PCLS IIR Filters with Simultaneous Frequency Response Magnitude and Phase Related Specifications

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Abstract

This paper presents the peak-constrained least-squares (PCLS) approach to designing IIR filters. PCLS IIR digital and analog filters are introduced that meet simultaneous specifications on the frequency response magnitude and phase related quantities.

1. Introduction

Presently, most IIR filters are designed to meet the frequency response magnitude specifications using the minimax criterion or the least-squares criterion. Allpass sections are frequently used to "equalize" the phase response to make the group delay or phase delay approximately constant or to force the phase to fit some arbitrary curve. However, the use of allpass sections is inefficient because the number of independent filter coefficients in an allpass section is about one half of the total number of filter coefficients.

The new PCLS IIR filter design algorithm can design a filter that has an "equalized" phase quantity without the use of allpass equalizer sections and it can simultaneously meet the frequency response magnitude specifications. The new algorithm uses all of the filter coefficients available to optimize the filter.

The proposed algorithm is based on the peak-constrained least-squares optimality criterion. Motivations for this optimality criterion are provided in [1]-[2].

2. PCLS Optimization Formulation

The IIR digital filter transfer function is defined by the following equation

$$H(z) = k_0 \prod_{i=1}^{N} \frac{z^2 - 2r_{0i}\cos(\phi_{0i})z + r_{0i}^2}{z^2 - 2r_{pi}\cos(\phi_{pi})z + r_{pi}^2} \quad (1)$$

where k_0 is the gain constant, r_{0i} is the ith zero radius, ϕ_{0i} is the ith zero angle, r_{pi} is the ith pole radius, ϕ_{pi} is the ith pole angle, and N is the number of quadratic sections.

The equation for the frequency response magnitude is

$$\alpha(\mathbf{A}, \phi) = k_0 \prod_{i=1}^{N} \frac{\{1 - 2r_{0i}\cos(\phi - \phi_{0i}) + r_{0i}^2\}^{1/2}}{\{1 - 2r_{pi}\cos(\phi - \phi_{pi}) + r_{pi}^2\}^{1/2}} \cdot \frac{\{1 - 2r_{0i}\cos(\phi + \phi_{0i}) + r_{0i}^2\}^{1/2}}{\{1 - 2r_{pi}\cos(\phi + \phi_{pi}) + r_{pi}^2\}^{1/2}}, \quad (2)$$

where $\mathbf{A}$ is the coefficient vector defined as

$$\mathbf{A} = [r_{01}, \phi_{01}, r_{p1}, \phi_{p1}, \ldots, r_{0i}, \phi_{0i}, r_{pi}, \phi_{pi}, \ldots, k_o]^T. \quad (3)$$

The equation for the phase is

$$\theta(A, \phi) = \sum_{i=1}^{N} atan\left(\frac{r_{0,i}^2\sin(2\phi) - 2r_{0,i}\cos(\phi_{0,i})\sin(\phi)}{1 - 2r_{0,i}\cos(\phi_{0,i})\cos(\phi) + r_{0,i}^2\cos(2\phi)}\right) - atan\left(\frac{r_{p,i}^2\sin(2\phi) - 2r_{p,i}\cos(\phi_{p,i})\sin(\phi)}{1 - 2r_{p,i}\cos(\phi_{p,i})\cos(\phi) + r_{p,i}^2\cos(2\phi)}\right). \quad (4)$$

The equation for the phase delay is

$$\theta D(\mathbf{A}, \phi) = \frac{\theta(\mathbf{A}, \phi)}{\phi}. \quad (5)$$

The equation for the group delay is

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$$\tau(\boldsymbol{A},\phi) = \sum_{i=1}^{N} \frac{1-r_{pi}\cos(\phi-\phi_{pi})}{1-2r_{pi}\cos(\phi-\phi_{pi})+r_{pi}^2} + \frac{1-r_{pi}\cos(\phi+\phi_{pi})}{1-2r_{pi}\cos(\phi+\phi_{pi})+r_{pi}^2} - \frac{1-r_{0i}\cos(\phi-\phi_{0i})}{1-2r_{0i}\cos(\phi-\phi_{0i})+r_{0i}^2} + \frac{1-r_{0i}\cos(\phi+\phi_{0i})}{1-2r_{0i}\cos(\phi+\phi_{0i})+r_{0i}^2}. \quad (6)$$

The equation for the partial derivatives with respect to the parameters are:

$$\frac{\partial\alpha}{\partial k_0} = \frac{\alpha}{k_0}, \quad (7)$$

$$\frac{\partial\alpha}{\partial r_{0i}} = \alpha\frac{r_{0i}-\cos(\phi-\phi_{0i})}{1-2r_{0i}\cos(\phi-\phi_{0i})+r_{0i}^2} + \alpha\frac{r_{0i}-\cos(\phi+\phi_{0i})}{1-2r_{0i}\cos(\phi+\phi_{0i})+r_{0i}^2}, \quad (8)$$

$$\frac{\partial\alpha}{\partial \phi_{0i}} = \alpha\frac{r_{0i}\sin(\phi-\phi_{0i})}{1-2r_{0i}\cos(\phi-\phi_{0i})+r_{0i}^2} - \alpha\frac{r_{0i}\sin(\phi+\phi_{0i})}{1-2r_{0i}\cos(\phi+\phi_{0i})+r_{0i}^2}, \quad (9)$$

$$\frac{\partial\theta}{\partial\phi_{0i}} = 2\frac{r_{0i}\sin(\phi_{0i})\sin(\phi)+r_{0i}^3\sin(\phi_{0i})\,\cdot}{(1-2r_{0i}\cos(\phi+\phi_{0i})+r_{0i}^2)\,\cdot} \quad (10)$$
$$\frac{(\cos(2\phi)\sin(\phi)-\cos(\phi)\sin(2\phi))}{(1-2r_{0i}\cos(\phi-\phi_{0i})+r_{0i}^2)},$$

$$\frac{\partial\theta}{\partial r_{0i}} = 2\frac{r_{0i}^2\cos(\phi_{0i})\cos(\phi)\,(\cos(2\phi)-}{(1-2r_{0i}\cos(\phi+\phi_{0i})+r_{0i}^2)\,\cdot} \quad (11)$$
$$\frac{\sin(2\phi)+r_{0i}\sin(2\phi)-\cos(\phi_{0i})\sin(\phi_{0i})}{(1-2r_{0i}\cos(\phi-\phi_{0i})+r_{0i}^2)},$$

$$\frac{\partial\tau}{\partial r_{pi}} = \frac{(1+r_{pi}^2)\cos(\phi-\phi_{pi})-2r_{pi}}{(1-2r_{pi}\cos(\phi-\phi_{pi})+r_{pi}^2)^2} + \frac{(1+r_{pi}^2)\cos(\phi+\phi_{pi})-2r_{pi}}{(1-2r_{pi}\cos(\phi+\phi_{pi})+r_{pi}^2)^2}, \quad (12)$$

$$\frac{\partial\tau}{\partial\phi_{pi}} = \frac{r_{pi}(1-r_{pi}^2)\sin(\phi-\phi_{pi})}{(1-2r_{pi}\cos(\phi-\phi_{pi})+r_{pi}^2)^2} - \frac{r_{pi}(1-r_{pi}^2)\sin(\phi+\phi_{pi})}{(1-2r_{pi}\cos(\phi+\phi_{pi})+r_{pi}^2)^2}, \quad (13)$$

and similarly for $\partial\alpha/\partial r_{pi}, \partial\alpha/\partial\phi_{pi}, \partial\theta/\partial r_{pi}, \partial\theta/\partial\phi_{pi}, \partial\tau/\partial r_{0i}$ and $\partial\tau/\partial\phi_{0i}$, but with the signs changed. The partial derivatives for the phase delay are equal to the partial derivatives of the phase divided by θ. The gradients of the frequency response magnitude and the phase related quantities with respect to the coefficient vector are made up of the above partial derivatives.

The total weighted squared error (which we also call the error energy) of an IIR digital filter is

$$\epsilon(\boldsymbol{A}) = \sum_{i=1}^{N_B} \int_{f_{L,i}}^{f_{U,i}} W_i(f)\,[\alpha(f)-\alpha_{D,i}(f)]^2 df. \quad (14)$$

N_B is the number of frequency bands. f is the frequency. $f_{L,i}$ is the ith frequency band lower edge frequency, and $f_{U,i}$ is the ith frequency band upper edge frequency. $\alpha_{D,i}(f)$ is the desired frequency response magnitude and $W_i(f)$ is the weighting function for the ith frequency band.

The gradient of the error energy is given by

$$\nabla_{\boldsymbol{A}}\epsilon(\boldsymbol{A}) = \sum_{i=1}^{N_B} 2\int_{f_{L,i}}^{f_{U,i}} W_i(f)\,[\alpha(f)-\alpha_{D,i}(f)]\,\nabla_{\boldsymbol{A}}\alpha(\boldsymbol{A})\,df. \quad (15)$$

According to the PCLS optimality criterion we have the following nonlinear optimization problem.

Minimize: $\epsilon(\boldsymbol{A})$ (16)

Subject to: $\boldsymbol{s}(\boldsymbol{A}) \geq 0.0$

$\boldsymbol{s}(\boldsymbol{A})$ is the nonlinear constraint error vector. The constraints are used to control the ripples in the frequency response magnitude and group delay.

The Kuhn-Tucker conditions for optimality are

$$\nabla_A \epsilon(\mathbf{A}) + \sum_{i=1}^{NC} u_i \nabla_A s_i(\mathbf{A}) = 0.0$$
$$s_i(\mathbf{A}) \geq 0.0 \qquad (17)$$
$$u_i \geq 0.0$$
$$u_i s_i(\mathbf{A}) = 0.0 \quad i=1,NC \ .$$

NC is the total number of inequality constraints. u_i is the Kuhn-Tucker multiplier corresponding to the ith constraint. $s_i(\mathbf{A})$ is the value of the constraint error of the ith constraint equation in $\mathbf{s}(\mathbf{A})$.

3. The RGME$_I$ Algorithm

We use the GME algorithm [3] combined with the RQP method [4] to find the solution to the nonlinear minimization problem formulated later in this Section. This new combination is called the recursive generalized multiple exchange (RGME$_I$). The "I" subscript denotes the fact that it is specialized for designing IIR filters. The following equation is used to find the new coefficient vector on each iteration.

$$\mathbf{A}_{k+1} = \mathbf{A}_k + b_k \mathbf{S}_k \quad (18)$$

k is the iteration index. $\mathbf{S}_k$ is called the step direction vector and is determined by solving the following quadratic programming problem.

$$\text{Minimize: } \frac{1}{2}\mathbf{S}_k^T \mathbf{H}_k \mathbf{S}_k + \nabla_A \epsilon(\mathbf{A}_k)^T \mathbf{S}_k \qquad (19)$$
$$\text{Subject to: } \mathbf{B}_k \mathbf{S}_k - \mathbf{a}_k \geq 0.0 \ .$$

$\mathbf{H}_k$ is a positive definite matrix and is updated as explained in [4]. The matrix $\mathbf{B}_k$ and the vector $\mathbf{a}_k$ are derived from the linearized constraint equations as explained later in this Section.

b_k is called the step size and is found by performing a one dimensional minimization of the Lagrangian function.

The frequency response magnitude, phase, phase delay and group delay are linearized using a first order Taylor series about $\mathbf{A}_k$, the coefficient vector for the kth iteration of the RGME$_I$ algorithm. The linearized frequency response magnitude and group delay equations are shown below:

$$-\nabla_A \alpha(\mathbf{A}_k)^T(\mathbf{A}-\mathbf{A}_k) - [\alpha(\mathbf{A}_k) - \alpha_D - \delta_\alpha] \geq 0.0, \quad (20)$$

$$\nabla_A \alpha(\mathbf{A}_k)^T(\mathbf{A}-\mathbf{A}_k) - [-\alpha(\mathbf{A}_k) + \alpha_D - \delta_\alpha] \geq 0.0, \quad (21)$$

$$-\nabla_A \tau(\mathbf{A}_k)^T(\mathbf{A}-\mathbf{A}_k) - [\tau(\mathbf{A}_k) - \tau_D - \delta_\tau] \geq 0.0, \quad (22)$$

$$\nabla_A \tau(\mathbf{A}_k)^T(\mathbf{A}-\mathbf{A}_k) - [-\tau(\mathbf{A}_k) + \tau_D - \delta_\tau] \geq 0.0. \quad (23)$$

If we let

$$\mathbf{S} = \mathbf{A} - \mathbf{A}_k \quad (24)$$

then the linearized frequency response magnitude inequality constraint equations and the linearized group delay inequality constraint equations can be combined and written in matrix format as

$$\mathbf{BS} - \mathbf{a} \geq 0. \quad (25)$$

$\mathbf{B}$ is the constraint normal matrix and $\mathbf{a}$ is the constraint right-hand-side (RHS) vector. The GME algorithm is used to solve the quadratic programming problem after the constraints have been linearized.

4. Examples

Before we look at the examples, some definitions and notation are in order. Two important performance measures of a lowpass IIR filter are the peak stopband gain and the passband to stopband energy ratio, as discussed in [2]. These are called DB_s and PSR, respectively, and are both measured in dB. In general, a small DB_s and a large PSR are desirable. In the examples that follow we will show that tradeoffs can be made between PSR, DB_s, and the group delay specifications for IIR filters designed using the RGME$_I$ algorithm.

Example 1. We use Deczky's [5] Example 1 as our first example Filter 1-i. Deczky equalizes the group delay of a minimax elliptic filter using an allpass filter. Filter 1-i is specified to have a passband edge frequency of 0.25 cycles/sample, a stopband edge frequency of 0.30 cycles/sample, a passband ripple of 0.5 dB, and a DB_s of -32.0 dB. The resulting PSR for this Deczky filter is 35.8 dB. The frequency response for this Deczky filter is shown as a thin solid line in Figures 1 through 3.

Filter 1-ii was designed using the RGME$_I$ algorithm with a specified group delay of 16.00 ±0.25 samples. The remaining specifications for Filter 1-ii are the same as for Filter 1-i. The number of quadratic sections (6) is the same for both filters. The PSR for this filter is 39.73 dB. The frequency response for Filter 1-ii is shown as a thick solid line in Figures 1 through 3.

Example 2. This analog lowpass LC ladder filter is taken from Reference [6], page 162, Figure 7.6. The passband edge frequency is 1.0 radians/second, the stopband edge frequency is 1.13257005 radians/second, the maximum passband loss is 0.043647 dB, the stopband loss is 32.0 dB, the desired group delay is 18.00 ± 2.00 seconds. The frequency response for Example 2 is shown in Figures 4 ,5, and 6.

5. PCLS Tradeoff Theorem

In Reference [1]-[3] we discuss the concept of a tradeoff curve. A tradeoff curve is a plot of the maximum stopband gain versus stopband energy for a set of any type of filters. A tradeoff curve has the least-squares filter at one end of the curve and the minimax filter at the other end of the curve. The minimax filter has the smallest maximum stopband error and the largest stopband error energy, while the least-squares filter has the largest maximum stopband error and the smallest stopband error energy. As in all filter tradeoff curves, the tradeoff curve for a IIR filter has zero slope at the least-squares filter and approaches infinite slope at the minimax filter.

In Reference [7] we presented the PCLS Tradeoff Theorem (PTT) as stated below.

PAT: ϵ versus δ tradeoff curves for all types of optimal PCLS filters must monotonically decrease and terminate with zero slope. ϵ is the stopband energy and δ is the maximum stopband error.

PAT proof: The feasible set for $\delta=\delta_a$ must be a subset of the feasible set for $\delta=\delta_b$ if $\delta\leq\delta_b$. Therefore, $\epsilon(\delta_b)\leq\epsilon(\delta_a)$ if $\delta_a\leq\delta_b$ and both solutions are optimal. This proves that ϵ must be a monotonically decreasing function of δ for optimal PCLS filters. The slope at the least-squares solution must be zero because all of the δ inequality constraints become inactive at the least-squares solution and their Kuhn-Tucker multipliers vanish.

6. Conclusion

A new nonlinear programming algorithm based on the recursive use of the GME algorithm was introduced. It was called the $RGME_I$ algorithm. The $RGME_I$ algorithm was used to design PCLS IIR filters with simultaneous specifications on the frequency response magnitude and phase related quantities. Classic IIR filter examples from the research literature were discussed in detail.

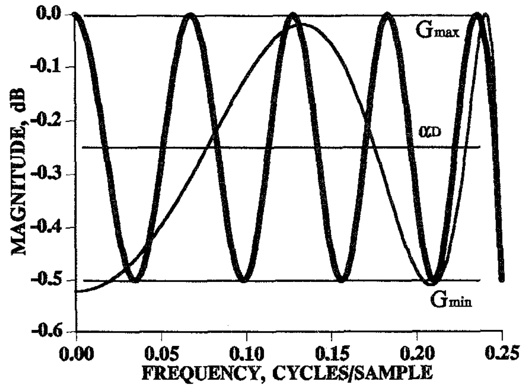

Figure 1. Frequency response magnitude in the passband for Example 1.

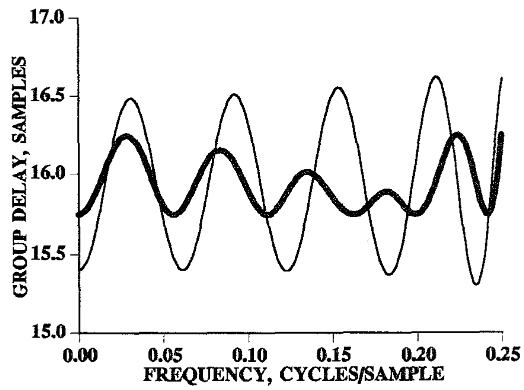

Figure 2. Group delay in the passband for Example 1.

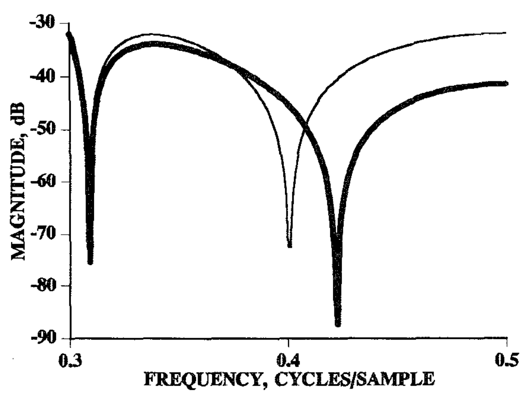

Figure 3. Frequency response magnitude in the stopband for Example 1.

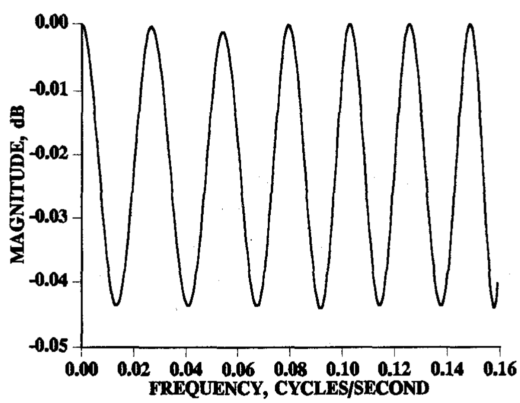

Figure 4. Frequency response magnitude in the passband for Example 2.

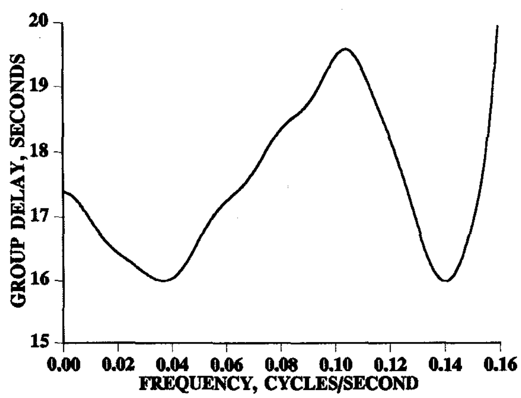

Figure 5. Group delay in the passband for Example 2.

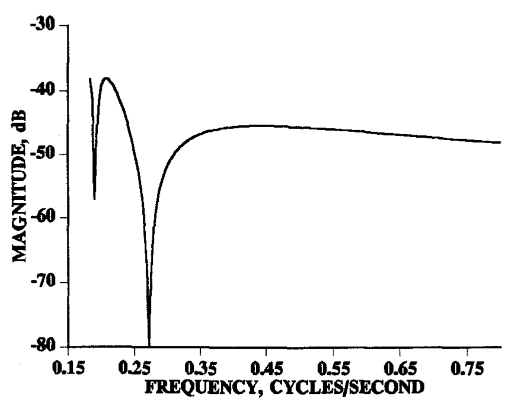

Figure 6. The frequency response magnitude in the stopband for Example 2.

7. References

[1] J.W. Adams, "A New Optimal Window," *IEEE Transactions on Signal Processing*, vol. 38, pp. 1753-1769, August 1991.

[2] J.W. Adams, "FIR digital filters with least-squares stopbands subject to peak-gain constraints," *IEEE Transactions on Circuits and Systems*, vol. CAS-39, pp 376-388, April 1991.

[3] J.W. Adams, J.L. Sullivan, R. Hashemi, C. Ghadimi, J. Franklin and B. Tucker, "New Approaches to Constrained Optimization of Digital Filters," *Proceedings of the IEEE International Symposium on Circuits and Systems*, pp. 80-83, Chicago, May 1993.

[4] M.J.D. Powell, "A fast Algorithm for Nonlinearly Constrained Optimization Calculations," in: Numerical analysis, Dundee, 1977, *Lecture Notes in Mathematics 630*, pp. 144-157, Springer Verlag, Berlin, 1978.

[5] A.G. Deczky, "Synthesis of Recursive Digital Filters Using the Minimum p-Error Criterion," *IEEE Trans. Audio Electroacoust.*, Vol. AU-20, pp. 257-263, Oct. 1972.

[6] Erich Christian, *LC-Filters*, John Wiley and Sons, 1983.

[7] J.W. Adams and J.L. Sullivan, "Peak-Constrained Least-Squares Optimization," *IEEE Transactions on Signal Processing*, To be Published.