	
	37	.		1	1	1	1	1	.	1	No. 1 C_8
	38		1	1	1	1	1	1		1	26 C_8
	39	1		1	1	1	1	1		1	No. 3 C_8
	40	1	1	1	1	1	1	1	.	1	= 19
13b		1	1	*c*	*d*	.	.	.	.	1	
	41	1	1	.		.	.	.	.	1	= 21
	42	1	1		1	.	.	.	.	1	No. 4 C_6
	43	1	1	1		.	.	.		1	unr.
	44	1	1	1	1	.	.	.	.	1	= 9

The ambit method can be applied to case 1a, 1b, . . . and for each no., its subties $C_k{}^K$ are listed in the following. Then, only 11 pseudoties are determined to be linearly independent. 4 pseudoties are realizable cases.

Subties C_k^{κ}:

TABLE VII

	Unrealizable Cases		Realizable Cases
1	4 . . 1 1 . 1	No. 1	4 . . . . . . 1 1 1
	6 1 1 1 1 1 1 1 1 1		6 1 1 1 1 1
	7 1 1 . . . 1 . 1		7 1 1 . . 1 1 1 1
	8 . . 1 1 . . . 1 1		8 . . 1 1 1 1 1 . 1
2	4 . . 1 1 . 1 1	No. 2	4 . . . . 1 . 1 1 1
	6 1 1 1 1 1 1 1 . 1		6 1 1 1 1 1 . . . 1
	7 1 1 . . . 1 1 1		7 1 1 . . . 1 . 1 1
	8 . . 1 1 1		8 . . 1 1 . 1 1 . 1
3	4 . . . 1 1 1 . 1	No. 3	4 1 . . . . 1 1 1
	6 1 1 1 1 1 1 . 1 1		6 . 1 1 1 1
	7 1 1 1		7 . 1 . . 1 1 . 1 1
	8 . . 1 1 . . 1 1 1		8 1 1 1 1 1 1 . 1
4	4 . . . . 1 1 1 1 1	No. 4	4 . . 1 . . . 1 1 1
	6 1 1 1 1 1 1 . . 1		6 1 1 . 1 1
	7 1 1 . . . . . 1 1		7 . . . 1 1 1 1 . 1
	8 . . 1 1 . . 1 . 1		8 1 1 1 . 1 1 . 1 1
6	4 . . . . . 1 1 1 1		
	6 1 1 1 1 . 1 . . 1		
	7 1 1 . . 1 . . 1 1		
	8 . . 1 1 1 . 1 . 1		
26	4 . 1 . . . . 1 1 1		
	6 1 . 1 1 . . . 1		
	7 1 . . . 1 1 . 1 1		
	8 . 1 1 1 1 1 1 . 1		
43	4 1 1 1 . . : . . 1		
	6 . . . 1 . . 1 1 1		
	7 . . 1 . 1 1 1 . 1		
	8 1 1 . 1 1 1 . 1 1		

Case 1a

```
                              x'   y' z  z' S1
        1 . . 1 1 . 1 . 1   1 . . 1 . 1 g' h . w
C_q^K = . 1 1 . 1 . . 1 1   . . 1 1 . . g h' 1 x
        . 1 . 1 . 1 . . 1   . 1 . 1 . 1 . . 1 w'
        . . . . 1 1 g h 1   . . . . 1 1 g h 1 y
                            .   . . . .
                            2   3 4 5 1
```

Add C_3 to C_2, then add C_4 to C_1 and C_2.

3 is unrealizable independent of g and h as seen from Fig. 8.

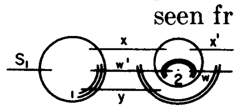


Fig. 8.

Case 1b

```
                              x' y  y' z  S1
        1 . . 1 1 . 1 . 1   1 . . 1 1 . 1 . 1 w
        . 1 1 . 1 . . 1 1   . . 1 1 e' f' 1 . 1 x
C_q^K = . 1 . 1 . 1 . . 1   . 1 . 1 . 1 . . 1 w'
        . . . . e f 1 1 1   . . . . e f 1 1 1 z'
                                . . . .   .
                                2 4 5 3   1
```

This is semi-diagonalized by adding C_3 and C_4 to C_2.

See Figs. 9 and 10.

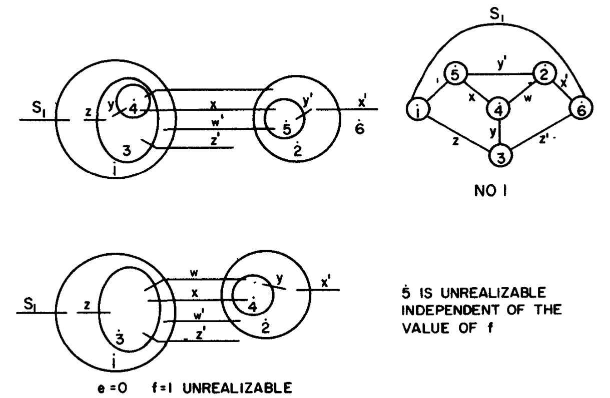


Fig. 9.

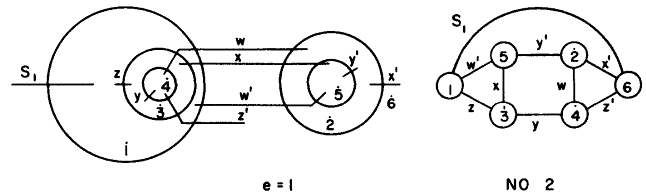


Fig. 10.

ef = . . realizable No. 1
= . 1 unrealizable as seen from Fig. 9
= 1 . realizable No. 2
= 1 1 unrealizable as seen from Fig. 10

Case 6

```
C_k^K = 4 1 1 1 1 . . g  h  1     gh = . . No. 1
        6 . . . . . . g' h' 1          . 1 C6 < C1
        7 . . 1 1 1 1 g' h  1          1 . C6 < C2
        8 1 1 . . 1 1 g  h' 1          1 1 C6 < Cg
```

No. 9 of Case 6 is identical to No. 1. C_6. All other 3 are smaller than one of C_g^{κ}. Hence, it is unrealizable.

Case 9b

```
             a b
Only No. 26  . 1
     No. 27  1 .
```

are unknown.

```
                          w w'      y   z   S'
        1 . . 1 1 . 1 . 1 1 1 . 1 1 . 1 . 1 x'
        . 1 1 . 1 . . 1 1 a b' 1 . 1 . 1 . . x
C_q^K = . 1 . 1 . 1 . . 1 1 1 . . 1 1 1 . . y'
        a b . . . . 1 1 1 a b . . . . 1 1 1 z'
                          . .       .   .   .
                          4 5       3   2   1
```

This is semi-diagonalized by adding C_1 to C_3 and C_4 to C_2.

The ambit 5 is unrealizable for No. 26 as seen from Fig. 11.

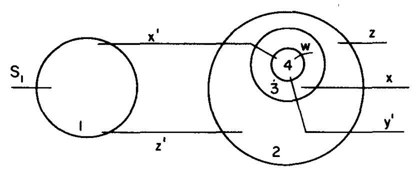


Fig. 11.

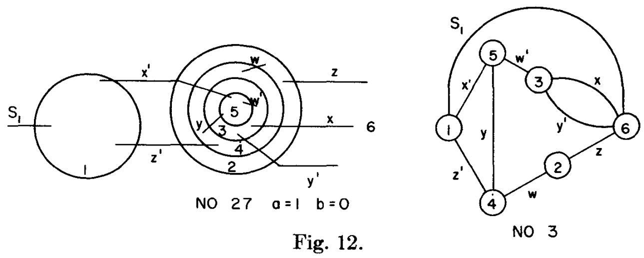


Fig. 12.

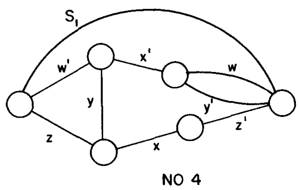


Fig. 13.

No. 27 is realizable and gives No. 3 of the solutions as seen in Fig. 12.

The first two connections Nos. 1 and 2 of Figs. 10 and 12 do not change by the transformation t_1. However, No. 3 changes to No. 4 of Fig. 13 by t_1. C_4^κ of No. 4 is obtained from C_4^κ of No. 3 by t_1.

Case 13b

No. 43

											w'	x	x'	y				S	
	1	.	.	1	1	.	1	.	1	.	1	1	1	1	.	1	.	.	z
	.	1	1	.	1	.	.	1	1	.	1	1	.	1	.	.	1	1	z'
$C_q^\kappa =$	.	1	.	1	.	1	.	.	1	.	1	.	1	.	1	.	.	1	y'
	1	1	1	.	.	.	.	.	1	1	1	1	.	.	.	.	.	1	w
											$\dot{2}$	$\dot{3}$	$\dot{4}$	$\dot{5}$				$\dot{1}$	

This is semi-diagonalized by adding C_4 to C_1.

In Fig. 14, the ambit $\dot{4}$ is unrealizable.

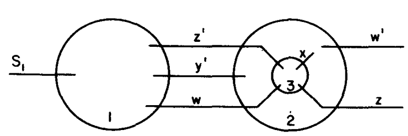


Fig. 14.

The algebraic ambit-realization is as follows.

Case 1b: $ef = \,.\;.$ (Fig. 9, No. 1)

$\dot{1}$: $(S\,w\,w'\,x\,z')\;S\,w\,w'\,x\,z'$

$\dot{2}$: $(S\,w\,w'\,x\,z')\;(w\,w'\,x\,x')\;x'\,S\,z'$
$= (S\,w\,w'\,x\,z')\;(w\,w'\,x\,x')\;S\,x'\,z'$

$\dot{3}$: $(S\,w'\,(w\,x\,z\,z')\,z)\;(w\,w'\,x\,x')\;S\,x'\,z'$
$= (S\,w'\,z\,(w\,x\,z\,z'))\;(w\,w'\,x\,x')\;S\,x'\,z'$

$\dot{4}$: $(S\,w'\,z\,((w\,x\,y)\,y\,z\,z'))\;(w\,w'\,x\,x')\;S\,x'\,z'$

$\dot{4}'$: $(S\,w'\,z\,(w\,x\,z\,z'))\;((w\,x\,y)\,w'\,x'\,y)\;S\,x'\,z'$

$\dot{5}$ (possible only from $\dot{4}$):

$(S\,w'\,z\,((w\,x\,y)\,y\,z\,z'))\;((w'x\,y')\,w\,x'\,y')\,S\,x'\,z'$
$(S\,w'\,z)\;(w\,x'\,y')\;(y\,z\,z')\;(w\,x\,y)\;(w'\,x\,y')\;(S\,x'\,z')\,.$

The coboundaries of the individual nodes are contained in the parenthesis.

Case 10a, No. 30, x, x', z and z' are semi-diagonalized by adding C_3 to C_1, C_4 to C_3. (Fig. 10, No. 2)

$\dot{1}$: $(S\,x'\,z')\;S\,x'\,z'$

$\dot{2}$: $(S\,x'\,z')\;S\,x'\,(z'\,w\,x\,z)\;w\,x\,z$
$= (S\,x'\,z')\;(w\,x\,z\,z')\;S\,w\,x\,x'\,z$

$\dot{3}$: $(S\,x'\,z')\;((w\,x\,z\,z')\,w\,x\,x'\,w')\;S\,z\,w'$

$\dot{4}$: $(S\,x'\,z')\;((w\,y\,z'\,(x\,y\,z))\,w\,w'\,x\,x')\;S\,w'\,z$
$= (S\,x'\,z')\;(((x\,y\,z)\,w\,y\,z')\,w\,w'\,x\,x')\;S\,w'\,z$

$\dot{5}$: $(S\,x'\,z')((((x\,y\,z)\,w\,y\,z')\,w\,x'\,y')\,w'\,x\,y')\;S\,w'\,z$
$(S\,x'\,z')\;(w\,y\,z')\;(w'\,x\,y')\;(x\,y\,z)\;(w\,x'\,y')\;(S\,w'\,z)\,.$

7. The addition of 2 pseudoties in S_1.

					1	1	.	.	1	1	.	1	.	1
			C_q^κ →		2	.	1	1	.	1	.	.	1	1
			C_k^κ		3	.	1	.	1	.	1	.	.	1
			↓		4	a	b	c	d	e	f	g	h	1
1	2	3	4	5	5	m	n	p	q	r	s	t	u	1
x	x	x			6	1	.	1	.	.	1	1	1	1
x	x		x		7	a'	b'	c'	d'	e	f	g'	h'	1
x		x	x		8	a'	b'	c	d	e'	f'	g'	h	1
	x	x	x		9	a	b	c'	d'	e'	f'	g	h'	1
x	x			x	10	m'	n'	p'	q'	r	s	t'	u'	1
x		x		x	11	m'	n'	p	q	r'	s'	t'	u	1
	x	x		x	12	m	n	p'	q'	r'	s'	t	u'	1
x			x	x	13	$1+a+m$	$b+n$	$c+p$	$1+d+q$	$1+e+r$	$f+s$	$1+g+t$	$h+u$	1
	x		x	x	14	$a+m$	$1+b+n$	$1+c+p$	$d+q$	$1+e+r$	$f+s$	$g+t$	$1+h+u$	1
		x	x	x	15	$a+m$	$1+b+n$	$c+p$	$1+d+q$	$e+r$	$1+f+s$	$g+t$	$h+u$	1
x	x	x	x	x	16	$1+a+m$	$b+n$	$1+c+p$	$d+q$	$e+r$	$1+f+s$	$1+g+t$	$1+h+u$	1

The base vectors C_q^κ and the subties C_k^κ must satisfy the following conditions:

(1) Vectors C_4 and C_5 are added and all subties C_k are either pseudoties or multiple loops.

(2) The above 12 vectors C_4, C_5 and C_k are not smaller than any of the 3 original short circuit relay-loop vectors C_1, C_2 and C_3. These conditions are necessary for realization.

The minimum number of branches in a multiple loop can be determined by the following reasoning. The minimum number of branches in C_q^κ is 4 (in C_3). However, a pseudotie can have only 3 branches as in C_4 of No. 1 of Fig. 9. On the other hand, the minimum number of contact branches of a local loop consisting only of contact branches is 2 as xy' in No. 3 of Fig. 12. Therefore, in this example, the minimum number of total branches of a multiple loop is $4 + 2 = 6$.

Since the number of variables is 4, every multiple loop has at least one pair of make and break contacts of one relay, that is, all multiple loops will be included in the subties C_k^κ if all vectors satisfying algebraic pseudotie conditions are considered.

In *6*, all possible vectors satisfying the algebraic pseudotie condition are already obtained for single vectors. Therefore, if any pair of these vectors further satisfies the algebraic pseudotie condition for C_{13}, C_{14}, C_{15} and C_{16}, and the resulting base vectors C_q^κ are realizable, then a network of solution can be obtained. For this purpose, the following vectors were chosen, one from each of 11 sets of the subties of Table VII.

TABLE VIII

		w	w'	x	x'	y	y'	z	z'	S_1
1	No. 3–4	1	.	.	.	.	.	1	1	1
2	26–4	.	1	.	.	.	.	1	1	1
3	No. 4–4	.	.	1	.	.	.	1	1	1
4	43–6	.	.	.	1	.	.	1	1	1
5	No. 2–4	.	.	.	.	1	.	1	1	1
6	6–4	.	.	.	.	.	1	1	1	1
7	No. 1–4	.	.	.	.	.	.	1	1	1
8	2–7	1	1	.	.	.	.	1	1	1
9	3–8	.	.	1	1	.	.	1	1	1
10	4–4	.	.	.	.	1	1	1	1	1
11	1–6	1	1	1	1	1	1	1	1	1

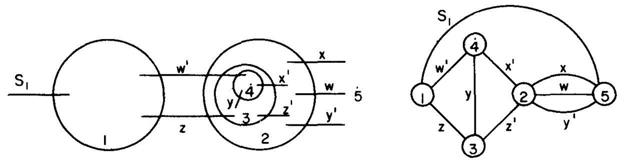
Out of ${}_{11}C_2 = 55$ combinations, it is not so difficult to find all possible combinations for which the new 4 subties $C_{13} - C_{16}$ satisfy the algebraic pseudotie condition. C_1C_3, C_1C_6 and C_3C_6 are the only possible combinations, and these build pseudoties at wx, wy' and xy', respectively.

wx, 1, 3:

												x		y			z'	S_1	
	1	.	.	1	1	.	1	.	1	1	.	1	.	.	.	.	.	.	w
	.	1	1	.	1	.	.	1	1	.	1	1	.	1	.	.	1	1	w'
$C_q^\kappa =$	.	1	.	1	.	1	.	.	1	.	.	1	.	.	1	.	.	.	y'
	1	.	.	.	.	.	1	1	1	.	.	.	1	1	.	.	1	.	x'
	.	.	1	.	.	.	1	1	1	.	.	1	.	.	.	1	1	1	z
												.			.		.	.	
Semi-diagonalization:												2			4		3	1	

1. Add C_1 to C_4, C_2 to C_3.
2. Add C_4 to C_1 and C_3.
3. Add C_5 to C_1.

This is realized in Fig. 15 as No. 5.



NO 5

Fig. 15.

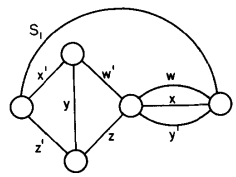


Fig. 16.

The other two: wy': 1, 6 and xy': 3, 6 generate only the networks of the same type. t_1 changes No. 5 to No. 6 of Fig. 16. Thus, these 6 networks are the only possible single contact realizations.

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Discussion

H. G. Boehm (IBM): Can your method be programmed on a computer?

Mr. Okada: Yes, we are considering it, but as yet we have not made a program of it. If the number of variables increases, calculations by these computers seems to be the only possible method.

Mr. Boehm: How long would it take to prove Moore's table as minimal?

Mr. Okada: I don't know. Unfortunately we could not finish. The proof is not so difficult, but to exhaust all possible minimum solutions, which means determination of all possible connections would take a long time. We are now doing research on this particular program.

G. G. Murray (RCA): Existing minimization methods are limited to series-parallel nets. However, these methods are quite useful. The method described in the present paper presumably gives a true minimum but is it a practical procedure? How many Boolean variables can be handled?

Mr. Okada: As soon as the number of variables increases, the juxtaposition of all possible Boolean functions is not so easy, but with a computer will be possible.

S. Sharin (RCA): How can this method be used to advantage to design code translators? Can the logic element used — relays, diode circuitry, and so forth — be introduced as a factor in getting a direct solution?

Mr. Okada: As a general principle this principle is applicable for design of any switching circuit. As I mentioned at the beginning this method can also be used to design series-parallel connections.

K. Enslein (Brooks Research): How does this new method differ from that which resulted in the Harvard tables?

Mr. Okada: According to my knowledge, the late Dr. Roderick Gould gave the best results, especially in the improvement of Dr. Moore's table. But Dr. Gould's method was not mathematically rigorous, and in most cases he gave only one solution; but, as I showed, all possible solutions can be given by our method. Our ambit method of realizing connections from a cut-set matrix is the dual of Gould's method of realization from loop matrices, which needs three kinds of operations: parenthesis, brackets and braces. In our algebraic ambit method, one kind of operation is sufficient as shown at the end of Paragraph 6 of Example 2. Also, we gave the maximum permissible number of pseudocuts or pseudoties which can be added, as shown in Paragraph 5 of Example 2. Generally, we use tie-set matrices and cut-set matricies simultaneously. We are planning to publish a paper on the "ambit realization."