A Combined Analog-Digital Differential Analyzer

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Introduction

THE ELECTRONIC analog computer, although very useful in solving many problems, and particularly useful in solving dynamic problems described by differential equations, suffers from limitations of accuracy and dynamic range. The digital differential analyzer, although capable of providing any required degree of accuracy or dynamic range, is slow in operation and subject to possible instability of solution due to quantization and the use of finite difference calculus in integration. By combining analog and digital techniques, it is possible to combine the analog advantages of high speed and continuous representation of variables with the digital capability of high precision and large dynamic range.

Dependent variables in such a combined system are represented by two quantities, a digital number, representing the more significant part, and an electrical voltage representing the less significant part. As in the electronic analog computer, the independent variable is always time. Let us consider what form some of the required computer components, such as integrators and multipliers, would take in such a combined system.

Integrator

Assume we wish to obtain the following:

$$y = y_0 + \frac{1}{T}\int_0^t x\,dt \qquad (1)$$

where x and y are functions of the time, and T is the "time constant" of the integration. As in the digital differential analyzer, it is necessary that the problem be scaled so that the maximum value of all dependent variables will not exceed unity. Let each of the two dependent variables x and y consist of a digital part and an analog part, denoted by the subscripts D and A, respectively. Thus, we have:

$$x = x_D + x_A \qquad (2)$$

$$y = y_D + y_A \qquad (3)$$

$$y = y_{0D} + y_{0A} + \frac{1}{T}\int_0^t (x_D + x_A)dt \qquad (4)$$

Let us assume time to be divided into discrete equal intervals of duration Δt, and that the digital parts of x and y can change only at times which are integral multiples of Δt. We may then write for the value of y at a time t somewhere in the nth interval:

$$y = y_{0D} + y_{0A} + \frac{1}{T}\left[\sum_{i=1}^{n-1}(x_D)_i\Delta t + (x_D)_n\{t-(n-1)\Delta t\} + \int_0^t x_A dt\right] \qquad (5)$$

where $(x_D)_i$ is the value of x_D during the ith interval Δt. Fig. 1 shows a curve of x as an arbitrary function of t. The area under this curve from $t = 0$ to any arbitrary t would equal y^{T} in equation (5), assuming that the first two terms (y_{0D} and y_{0A}) on the right of equation (5) are *zero*. The first term in the bracketed expression, represented by area 1, is the integral of the digital part of x up to the time $(n-1)\Delta t$. The second term in the bracket expression, represented by area 2, is the integral of the digital part of x between $(n-1)\Delta t$ and t. The third term, represented by area 3, is the integral of the analog part of x from $t = 0$ to t.

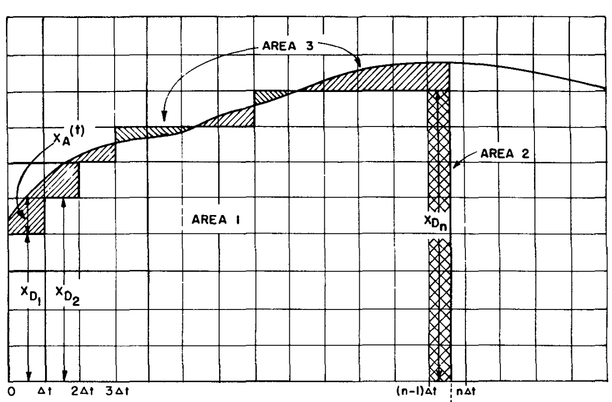

Fig. 1—Diagram of integration method.

Fig. 2 is a block diagram of an integrator unit. It contains an input digital register x_D, a digital register R, two digital-to-analog converters, a conventional analog integrator, a special resettable analog integrator, an analog summer, and a comparator unit. The register y_D shown on the far right of the figure is the input register of the next component to which this unit might be connected in solving a problem. E is the analog reference voltage supplied to the digital-to-analog converters, and α is the digital equivalent of the reference voltage E, chosen for any given problem so as to provide the desired compromise between speed of solution and precision, but subject to the limitation that $|dx/dt|_{max}$ should not exceed $\alpha/\Delta t$. The number of digits required in the x_D and R registers will depend upon the minimum value of α for which provision is to be made; the minimum value of α will be one in the least significant digit of the x_D register.

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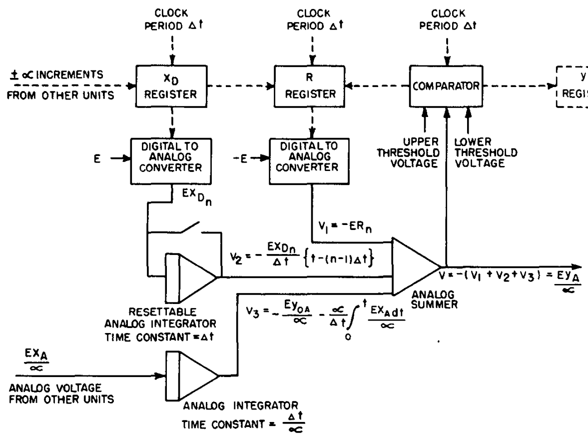

Fig. 2—Integrator unit.

At the beginning of each Δt period, the values x_D and R are sampled and converted to analog voltages which are held constant during the period, unaffected by future changes in x_D or R which occur during the period. The value of x_D is then algebraically added to the R register. The voltage V_1, which represents that portion of the prior summation of $(x_D)_i\,\Delta t$ which is of analog magnitude is given during the *nth* interval Δt by:

$$V_1 = -ER_n \tag{6}$$

The voltage V_2, which provides integration of the current x_D value within the *nth* interval Δt, and which is reset to *zero* at the end of this interval, is given by:

$$V_2 = \frac{E(x_D)_n\{t - (n - 1)\Delta t\}}{\Delta t}\,dt \tag{7}$$

The voltage V_3, which results from the purely analog integration of the continuously varying analog part of x, is given by:

$$V_3 = \frac{-Ey_{0A}}{\alpha} - \frac{\alpha}{\Delta t}\int_o^t \frac{Ex_A}{\alpha}\,dt \tag{8}$$

These three voltages are added in the analog summer to give voltage V. The analog part of the output of the integrator is equal to:

$$\frac{Ey_A}{\alpha} = V = -(V_1 + V_2 + V_3) \tag{9}$$

If, at any time during a period Δt, the voltage V at the output of the analog summer exceeds a predetermined upper threshold, this is sensed by the comparator and, during the next Δt interval immediately following the addition of x_D to R, unity is subtracted from the R register and the number α is added to the input register of the following unit (y_D in Fig. 2). Conversely, if the voltage V falls below a predetermined lower threshold, unity is added to the R register and the number α is subtracted from the input register of the following unit.

The time constant T of this integrator unit is equal to $\Delta t/\alpha$, as can be seen from the following. Assume that from time O up to a time t during the *nth* interval Δt, the comparator has caused N subtractions of unity from the R register, and the addition of N to the y_D register. The contents of the R register at this time is:

$$R = \sum_{i=1}^{n-1} (x_D)_i - N \tag{10}$$

and the value of y_D is given by

$$y_D = y_{OD} + N\alpha \tag{11}$$

Substituting equations (6), (7), (8) and (10) into (9) and solving for y_A, we obtain:

$$y_A = \alpha \sum_{i=1}^{n-1} (x_D)_i - N\alpha + \frac{\alpha}{\Delta t}(x_D)_n\,\{t - (n-1)\Delta t\} + y_{OA} + \frac{\alpha}{\Delta t}\int_o^t x_A dt \tag{12}$$

Adding equations (11) and (12), we have

$$y = y_D + y_A = y_{OD} + y_{OA} + \frac{\alpha}{\Delta t}\left[\sum_{i=1}^{n-1}(x_D)_i\,\Delta t + (x_D)_n\,\{t-(n-1)\Delta t\} + \int_o^t x_A dt\right] \tag{13}$$

Equation (13) is seen to be identical to equation (5) if $T = \Delta t/\alpha$.

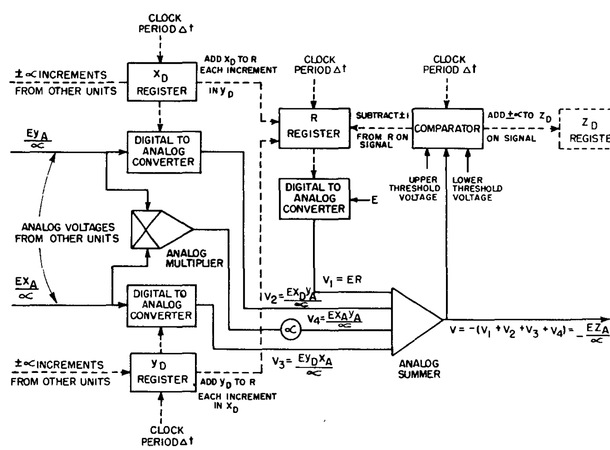

Fig. 3—Multiplier unit.

Multiplier

Let us now investigate the form taken by a combined analog-digital multiplier. Suppose we wish to obtain the product $z = xy$. Assuming, as before, that each variable consists of a digital part and an analog part, we have:

$$z_D + z_A = x_D y_D + x_A y_D + x_D y_A + x_A y_A \tag{14}$$

where the subscripts D and A signify the digital and analog parts, respectively. Assume, as before, that time is divided into equal intervals of duration Δt, and that the digital parts x_D and y_D can change only at times which are integral multiples of Δt. Fig. 3 is a block diagram of a multiplier unit. It has three digital registers for x_D, y_D, and R, three digital-to-analog converters, an analog summer, an analog multiplier, and a comparator unit. As before, E is the analog reference voltage and α is the digital value of the reference voltage E, chosen for any given problem so as to provide the desired compromise between speed of solution and precision, subject to the condition that neither $|dx/dt|_{\max}$ nor $|dy/dt|_{\max}$ should exceed $\alpha/\Delta t$.

At the beginning of each period Δt, the values of x_D, y_D, and R are sampled and converted to voltages which are held constant during the period. If, during the period, x_D receives an increment (or decrement) α from another unit, y_D is added to (or subtracted from) R; and if y_D receives an increment (or decrement) α from another unit, x_D is added to (or subtracted from) R. If both x_D and y_D change during Δt, the additions to R must either be done serially, using the new x_D or y_D obtained after each addition to R, for the next addition to R, or some other system must be used to obtain a true digital product $x_D y_D$. The quantity $x_D y_D$ can contain twice as many digits as x_D or y_D; the more significant part will be of digital magnitude, and appear in z_D, the input register of the following unit; and the less significant part will be of analog magnitude, and remain in the R register. The reference voltage E is applied to the digital-to-analog converter connected to register R, producing an output voltage $V_1 = ER$; the input voltage $y_A E/\alpha$ is applied to the converter connected to the register x_D producing an output voltage $V_2 = Ex_D y_A/\alpha$, and the input voltage Ex_A/α is applied to the converter connected to register y_D producing an output voltage $V_3 = Ey_D x_A/\alpha$. An analog multiplier is connected to the two analog inputs Ey_A/α and Ex_A/α. Its output, attenuated by α, produces a voltage $V_4 = Ex_A y_A/\alpha$. An analog summer sums the voltages V_1, V_2, V_3, and V_4 to produce a voltage V equal to $-Ez_A/\alpha$.

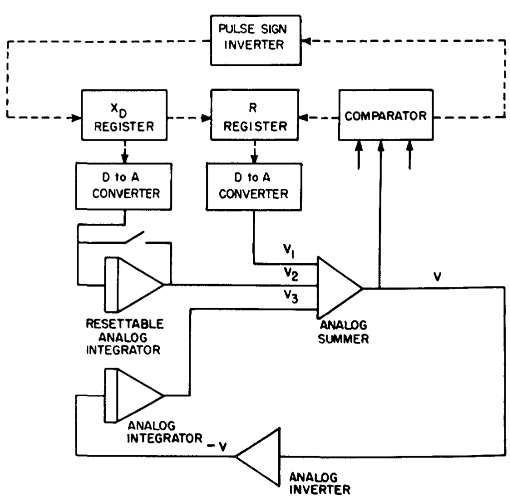

Fig. 4—Integrator used to solve: $\dot{x} = -x$.

During the next Δt after the voltage V exceeds (or falls below) predetermined threshold voltages, unity is subtracted from (or added to) the R register and the number α is added to (or subtracted from) the input register of the following unit (z_D in Fig. 3).

It should be noted that for small values of α the analog multiplier may be omitted, producing a maximum error of α. For values of α less than the resolution of the analog components, say .001 or less, this error is negligible.

If one of the factors to be multiplied is a constant, the equipment required is simplified, since only one digital register needs to be capable of accepting increments, and the R register receives additions from only one other register. If the factor is a purely digital quantity, one of the digital-to-analog converters and the analog multiplier may be omitted.

Summing

Summing may most easily be done by permitting each integrator or multiplier unit to accept digital increments and analog voltages from several units. For example, in the integrator of Fig. 2, if the $\pm\alpha$ increments from a number of other units are connected to its x_D register, and if the sum of the increments put out by these units is $N\alpha$ during any period, the increment in x_D would equal $N\alpha$. The analog outputs from the other units would each be connected to an input summing resistor in the analog integrator.

In the case of the multiplier unit, if the $\pm\alpha$ increments from a number of other units are connected to its x register, and if the sum of the increments put out by these units is $N\alpha$, y_D would be summed into the R register N times. The analog outputs from the other units would be connected to inputs of an analog summer whose output would form the analog input $x_A E/\alpha$ to the multiplier.

Solution of Simple Differential Equations

Examples of the operation of this proposed combined system can be seen from following in detail how some simple differential equations would be solved.

Let us consider first the differential equation

$$\dot{x} = -x \tag{15}$$

Fig. 4 shows a block diagram of how a single integrator unit with output fed back into its input would solve this equation. The voltages V_1, V_2, V_3, and V are those defined in equations (6) to (9).

Differentiating equations (6) to (9), we obtain, since $\dot{V}_1 = 0$

$$\dot{V} = \frac{Ex_D}{\Delta t} - \dot{V}_3 \tag{16}$$

From the interconnections of Fig. 4, the following expression must hold:

$$\dot{V}_3 = \frac{\alpha}{\Delta t} V \qquad (17)$$

V will then be given by the following differential equation:

$$\dot{V} + \frac{\alpha}{\Delta t} V = \frac{Ex_D}{\Delta t} \qquad (18)$$

Subject to the initial conditions that at $t = 0$, $x_D = x_{OD}$, and $-V = Ex_{OA}/\alpha$ the differential equation will have the following solution:

$$V = \frac{Ex_D}{\alpha} - \frac{E(x_{OD} + x_{OA})}{\alpha} e^{-\frac{\alpha}{\Delta t} t} : \qquad (19)$$

and x will be given by

$$x = x_D - \frac{\alpha V}{E} = (x_{OD} + x_{OA}) e^{-\frac{\alpha}{\Delta t} t} : \qquad (20)$$

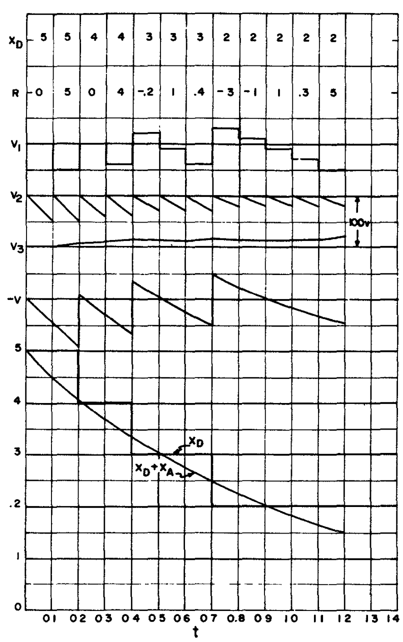

Fig. 5—Time history of solution of $\dot{x} = -x$.

Fig. 5 shows in detail the quantities that would appear in the x_D and R registers, the voltages V_1, V_2, V_3, and V as functions of the time, using the following parameters:

$E = 100$ volts $\qquad \alpha = 0.1 \qquad \Delta t = 0.1$ sec

$x_{OD} = 0.5 \qquad x_{OA} = 0$

The threshold values on V for initiating decrements to the x_D register and subtracting one from the R register are plus and minus 50 volts. In this example, since $\alpha = 0.1$, the precision obtained in solving this equation should be 10 times that which would be obtained in solving this on a purely analog computer with analog components of equivalent precision to those of the combined system.

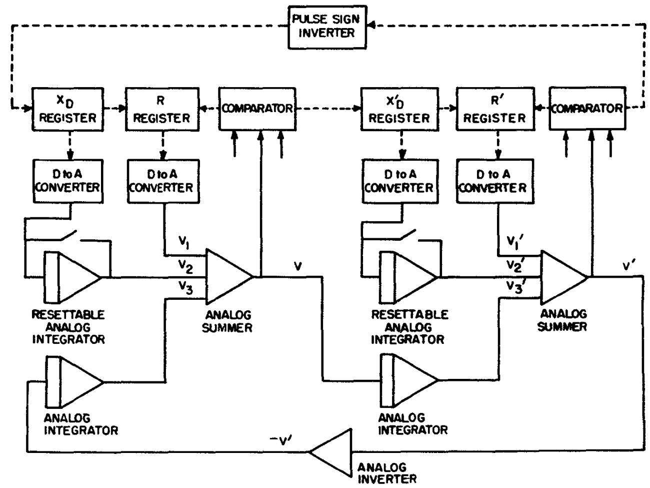

Fig. 6—Two integrators interconnected to solve: $\dot{x}' = x$, $\dot{x} = -x'$.

Another example of the operation of the proposed combined system is the solution of the following pair of simple differential equations:

$$\dot{x}' = x \qquad (21)$$

$$\dot{x} = -x' \qquad (22)$$

Fig. 6 shows a block diagram of how two integrator units would be interconnected to solve these equations. The voltages V_1, V_2, V_3, V are those defined by equations (6) to (9), and occur in the integrator containing x; the primed voltages are those which occur in the integrator containing x'. Differentiating equations (6) to (9), we have, since $\dot{V}_1 = \dot{V}'_1 = 0$:

$$\dot{V} = \frac{x_D E}{\Delta t} - \dot{V}_3 \qquad (23)$$

$$\dot{V}' = \frac{x_D E}{\Delta t} - \dot{V}'_3 \qquad (24)$$

From the interconnections of Fig. 6, the following expressions are seen to hold:

$$\dot{V}_3 = + \frac{\alpha}{\Delta t} V' \qquad (25)$$

$$\dot{V}'_3 = - \frac{\alpha}{\Delta t} V \qquad (26)$$

The quantities V and V' therefore will be given by the following differential equations:

$$\dot{V} = \frac{x_D E}{\Delta t} - \frac{\alpha}{\Delta t} V' \qquad (27)$$

$$\dot{V}' = \frac{x'_D E}{\Delta t} + \frac{\alpha}{\Delta t} V \qquad (28)$$

Subject to the condition that at $t = 0$, $x_D = x_{0D}$,

$x'_D = x'_{0D}$, $V = x'_{OA}E/\alpha$, $-V' = x_{OA}E/\alpha$; these differential equations will have the following solution:

$$V = -\frac{E}{\alpha}x'_D + \frac{E}{\alpha}(x'_{OD} + x'_{OA})\cos\frac{\alpha t}{\Delta t} + \frac{E}{\alpha}(x_{OD} + x_{OA})\sin\frac{\alpha t}{\Delta t} \quad (29)$$

$$V' = \frac{E}{\alpha}x_D - \frac{E}{\alpha}(x_{OD} + x_{OA})\cos\frac{\alpha t}{\Delta t} + \frac{E}{\alpha}(x'_{OD} + x'_{OA})\sin\frac{\alpha t}{\Delta t} \quad (30)$$

and the quantities x and x' will be given by the following expressions:

$$x = x_D - \frac{\alpha}{E}V' = (x_{OD} + x_{OA})\cos\frac{\alpha t}{\Delta t} - (x'_{OD} + x'_{OA})\sin\frac{\alpha t}{\Delta t} \quad (31)$$

$$x' = x'_D + \frac{\alpha}{E}V = (x'_{OD} + x'_{OA})\cos\frac{\mathrm{o}t}{\Delta t} + (x_{OD} + x_{OA})\sin\frac{\alpha t}{\Delta t} \quad (32)$$

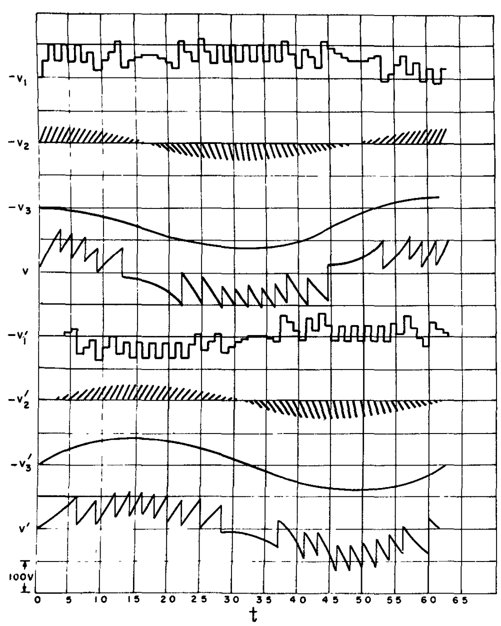

Fig. 7—Time history of solution of $\dot{x}' = x, \dot{x} = -x'$.

Fig. 7 shows in detail the voltages which would appear as a function of the time when solving the above equation, using the following values of the various parameters:

$E = 100$ volts $\alpha = 0.1$ $\Delta t = 0.1$ sec
$x_{OD} = 0.5$ $x_{OA} = 0$ $x'_{OD} = 0$ $x'_{OA} = 0$

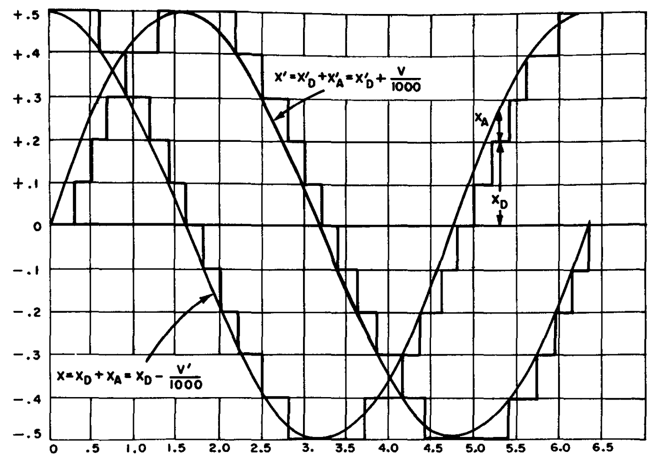

Fig. 8—Plot of x and x' as functions of time.

As in the previous example, the threshold values on V and V' for initiating positive or negative increments to the x'_D or the x_D register, are plus and minus 50 volts, respectively. Fig. 8 shows the results of combining the digital and analog portions. The stepped curves are the digital part only; the smooth curves are the sum of the digital and analog parts. As in the first example, the precision attained should be 10 times that which would be obtained solving these equations on an analog computer with analog components of precision equivalent to those used in the combined system.

Component Requirements

Now a few words on the characteristics needed in the hardware to realize the above components. For a maximum speed-precision product to be obtained, the value of Δt should be as small as possible consistent with hardware limitations. The smaller Δt is made, however, the greater the bandwidth required in the operational amplifiers, since the analog voltages must be capable of a full scale voltage excursion E during the time Δt. The digital-to-analog converters should be capable of holding their output values constant during each period of Δt and equal to its value at the beginning of the period, and then rapidly changing to its new value at the beginning of the next period. In the case of the integrator, the necessary additions of x_D to R, subtractions of ± 1 from R, and incrementing the input registers of following units and, in the case of the multiplier, the additions of x_D and y_D to R, subtractions of ± 1 from R, and incrementing the input registers of following units must all be completed within the period Δt. In the integrator unit, the resettable analog integrator might well consist of two analog integrators with switching between them so that each is used to integrate during alternate Δt periods while the other is being reset.

Discussion and Conclusions

There are a number of problems associated with this combined system that have not yet been investigated. One of these, which this computer has in common with both analog and digital differential analyzers, is that of proper scaling so as to prevent overflowing of the digital registers or saturation of the analog integrators. Another is the choice of the threshold voltages for the comparator units. It is possible that the best value for both upper and lower threshold voltages should be *zero* — requiring a positive or negative digital increment to be sent to the next unit each Δt, depending upon the sign of the voltage V. The problem of scaling and choice of thresholds are interrelated. and it should be possible to exercise some control over the overloading of analog integrators by proper choice of the threshold voltages.

The number of digits to be carried in the digital registers depends, of course, on the minimum value of α for which provision is to be made. In general, the R registers should contain one more binary digit than the other registers to prevent overflow under conditions where a large x_D of the same sign as R is added to R.

For any particular problem, the value of α to be chosen depends upon the particular compromise between precision and speed of solution desired. As a simple illustration, consider integration of the function $x = A \sin \omega t$, and assume Δt equals .001 second, $\alpha = .001$, and A is 1. Since the maximum time rate of change of this function $A\omega$ should not exceed $\alpha/\Delta t$, the highest frequency representable at full-scale amplitude would be $\omega = \alpha/A\Delta t = 1$ radian per second, and the precision (assuming an analog resolution of .001) would be one part in one million. If we chose $\alpha = .1$, the highest frequency representable at full scale amplitude would be 100 radians per second, and the precision would be one part in ten thousand.

The combined analog-digital differential analyzer of the type described shows promise of overcoming some of the limitations of present designs of differential analyzers which use purely analog or purely digital techniques. Also if analog components of sufficient bandwidth are available, the precision-speed product of this combined system should be greater than that possible with a parallel digital differential analyzer having equal length digital registers and equal iteration rate, by a factor dependent upon the resolution of the analog components — perhaps a factor of one thousand.

The greatest usefulness of the proposed system is believed to be on problems where the precision required is of the order of 10 to 100 times that obtainable by analog methods, yet requiring the real-time speed of analog methods. In this case, integrators and multipliers of the combined system would contain short digital registers and only moderate requirements would be put on the speed of switching circuits and the bandwidth of the analog components.

Work is under way at the National Bureau of Standards to construct breadboards of two integrator and two multiplier units, each capable of receiving inputs from two other units. The digital registers and digital-to-analog converters are being constructed from transistorized digital building block packages developed at NBS, and the analog components from commercially available wide band amplifiers. These units are planned to have 7 bit plus sign input registers, 8 bit plus sign "R" registers, an analog reference voltage of 10 volts, and operating with a Δt of one millisecond.

Acknowledgment

The author wishes to acknowledge the assistance of the Navy Bureau of Aeronautics in supporting the construction of the breadboard units for evaluating the proposed computer system.

Discussion

M. Rubinoff: I note there is no analog to digital conversion in the drawings. Is this true for the entire system?

Dr. Skramstad: This is true of the entire system. At no time do we have to convert an analog voltage to a digital quantity. Going from digital to analog is the easy way to go.

Mr. Rubinoff: I believe you said something about — I hope I am quoting you correctly — this system has 10 times the precision of an analog computer. What does this mean, 10 times as many digits? How does this compare with the DDA with greater precision?

Dr. Skramstad: The factor 10 applies only to the example shown; the actual factor depends upon the value chosen for α. In the example given, assume that 100 volts could be divided to a resolution of a part in a thousand, say, so that you might say that a tenth of a volt would represent, in a way, the least significant digit. Now in the examples I showed, the analog voltages would go through this 100 volts 10 times as the variable goes from zero up to unity. The effective resolution would thus be a part in 10,000.

D. Cohen (Airborne Inst. Lab.): Does the comparator have to be more accurate than in the pure analog computer to achieve the 10 times resolution?

Dr. Skramstad: The comparator does not have to be precise; it only detects that a theshold voltage has been exceeded sometime during the interval.

Mr. Labin (AERO, France): Would the cooperation of a digital block to the differential analyzer allow treatment of non-linear equations, at least approximately?

Dr. Skramstad: I think it would. I have not had time to go into what various non-linear components would be like in this system. I believe non-linear components could be devised that would work with the system.

N. Nesenoff (Republic Aviation): Could you indicate the percentage of equipment saved by the combined analog-digital system when compared to the equipment required by a purely digital system of equal accuracy?

Dr. Skramstad: This would be a hard one to put a figure on. I think the big advantage here is that you still have what you might call real time speed capability for real time simulation of physical systems. As you all know, analog computers are very useful in real time simulation of dynamic missile systems and aircraft systems, but often there are a number of variables that have to be carried to higher precision than the analog is capable. This system would allow you to carry those variables in the combined systems where additional precision is required.

Mr. Nesenoff: Did you ever build a computer of this type?

Dr. Skramstad: No, this is just an idea. We are in the process of building up a breadboard to try out the idea. We are planning a device carrying 7 bits plus sign in the X register, 8 bits plus sign in the R register, and voltage of 10 volts. I hope to be able to report how this works at a later date.

F. Verzuh (*MIT*): Comment on the solution of general integration: $W = I/K \int udv$ where v = time using your device. (*Chairman*: I think he is getting back to the fact that digital differential analysers will work with any variable whereas the analog computer insists that the independent variable in the integration is time.)

Dr. Skramstad: This is the limitation of this system, just as it is a limitation of the electronic analog computer. You have to devise the program in such a way as to associate time as the end variable.

Mr. Rubinoff: So it loses the advance that the digital has?

Dr. Skramstad: That is right. The multiplying device, however, is working only in dependent variables. For problems not involving integration, you are free of this limitation.

D. A. Bourne (*IBM*): Please summarize what you consider the most important advantages of a hybrid computer.

Dr. Skramstad: Well, I think I just mentioned that in the answer to another question. I think the greatest usefulness would be in simulation of physical systems where the precision of the analog computer is not quite great enough, and thus by the addition of short digital registers and digital components to an analog system, it would be able to handle these cases.

E. M. Ginsberg (*Burroughs Corp.*): Please comment on the accumulation and propagation of errors due to tolerances in the two systems and incurred by overlapping at different precision points.

Dr. Skramstad: Any errors due to drift in amplifiers in the analog portion can affect only the precision of the analog part, and these errors would be decreased in the overall solution by the factor Alpha.

Mr. Ginsberg: Please comment on any effect in slew time with this system as compared to a DDA.

Mr. Rubinoff: Slew time is the rate at which numbers can be changed in the differential analyzer, which limits the simple DDA.

Dr. Skramstad: I think some of the same techniques used in DDA would apply directly to this computer. I would think it would be a similar problem to that of the DDA.

K. Enslein (*Brooks Research*): How about the possibility of two different Deltas, one coarse and one fine?

Dr. Skramstad: This is something I haven't considered as yet. It might be worth looking into.

M. Tayyabkhan (*Union Carbide*): Would the use of floating point arithmetic in the digital registers solve some of the scaling problems of analog computers?

Dr. Skramstad: Well, now there is a possibility here. Again, this has not been investigated so that I cannot give a definite answer to that one.