CONVERGENCE PROPERTIES OF THE FREQUENCY-DOMAIN BLOCK-LMS ADAPTIVE ALGORITHM

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ABSTRACT

Convergence properties of the unconstrained frequency-domain block LMS adaptive algorithm are analyzed. The learning characteristics of the unconstrained case are compared with the constrained case via computer simulation. It is shown that the unconstrained algorithm has a slower convergence rate and smaller stable range of step size than that of the constrained algorithm.

1. INTRODUCTION

The block-based LMS algorithm (BLMS) [3] is an efficient adaptive filtering algorithm aimed at increasing convergence speed and reducing the computational complexity. The basic principle of the BLMS algorithm is that the filter coefficients remain unchanged during the processing of each data block, and are updated only once per block. The updating equation of the block LMS algorithm is as follows:

$$w_{k+1} = w_k + \mu \sum_{m=0}^{L-1} x(kL+m)e(kL+m). \quad (1)$$

In equation (1), k refers to the $k-th$ block of input data, $n = kL + m$ denotes the index of the incoming data, and L is the block length. Due to the fact that in the block LMS algorithm the computation of the filter output and of the gradient itself are represented by linear convolution and correlation, they can be implemented efficiently using the FFT. Based on the convolutional property of the FFT and the overlap-save method, Clark et al. [3] developed a frequency domain block LMS algorithm (FBLMS). In order to further decrease the computational complexity of the FBLMS algorithm, Mansour and Gray [4] proposed a simplified form of the block LMS algorithm which they referred to as the unconstrained frequency domain block LMS algorithm. By ignoring the gradient constraint, the unconstrained FBLMS algorithm saves one forward and one inverse FFT, thereby decreasing the computational complexity dramatically. The block diagram of the constrained and unconstrained FBLMS algorithms is shown in Figure 1.

Although some analyses of the convergence rate of the unconstrained FBLMS algorithm have been reported previously in [4, 5, 6], certain basic questions remain unanswered. For example, there has been no direct comparison between the convergence rates of the constrained and unconstrained FBLMS algorithms. In other words, what is the impact on the convergence rate of removing the gradient constraint from the constrained FBLMS algorithm? How will the stable range of the step size of the unconstrained FBLMS algorithm change? Mansour and Gray [4] proved the almost sure asymptotic exponential convergence of the unconstrained algorithm, but they did not compare it with the constrained FBLMS algorithm. Also, references [5, 6] presented convergence analyses of the FBLMS algorithm, but these results did not present a conclusive comparison of convergence rate between the constrained and unconstrained algorithms.

In this paper we will analyze and compare the convergence rates of the constrained and unconstrained FBLMS algorithms. By transforming the two algorithms into the time domain, it is shown that the two algorithms are equivalent to time domain adaptive filters with different number of taps. It is this effective difference in filter length that results in different performance characteristics of the two algorithms. It will also be shown that, although the unconstrained FBLMS algorithm corresponds to a longer filter, the "wrap-

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around" error prevents it from effectively performing as a higher order filter.

In section 2, the constrained and unconstrained FBLMS algorithms are summarized. A convergence analysis and a comparison of the two algorithms is given in section 3. In section 4 computer simulation results are presented to substantiate the theoretically predicted performance.

2. ANALYSIS OF FREQUENCY-DOMAIN BLOCK LMS ALGORITHMS

In this section we briefly summarize the constrained and unconstrained FBLMS algorithms and their performance. Throughout this paper, Capital letters are used to denote the frequency-domain variables and lowercase letters to denote the time-domain variables. Boldface letters denote vectors and matrices. The basic principle of the algorithm is to use the DFT to calculate the filter output(linear convolution) and the block gradient(linear correlation).

Because the DFT-based calculation actually implements circular convolution, in order to obtain a result that is equivalent to linear convolution, a data vector $\mathbf{x}_k$ that has a larger dimension than the filter length must be used. Assume the filter length is N. Then we formulate a length 2N input data vector $\mathbf{x}_k$ as:

$$\mathbf{x}_k = [\ x(2Nk),\ x(2Nk+1),\ ....,\ x(2Nk+2N-1)\ ]^t, \tag{2}$$

where the last N samples in $\mathbf{x}_k$ are taken from the current block of input data and the first N samples are taken from the previous input data block. In order to maintain consistent dimensions, the filter weight vector must also be increased to length $2N$ by appending N zeros to the original filter weights, as indicated below:

$$\mathbf{w}_k = [\ \overbrace{w_k(0),\ w_k(1),\ \cdots,\ w_k(N-1)}^{\mathbf{w}_{k,c}},\ \overbrace{0\ \cdots\ 0}^{N}\ ]^t. \tag{3}$$

If $\mathbf{F}$ denotes the $2N \times 2N$ DFT matrix with elements $F_{mn} = exp(-j2\pi mn/2N)$, then the filter coefficients are:

$$\mathbf{W}_k = \mathbf{F}\mathbf{w}_k \tag{4}$$

The output of the adaptive filter in the frequency domain is:

$$\mathbf{Y}_k = \mathbf{X}_k\mathbf{W}_k, \tag{5}$$

where $\mathbf{X}_k = diag(\mathbf{F}\mathbf{x}_k)$. Because the last N elements of $\mathbf{F}^{-1}\mathbf{Y}_k$ coincide with the results of linear convolution, only these elements should be used in calculating the error vector. Define:

$$\mathbf{d}_k = [\ \overbrace{0\ \cdots\ 0}^{N}\ \overbrace{d_k(0)\ d_k(1)\ \cdots\ d_k(N-1)}^{\mathbf{d}_{k,c}}\ ]^t, \tag{6}$$

$$\mathbf{y}_k = [\ \overbrace{0\ \cdots\ 0}^{N}\ \overbrace{\textit{last N elements of } \mathbf{F}^{-1}\mathbf{Y}_k}^{\mathbf{y}_{k,c}}\ ]^t. \tag{7}$$

Then the error signal in the frequency domain is defined as:

$$\mathbf{E}_k = \mathbf{F}[\mathbf{d}_k - \mathbf{y}_k]. \tag{8}$$

In order to guarantee that there are only N nonzero terms in the impulse response of the adaptive filter, a gradient constraint should be used to force the last N terms of the gradient in the time domain to be zeros, as shown in Figure 1. The updating equation for the constrained FBLMS is:

$$\mathbf{W}_{k+1} = \mathbf{W}_k + \mu\mathbf{F}(\nabla(k)^t, 0....0)^t, \tag{9}$$

where $\nabla(k) =$ *the first N elements of* $\mathbf{F}^{-1}\mathbf{X}_k^H\mathbf{E}_k$ and H denotes the complex conjugate transpose. By simply removing the constraint, the unconstrained FBLMS algorithm is produced [4]:

$$\mathbf{W}_{k+1} = \mathbf{W}_k + \mu\mathbf{X}_k^H\mathbf{E}_k. \tag{10}$$

With the constraint removed, the computational complexity is significantly reduced.

In order to formulate the constrained and unconstrained FBLMS algorithms mathematically, define the two matrices $\mathbf{h}_u$ and $\mathbf{h}_l$ as follows:

$$\mathbf{h}_u = \begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{vmatrix} \quad \textit{and} \quad \mathbf{h}_l = \begin{vmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{vmatrix}. \tag{11}$$

Then the vector $\mathbf{y}_k$ can be represented as:

$$\mathbf{y}_k = \mathbf{h}_l\mathbf{F}^{-1}\mathbf{X}_k\mathbf{W}_k. \tag{12}$$

Because $\mathbf{F}^{-1}\mathbf{Y}_k = \mathbf{F}^{-1}\mathbf{X}_k\mathbf{W}_k$ is actually the result of the circular convolution of the two sequences $\mathbf{x}_k$ and $\mathbf{w}_k$, we can formulate the circular matrix χ_k as:

$$\chi_k^t = \begin{vmatrix} x(2Nk) & \dots & x(2Nk+2N-1) \\ x(2Nk+2N-1) & \dots & x(2Nk+2N-2) \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ x(2Nk+2) & \dots & x(2Nk+1) \\ x(2Nk+1) & \dots & x(2Nk) \end{vmatrix} \tag{13}$$

The output of the FBLMS adaptive filter can be written as:

$$\mathbf{F}^{-1}\mathbf{Y}_k = \mathbf{F}^{-1}\mathbf{X}_k\mathbf{W}_k = \chi_k\mathbf{w}_k. \tag{14}$$

Because χ_k is a circulant matrix, it can be shown [1] that $\mathbf{F}\chi_k\mathbf{F}^{-1}$ is a diagonal matrix and its diagonal elements are equal to the discrete Fourier transform of the column vector $\mathbf{x}_k$. That is

$$\mathbf{X}_k = diag(\mathbf{F}\mathbf{x}_k) = \mathbf{F}\chi_k\mathbf{F}^{-1} \tag{15}$$

By substituting $\mathbf{E}_k$, $\mathbf{y}_k$ and $\mathbf{X}_k$ from the equations (8), (12) and (15) into (9) and (10) we obtain the weight updating equations for the constrained and unconstrained FBLMS algorithms respectively as:

$$\mathbf{W}_{k+1,c} = \mathbf{W}_{k,c} + \mu \mathbf{F} \mathbf{h}_u \mathbf{F}^{-1} \mathbf{X}_k^H \mathbf{E}_k, \quad (16)$$

and

$$\mathbf{W}_{k+1,u} = \mathbf{W}_{k,u} + \mu \mathbf{X}_k^H \mathbf{E}_k. \quad (17)$$

Transforming the two updating equations (16) and (17) into the time domain produces the corresponding updating equations for the constrained and unconstrained FBLMS algorithms in time domain:

$$\mathbf{w}_{k+1,c} = \mathbf{w}_{k,c} + \mu \chi_{k,c}^H [\mathbf{d}_{k,c} - \chi_{k,c} \mathbf{w}_{k,c}], \quad (18)$$

and

$$\mathbf{w}_{k+1,u} = \mathbf{w}_{k,u} + \mu \chi_{k,u}^H [\mathbf{d}_{k,c} - \chi_{k,u} \mathbf{w}_{k,u}], \quad (19)$$

where $\mathbf{w}_{k,c}$ is an $N \times 1$ vector, and where $\mathbf{d}_{k,c}$ is defined in (6). The matrices $\chi_{k,c}$ and $\chi_{k,u}$ are the $N \times N$ submatrix from the left-lower corner, and the $N \times 2N$ sub-matrix from the lower part of the matrix χ_k, respectively. The details of achieving (18) and (19) can be found in [7].

The updating equations (18) and (19) truly reflect the different characteristics of the two algorithms. The equation (18) is actually the updating equation of the time domain BLMS algorithm. So the constrained FBLMS algorithm implements a length-N BLMS algorithm. Unlike the constrained FBLMS algorithm, the unconstrained FBLMS algorithm implements a length-2N adaptive filter in time domain and its data matrix $\chi_{k,u}$ is circularly generated from the reversed data vector $\mathbf{x}_k$.

It has been shown [4] that the optimum solution of the unconstrained algorithm is $\mathbf{W}_{opt,u} = \mathbf{R}_u^{-1} \mathbf{p}_u$. Here $\mathbf{R}_u = E[\chi_k^H \mathbf{h}_l \chi_k]$ and $\mathbf{p}_u = E[\chi_k^H \mathbf{d}_k]$. It can also be easily shown that the optimum solution of the constrained FBLMS algorithm is $\mathbf{w}_{opt,c} = \mathbf{R}_c^{-1} \mathbf{p}_c$ with $\mathbf{R}_c = E[\chi_{k,c}^H \chi_{k,c}]$ and $\mathbf{p}_c = E[\chi_{k,c}^H \mathbf{d}_{k,c}]$. Assume $\mathbf{v}_{k,c} = \mathbf{w}_{k,c} - \mathbf{w}_{opt,c}$ and $\mathbf{v}_{k,u} = \mathbf{w}_{k,u} - \mathbf{w}_{opt,u}$. By substituting $\mathbf{v}_{k,c}$ and $\mathbf{v}_{k,u}$ into the equations (18) and (19) and taking the expectation on both sides of these two equations we can get:

$$E[\mathbf{v}_{k+1,c}] = (\mathbf{I} - \mu \mathbf{R}_c) E[\mathbf{v}_{k,c}], \quad (20)$$

$$E[\mathbf{v}_{k+1,u}] = (\mathbf{I} - \mu \mathbf{R}_u) E[\mathbf{v}_{k,u}]. \quad (21)$$

From the above analysis we can see that the convergence performance of the two FBLMS algorithms are both determined by their correlation matrices, $\mathbf{R}_c$ and $\mathbf{R}_u$, and their equivalent filter lengths. The convergence to the optimum solution of the two algorithms has been verified in [3, 4, 6]. Here we focus mainly on the transient behaviors of the two algorithms.

From the property of the LMS algorithm [2], in order to guarantee the mean-squared error convergence, the step size parameter μ must satisfy the condition:

$$0 < \mu < \frac{2}{Total\ input\ power} = \frac{2}{Tr(\mathbf{R})} \quad (22)$$

and the average time constant is:

$$\tau_{avg} = \frac{1}{2\mu\lambda_{avg}} \quad (23)$$

with $\lambda_{avg} = \frac{1}{M}\sum_1^M \lambda_i$. For stationary input signals it can be easily shown that $Tr(\mathbf{R}_c) = 2Tr(\mathbf{R}_u)$. Also from $Tr(\mathbf{R}) = \sum \lambda_i$ we can see that the λ_{avg} is the same for both $\mathbf{R}_c$ and $\mathbf{R}_u$. Combining the above two conditions with the properties of the correlation matrices $\mathbf{R}_c$ and $\mathbf{R}_u$ we can conclude that the unconstrained FBLMS algorithm has a reduced stable range of step size compared to the constrained FBLMS algorithm and its best step size (in terms of achieving fastest convergence rate), is about one half that of the constrained FBLMS algorithm. The convergence rate of the unconstrained FBLMS algorithm is approximately one half that of the constrained FBLMS algorithm.

The equation (19) shows that the unconstrained FBLMS algorithm is equivalent to a length-2N adaptive filter. Is it capable of modeling higher order system than the constrained FBLMS algorithm? The answer is no. From the equation (19) we can see that the unconstrained FBLMS algorithm works on the circular data matrix $\chi_{k,u}$ which generates the "wrap-around" error. This error is included into the gradient of the unconstrained FBLMS algorithm during the learning process. So this error prevents the unconstrained FBLMS algorithm from modeling a higher order system any better than the constrained FBLMS algorithm.

3. COMPUTER SIMULATION RESULTS

In the first example, an 8-tap FIR lowpass filter was used to identify an 8-tap lowpass filter. White Gaussian noise and colored Gaussian noise were used as input signals in these two cases, respectively. In each case the constrained and unconstrained FBLMS algorithms were used in the identification. During the adaptation the best step sizes (as determined experimentally for fastest convergence) for each case was used. Figure 2 and 5 show the learning curves of the constrained and unconstrained FBLMS algorithms with white Gausian noise and colored Gausian noise respectively. From the two figures we see that the constrained FBLMS algorithm converges faster than the unconstrained FBLMS algorithm. The second example follows from the first one. Here for each different input, the best step size of the constrained FBLMS algorithm was used in updating of the unconstrained FBLMS algorithm. Figure 3 and 6 show the learning curves for the two different input signal cases. From the figures we can see that when the best step size of the constrained FBLMS algorithm is used the performance of the unconstrained FBLMS algorithm tends to be unstable. In the third example, an 8-tap FIR filter was used to identify an 12-tap FIR lowpass filter. Again the best step size was used for each different algorithm. Figure 4 and 7 show the simulation results. From the figure we can see that the noise floor of the unconstrained FBLMS algorithm is about 7dB lower than that of the constrained algorithm, but because of the "wrap-around" error it is unable to identify the higher system with much higher precision.

4. CONCLUSIONS

By formulating the time updating equations of the two FBLMS algorithms we identified what we believe some of the characteristic differences of the two algorithms. The constrained FBLMS algorithm actually implements a length-N time domain block LMS algorithm. The unconstrained FBLMS algorithm is equivalent to a length-2N adaptive filtering algorithm. The data matrix it works on is circularly generated from the reversed data vector $\mathbf{x}_k$. The ultimate effect of removing the gradient constraint from the constrained FBLMS algorithm is slower convergence, smaller range of stable step size, doubling the equivalent filter length, and the inability of modeling higher order signal model with higher accuracy than the constrained FBLMS algorithm.

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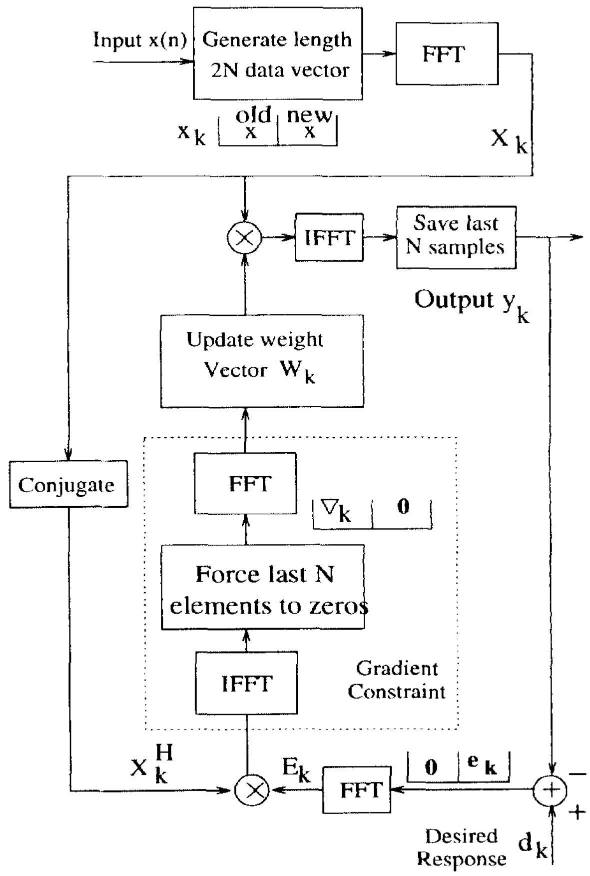

Figure 1: The block diagram of the constrained and unconstrained FBLMS algorithms.

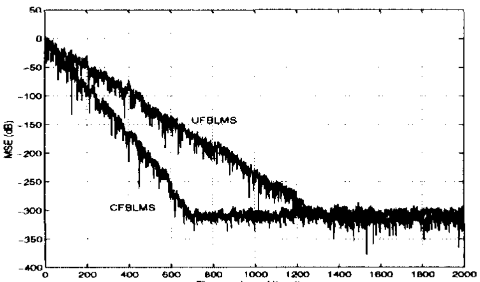

Figure 2. Learning Curves of the length-8 constrained(CFBLMS, $\mu = 0.035$) and unconstrained(UFBLMS, $\mu = 0.020$) adaptive filters with white Gausian noise input.

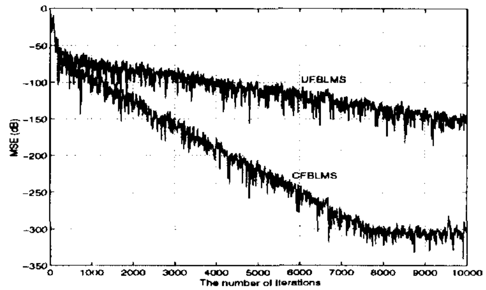

Figure 3. Learning Curves of the length-8 constrained(CFBLMS, $\mu = 0.055$) and unconstrained(UFBLMS, $\mu = 0.030$) adaptive filters with colored Gausian noise input.

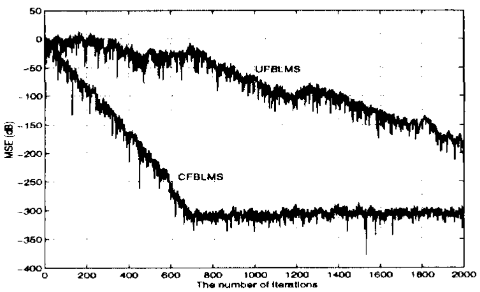

Figure 4. Learning Curves of the length-8 constrained(CFBLMS, $\mu = 0.035$) and unconstrained(UFBLMS, $\mu = 0.035$) adaptive filters with white Gausian noise input.

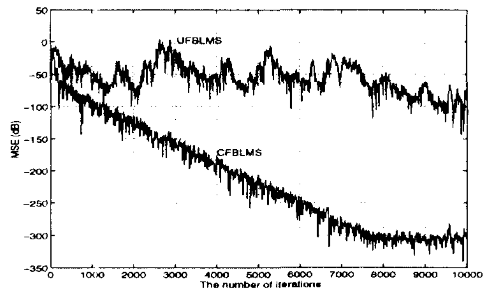

Figure 5. Learning Curves of the length-8 constrained(CFBLMS, $\mu = 0.055$) and unconstrained(UFBLMS, $\mu = 0.055$) adaptive filters with colored Gausian noise input.

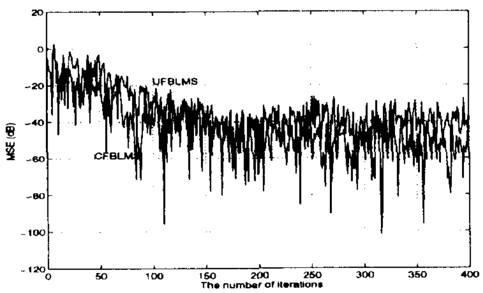

Figure 6. Learning Curves of the length-8 constrained(CFBLMS, $\mu = 0.035$) and unconstrained(UFBLMS, $\mu = 0.020$) adaptive filters with white Gausian noise input. The system being identified is length-12 lowpass FIR filter.

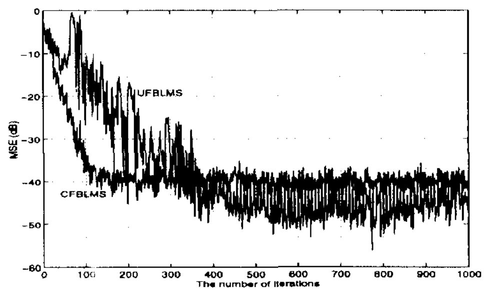

Figure 7. Learning Curves of the length-8 constrained(CFBLMS, $\mu = 0.055$) and unconstrained(UFBLMS, $\mu = 0.030$) adaptive filters with colored Gausian noise input. The system being identified is length-12 lowpass FIR filter.