The Frequency and Scale Content of Biological Signals

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Abstract— We discuss the frequency and scale content of biological signals. We argue that generally the frequency is not constant. We develop a model of a harmonic oscillator with non constant coefficients which produces an arbitrary instantaneous frequency law.

1. INTRODUCTION

About 1865, some fifty years after Fourier, it was realized that under many circumstances the spectrum of signal is a characterization of the source of the signal and the medium of propagation. This idea was first appreciated by Bunson and Kirchhoff has been the main tool for analysis of physical, chemical and biological signals. The reason Fourier analysis is so powerful is not a mathematical one but a physical one: The laws of nature are such that a frequency decomposition of waves is a fingerprinter of the source and medium.

For a signal, $f(x)$ the Fourier transform is

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int f(x)\, e^{-j\omega x}\, dx\,, \tag{1.1}$$

and the inverse formula is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int F(\omega)\, e^{j\omega x}\, d\omega\,, \tag{1.2}$$

The energy density spectrum, $P(\omega)$, is given by

$$P(\omega) = |F(\omega)|^2 \tag{1.3}$$

and indicates which frequencies exist in the signal and their relative intensity.

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The fundamental characteristic of the Fourier transform. If we collect the light of the sun and if someone on Jupiter collects the same light considerably later, we will both get the same energy density spectrum and hence agree that it is the same sun. The energy density spectrum of a function is identical to that of the translated function. In particular if we have two functions $f(x)$ and $f_{tr}(x) = f(x + x_0)$, one a translated version of the other, then their Fourier transforms are related by

$$F_{tr}(\omega) = e^{j\omega t_0}\, F(\omega) \tag{1.4}$$

where F_{tr} is the Fourier transform of $f_{tr}(x)$. Therefore, the energy density spectrums are the same

$$P_{tr}(\omega) = P(\omega) \tag{1.5}$$

In a loose way we can say that frequency analysis is insensitive to translations.

Fundamental concept of scale. To bring forth the main intuitive idea of scale we consider a number of simple situations. As we move closer to an object it gets bigger, however we still determine and say that it is the same object. If we have two photographs of the same person taken at the same time but from different distances we still say it is the same person. The mathematical tool that we want to study scale must be insensitive to magnification of the object. The reason we want that is because if indeed there is a difference between two objects we do not want mere magnification to interfere with that difference. This is analogous to the situation that when we hear two different sounds an hour apart and we want to compare them we don't want the fact that they were said an hour apart to interfere with the comparison; that is why the energy density spectrum is insensitive to translation. Similar-

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The scale transform. Recently a general approach to study scale has been developed by a number of investigators [1,3-13]. The scale transform, $D(c)$, is defined by

$$D(c) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x) \frac{e^{-jc\ln x}}{\sqrt{x}} dx \qquad (1.6)$$

It is the Mellin transform with the argument $-jc + \frac{1}{2}$. The inverse transformation is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int D(c) \frac{e^{jc\ln x}}{\sqrt{x}} dc \quad ; \quad x \geq 0 \qquad (1.7)$$

Notice that the function we are studying, $f(x)$ starts at $x = 0$. The energy density scale spectrum is given by

$$P(c) = |D(c)|^2 \qquad (1.8)$$

Now consider two functions $f(x)$ and $f_{sc} = \sqrt{a}\, f(ax)$, where f_{sc} is a magnified or reduced version of $f(x)$. We have magnification or reduction depending on the range of a. If $0 \leq a \leq 1$ then we have magnification. If $1 \leq a \leq \infty$ then we have reduction. It is simple to verify that the scale transform of the two functions are related by

$$D_{sc}(x) = e^{jc\ln a}\, D(c) \qquad (1.9)$$

Therefore

$$P_{sc}(c) = P(c) \qquad (1.10)$$

which shows that if two functions are just scaled versions of each other the scale transform can be used to tell us that; more importantly if there are inherent differences, that is differences other then mere magnification, the scale transform will pick them out and not be confused by the magnification factor. To use the well known mouse to elephant idea – if an elephant was a scaled version of a mouse then the scale transform of an elephant and mouse would identical. However if an elephant is not just a magnified mouse the scale transform will tell us that because it will be different for mouse and elephant.

Scale in Biology. The concept of scale has been fundamental in biology and arises in many contexts. In the oldest and simplest form is that certain biological parameters vary with mass and size of an animal. Perhaps the earliest introduction of scaling was by Galileo who considered the issue of how bone mass varies with size. How much a bone can support is proportional to the diameter. Galileo argued that as the linear size of an animal increases its weight goes as the cube of the size. But how much a bone can support is proportional to its diameter. Therefore bone mass has to increase faster than volume (mass) to compensate. If we were to continue increasing the mass of an animal we would have an animal that is all bone and no meat. Historically, the seminal book by Thompson *On Growth and Form*, stands as a landmark in the concept of scaling and transformation. At the present time there are numerous formulations of fundamental biological quantities whose scaling properties have been studied. Perhaps the most studied is metabolic rate. An equation which approximates the relationship between metabolic rate, T, (for mammals) with animal mass, M, is

$$T = cM^p \qquad (1.11)$$

If one argues that the metabolic rate is dominated by loss of heat then it should be crudely proportional to the surface area of the animal. Surface goes two thirds power of the volume, which in turn is directly proportional to mass. Therefore, one would expect that $p \sim 2/3$. However making this simplified argument does not make it correct. Of course it is deviations from the simplified argument which gives insight into what is going on. In fact the accepted experimental value for p is approximately 3/4. The reason for developing a mathematical framework for scaling is indeed it may allow a clear understand to what scaling is and will allow separation of different causes.

2. A MODEL FOR BIOLOGICAL SIGNALS

The fundamental model for many biological signals has been the damped harmonic oscillator. Many other nonlinear models have been proposed to study certain biological phenomena. We will take the approach of developing the equation given the signal. Let us first ask whether the harmonic has served well. The damped harmonic oscillator, with mass m, resistive force b, and spring constant k, is

$$m\ddot{x} + b\dot{x} + kx = 0 \qquad (2.1)$$

and the parameters m, b, k are directly related to the physical process involved. In standard form this equation is

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \qquad (2.2)$$

with

$$\omega_0 = \sqrt{\frac{k}{m}}\,, \qquad \gamma = \frac{b}{2m} \qquad (2.3)$$

If we assume γ, ω_0 are constant in time the general solution (for the damped case) is

$$x = A_0 e^{-\gamma t} \cos(\omega t + \alpha) \tag{2.4}$$

where

$$\omega^2 = \omega_0^2 - \gamma^2 = \frac{k}{m} - \frac{b^2}{4m^2} \tag{2.5}$$

and where α is a constant phase factor. The important issue for our considerations is that the frequency of vibration, ω, is a constant. Is this realistic? Does it fit with common biological signals? Consider the example of muscle sounds [2], to a first approximation the frequency is constant. But there certainly are deviations from constant frequency, the frequency is definitely changing in time. This changing frequency can be approximated by

$$\omega_i = \omega + \beta t \tag{2.6}$$

This introduces a new parameter, β whose physical significance is not clear, but which may be a characterization of the biological process. What is also not clear is the equation which generates the signal whose frequency increases.

Another example which does not follow the ideal harmonic oscillator equation are bat signals and other animal sounds. For the case of a bat signal the frequency appears to be hyperbolic in time,

$$\omega_i(t) = \frac{c}{t} \tag{2.7}$$

The question that now arises is what kind of a model, that is what kind of differential equation, can one construct to give such solutions. We shall consider harmonic oscillator equations with non constant coefficients,

$$\ddot{x} + 2\Gamma(t)\dot{x} + \eta^2(t)x = 0 \tag{2.8}$$

and see what they have to be so that we get specific instantaneous frequency laws. A signal with arbitrary amplitude, $A(t)$ and phase, $\varphi(t)$ is

$$x(t) = A(t) \cos \varphi(t) \tag{2.9}$$

The frequency is given by the derivative of the phase.

$$\omega_i(t) = \dot{\varphi}(t) \tag{2.10}$$

Since the frequency is changing in time it is termed the instantaneous frequency. For the case of a harmonic oscillator it is a constant; for the example given above where it is changing linearly it is called a chirp because that is the chirping sounds of birds.

It is convenient to define

$$\sigma(t) = \frac{\dot{A}}{A} = \frac{d}{dt} \ln A \tag{2.11}$$

This σ has physical significance however we will not discuss that here.

Differentiating with respect to time we have

$$\dot{x} = -A\omega_i \sin \varphi + \sigma x \tag{2.12}$$

$$\ddot{x} = -\omega_i^2 x - \frac{dA\omega_i}{dt} \sin \varphi + \dot{\sigma} x + \sigma \dot{x} \tag{2.13}$$

From the first equation

$$\sin \varphi = -\frac{1}{A\omega_i} (\dot{x} - \sigma x) \tag{2.14}$$

which upon substitution in Eq. (2.13) gives

$$\ddot{x} = -\omega_i^2 x + \frac{(\dot{x} - \sigma x)}{A\omega_i} \frac{dA\omega_i}{dt} + \frac{d}{dt}\sigma x \tag{2.15}$$

$$= -\omega_i^2 x + (\dot{x} - \sigma x) \frac{d \ln(A\omega_i)}{dt} + \frac{d}{dt}\sigma x \tag{2.16}$$

$$= \left(\frac{d \ln(A\omega_i)}{dt} + \sigma \right) \dot{x} + \left(-\omega_i^2 + \dot{\sigma} - \sigma \frac{d \ln(A\omega_i)}{dt} \right) x \tag{2.17}$$

$$= \frac{d \ln A^2 \omega_i}{dt} \dot{x} - \left[\omega_i^2 + \sigma^2 + \sigma \dot{\omega}_i / \omega_i - \dot{\sigma} \right] x \tag{2.18}$$

$$= \left[2\sigma + \dot{\omega}_i / \omega_i \right] \dot{x} - \left[\omega_i^2 + \sigma^2 + \sigma \dot{\omega}_i / \omega_i - \dot{\sigma} \right] x \tag{2.19}$$

Comparing this equation with Eq. (2.8) we have

$$\Gamma(t) = -\frac{1}{2} \frac{d \ln A^2 \omega_i}{dt} \tag{2.20}$$

$$= -\sigma - \frac{\dot{\omega}_i}{2\omega_i} \tag{2.21}$$

$$\eta^2(t) = \omega_i^2 + \sigma^2 + \sigma \dot{\omega}_i / \omega_i - \dot{\sigma} \tag{2.22}$$

Alternative Forms. Some alternative forms are convenient. If we take the amplitude modulation of the form

$$A(t) = A_0 e^{-f(t)} \tag{2.23}$$

then we have

$$\sigma = -\dot{f} \tag{2.24}$$

$$\eta^2(t) = \omega_i^2 + \dot{f}^2 - \dot{f} \dot{\omega}_i / \omega_i + \ddot{f} \tag{2.25}$$

$$\Gamma(t) = \dot{f} - \tfrac{1}{2} \dot{\omega}_i / \omega_i \tag{2.26}$$

If we take

$$A(t) = A_0 t^{-\lambda} = A_0 e^{-\lambda \ln t} \qquad (2.27)$$

then

$$\sigma = -\frac{\lambda}{t} \qquad (2.28)$$

$$\eta^2(t) = \omega_i^2 + \frac{\lambda(\lambda-1)}{t^2} - \frac{\lambda}{t}\frac{\dot{\omega}_i}{\omega_i} \qquad (2.29)$$

$$\Gamma(t) = \frac{\lambda}{t} - \frac{1}{2}\frac{\dot{\omega}_i}{\omega_i} \qquad (2.30)$$

3. SPECIAL CASES

We now specialize to two fundamental cases. First is the case where the amplitude is constant but the phase arbitrary and second where the amplitude arbitrary but the frequency is constant.

Constant Amplitude, Arbitrary Instantaneous Frequency. For constant amplitude we have $\sigma = 0$ and hence

$$\eta^2 = \omega_i^2, \qquad \Gamma(t) = -\frac{\dot{\omega}_i}{2\omega_i} \qquad (3.1)$$

In which case

$$\ddot{x} - \frac{\dot{\omega}_i}{\omega_i}\dot{x} + \omega_i^2 x = 0 \qquad (3.2)$$

This is a remarkably simple equation.

Constant Frequency, Arbitrary Amplitude. For this case we have that $\dot{\omega}_i = 0$ and we $\omega_i = \omega$. Then,

$$\Gamma(t) = -\sigma \qquad (3.3)$$

$$\eta^2(t) = \omega^2 + \sigma^2 - \dot{\sigma} \qquad (3.4)$$

and the equation of motion is

$$\ddot{x} - \sigma\dot{x} + (\omega^2 + \sigma^2 - \dot{\sigma})\, x = 0 \qquad (3.5)$$

4. PARTICULAR CASES

Let us now consider some particular combinations of phase and amplitude.

Constant Frequency. Taking the case where $f(t) = \gamma t$ we have

$$\eta^2(t) = \omega^2 + \gamma^2 \qquad (4.1)$$

$$\Gamma(t) = \gamma \qquad (4.2)$$

This is the standard case given by Eq. (2.2).

$$\omega_0^2 = \eta^2(t) = \omega^2 + \gamma^2 \qquad (4.3)$$

Now consider the case where

$$A = A_0 t^{-\lambda} \qquad (4.4)$$

in which case

$$\eta(t) = \omega^2 + \frac{\lambda(\lambda-1)}{t^2}, \qquad \Gamma(t) = \lambda/t \qquad (4.5)$$

and the differential equation takes the form of

$$\ddot{x} + \frac{2\lambda}{t}\dot{x} + \left[\omega^2 + \frac{\lambda(\lambda-1)}{t^2}\right] x = 0 \qquad (4.6)$$

Notice that we still get a constant frequency for the value $\lambda = 1$. For a general amplitude modulation where Γ is arbitrary we have

$$x = A_0 \exp\{-\int_0^t \Gamma(t)\, dt\} \cos(\omega t + \alpha) \qquad (4.7)$$

Thus we can get constant frequency with a very large variety of amplitude modulations.

Chirp. The phase is

$$\varphi(t) = \tfrac{1}{2}\beta t^2 + \omega t + \alpha \qquad (4.8)$$

with instantaneous frequency

$$\omega_i = \omega + \beta t \qquad (4.9)$$

Also,

$$\eta^2(t) = \omega_i^2 + \ddot{f} + \dot{f}^2 - \dot{f}\frac{\beta}{\omega_i} \qquad (4.10)$$

$$\Gamma(t) = \dot{f} - \frac{\beta}{2\omega_i} \qquad (4.11)$$

For the case where $f = \gamma t$ we have

$$\eta^2(t) = \omega_i^2 + \gamma^2 - \frac{\gamma\beta}{\omega_i} \qquad (4.12)$$

$$\Gamma(t) = \gamma - \frac{\beta}{2\omega_i} \qquad (4.13)$$

Now suppose the amplitude is constant, then $\gamma = 0$ and we have

$$x(t) = A_0 \cos \qquad (4.14)$$

In this case

$$\eta^2(t) = \omega_i^2 \qquad (4.15)$$

$$\Gamma(t) = -\frac{\beta}{2\omega_i} \qquad (4.16)$$

The differential equation is

$$\ddot{x} - \frac{\beta}{\beta t + \omega_0}\dot{x} + (\beta t + \omega_0)^2 x = 0 \qquad (4.17)$$

Hyperbolic instantaneous frequency. The phase and instantaneous frequency are

$$\varphi(t) = \beta \ln t + \alpha\,, \qquad \omega_i = \frac{\beta}{t} \qquad (4.18)$$

This gives

$$\frac{\dot{\omega}_i}{\omega_i} = -\frac{1}{t} \qquad (4.19)$$

$$\eta^2(t) = \omega_i^2 + \ddot{f} + \dot{f}^2 + \dot{f}/t \qquad (4.20)$$

$$\Gamma(t) = \dot{f} + \frac{1}{2t} \qquad (4.21)$$

For the case where $f = \gamma t$ we have

$$\eta^2(t) = \omega_i^2 + \gamma^2 + \gamma t \qquad (4.22)$$

$$\Gamma(t) = \gamma + \tfrac{1}{2}t \qquad (4.23)$$

For the constant amplitude case we have

$$\eta^2(t) = \omega_i^2 \qquad (4.24)$$

$$\Gamma(t) = \tfrac{1}{2}t \qquad (4.25)$$

and the differential equation is

$$\ddot{x} - t\dot{x} + \beta x/t = 0 \qquad (4.26)$$

5. CONCLUSION

The main result of this paper has been to obtain equations of motion whose solutions are signals of a given instantaneous frequency. This enlarges the number of parameters compared to the standard harmonic oscillator equation. In the above considerations we have assumed the driving force of the oscillator is zero. We now briefly address what happens when a driving force is incorporated. In general we can write

$$\ddot{x} + 2\Gamma\dot{x} + \eta^2 x = F(t) \qquad (5.1)$$

where $F(t)$ is the force per unit mass. Now instead of comparing Eq. (2.16) with Eq. (2.6) we leave Γ and η arbitrary and substitute for $\ddot{x}$ to obtain

$$F(t) = \left[2\Gamma + 2\sigma + \dot{\omega}_i/\omega_i\right]\dot{x} + \left[\eta^2 - \omega_i^2 - \sigma^2 - \sigma\dot{\omega}_i/\omega_i + \dot{\sigma}\right] x \qquad (5.2)$$

This is the main equation. By taking special cases a rich variety of possibilities arises.

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