

# Comparative Performance of LMSTDE and ETDE for Delay and Doppler Estimation

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## Abstract

While the LMSTDE interpolates filter coefficients to determine the time difference of a signal received at two separated sensors, the ETDE adapts and tracks the time delay explicitly. In this paper, the convergence dynamics of these two approaches are analyzed when the delay is a linearly time-varying function due to relative motion between the signal source and the receivers. The merits of these methods in terms of parameter estimation and computational load will be discussed. Simulation results show that ETDE outperforms LMSTDE for delay and Doppler estimation and its performance is comparable to the CRLB under noisy environment.

## I. Introduction

Estimating and tracking the time delay between signals received at two spatially separated sensors are of considerable interests in areas such as sonar, radar, biomedical engineering and communication [1]. For example in passive sonar, the bearing and the range of a signal source can be determined from the time delay measurements by triangulation [2]. Generalized cross correlation [3,4] is a conventional method for time delay estimation (TDE) but it requires *a priori* knowledge of the signal statistics in order to achieve optimal delay estimation. Other approaches for TDE using adaptive filtering technique [5-9] have also been investigated in which the estimation of the signal spectrum is not necessary. Basically, these methods determine the time delay iteratively and are capable of tracking nonstationary delays due to relative motion between the source and receivers or due to the time-varying characteristics of the transmission medium.

In this paper, the performance of two adaptive TDE systems, namely, the Least Mean Square Time Delay Estimator (LMSTDE) [5-7] and the Explicit Time Delay Estimator (ETDE) [8,9] are compared when the delay is linearly time-varied. Although the LMSTDE and the ETDE are quite similar since they both model the time delay by an FIR filter, they have some inherent differences. In LMSTDE, each filter weight is adapted simultaneously and independently, and the delay estimate is obtained by interpolating the filter coefficients. On the other hand, the ETDE adjusts the estimated delay directly

and thus no interpolation is required. Its filter weights are constrained to be samples of a *sinc* function and they are computed according to the estimated delay only.

The organization of this paper is as follows. The tracking behaviour, delay and Doppler estimation performance, and computational complexities of the LMSTDE and the ETDE are analyzed and discussed in Section II and Section III respectively. Simulation results of these two methods are presented in Section IV and finally, conclusions are drawn in Section V.

## II. Performance Analysis of the LMSTDE

The discrete-time outputs of two spatially separated sensors can be represented by

$$x(k) = s(k) + n_1(k) \quad (1a)$$

and

$$y(k) = s(k - D(k)) + n_2(k) \quad (1b)$$

where  $s(k)$ , the source signal,  $n_1(k)$  and  $n_2(k)$ , the corrupting noises, are all uncorrelated, white Gaussian random processes. Without loss of generality, we assume that the signal and the noise spectrum are band-limited between  $-\pi$  and  $\pi$  while the sampling period is unity. Our tasks are to estimate the nonstationary delay,  $D(k)$ , as well as the Doppler parameter, from  $x(k)$  and  $y(k)$ . When  $D(k)$  is a linearly time-varying function, it can be expressed as [10]

$$D(k) = \theta_0 + \theta_1 k \quad (2)$$

where  $\theta_0$  is the delay at  $k = 0$  and  $\theta_1$  is the Doppler time compression.

Figure 1 shows the system block diagram of the LMSTDE. The basic idea is to model the time difference by using an adaptive FIR filter,  $W(z)$ , which is given by

$$W(z) = \sum_{i=-P}^P w_i z^{-i} \quad (3)$$

where  $P$  should be chosen large enough to reduce the truncation error from interpolating the finite filter coefficients  $\{w_i(k)\}$ . It has been shown [11] that this truncation error decreases as  $P$  increases. For example, when  $P$  is 5, the largest possible error is 8.2% but it drops to 4% for  $P = 10$ . Each  $w_i(k)$  is adapted according to Widrow's LMS algorithm [12] as follows,

$$w_i(k+1) = w_i(k) + 2\mu_w e(k)x(k-i) \quad (4)$$

where

$$e(k) = y(k) - \sum_{i=-P}^P x(k-i)w_i(k)$$

and  $\mu_w$  is a parameter that controls convergence rate and stability of the algorithm.

Define  $\xi(t)$  as the output of  $\{w_i(k)\}$  after passing through an ideal low-pass filter with a normalized bandwidth of 0.5. We can write

$$\xi(t) = \sum_{i=-P}^P w_i(k) \text{sinc}(t-i) \quad (5)$$

where  $\text{sinc}(v) = \sin(\pi v)/(\pi v)$ . It has been shown [11] that the delay estimate,  $\hat{D}_w(k)$ , is given by the value of  $t$  at which the maximum of  $\xi(t)$  occurs.

Taking expectation on both sides of (4) and using (1) and (2), we obtain

$$E\{w_i(k+1)\} = \lambda E\{w_i(k)\} + 2\mu_w \sigma_s^2 \text{sinc}(i - \theta_0 - \theta_1 k) \quad (6)$$

$k = 0, 1, 2, \dots$

where

$$\lambda = 1 - 2\mu_w(\sigma_s^2 + \sigma_n^2)$$

and  $\sigma_s^2$  represents the signal power and  $\sigma_n^2$  denotes the power of  $n_1(k)$  and  $n_2(k)$ . To guarantee convergence,  $\lambda$  is bounded between  $-1$  and  $1$ . Multiplying the factor  $\lambda^{k-l}$ , where  $l = 1, 2, \dots$ , to the first  $k$  equations of (6) and then sum them together yields

$$E\{w_i(k)\} = 2\mu_w \sigma_s^2 \sum_{j=0}^{k-1} \lambda^{k-1-j} \text{sinc}(i - \theta_0 - \theta_1 j) + w_i(0)\lambda^k \quad (7)$$

Substituting (7) into (5) and choosing all the initial weights  $w_i(0)$  to be zero,  $\xi(t)$  becomes

$$\begin{aligned} \xi(t) &= 2\mu_w \sigma_s^2 \sum_{i=-P}^P \sum_{j=0}^{k-1} \lambda^{k-1-j} \text{sinc}(i - \theta_0 - \theta_1 j) \\ &\quad \cdot \text{sinc}(t-i) \\ &\approx 2\mu_w \sigma_s^2 \sum_{j=0}^{k-1} \lambda^{k-1-j} \text{sinc}(t - \theta_0 - \theta_1 j) \end{aligned} \quad (8)$$

which is a summation of  $k$  scaled  $\text{sinc}$  functions. We notice that each term peaks at  $t = \theta_0 + \theta_1 j$  and with a scaling factor of  $2\mu_w \sigma_s^2 \lambda^{k-1-j}$  for  $0 \leq j \leq k-1$ . Hence we may approximate the mean value of  $\hat{D}_w(k)$  by

$$\begin{aligned} E\{\hat{D}_w(k)\} &\approx \frac{\sum_{j=0}^{k-1} (\theta_0 + \theta_1 j) \lambda^{k-1-j}}{\sum_{j=0}^{k-1} \lambda^{k-1-j}} \\ &= \theta_0 + \theta_1 k + \frac{\theta_1 k \lambda^k}{1 - \lambda^k} - \frac{\theta_1}{2\mu_w(\sigma_s^2 + \sigma_n^2)} \end{aligned} \quad (9)$$

The third term is a transient term which converges to zero when  $k$  goes to infinity since  $|\lambda| < 1$ . The last term,  $\theta_1/(2\mu_w(\sigma_s^2 + \sigma_n^2))$ , contributes to the time lag in steady state. This time lag increases with  $\theta_1$  and decreases with  $\mu_w$ . In general, the smaller the time lag, the better the tracking of the time-varying delay. Upon convergence, (9) becomes a straight line with a slope of  $\theta_1$ . Fitting the trajectory of  $\hat{D}_w(k)$  onto a straight line, we expect an accurate estimate of  $\theta_1$  to be obtained from the slope of the line.

The steady state mean square error of  $\hat{D}_w(k)$ ,  $\varepsilon_w$ , equals to the sum of the variance of a static delay [5] and the square of the time lag, which is given by

$$\varepsilon_w = \frac{3\mu_w \sigma_s^2 (1 + 2SNR)(1 + SNR)}{\pi^2 SNR^3} + \left( \frac{\theta_1}{2\mu_w(\sigma_s^2 + \sigma_n^2)} \right)^2 \quad (10)$$

where  $SNR$  is the signal-to-noise ratio. The parameter  $\mu_w$  should be selected appropriately to attain an optimal value of  $\varepsilon_w$  since the first term of (10) is directly proportional to  $\mu_w$  while the second term is inversely proportional to  $\mu_w^2$ . It is intriguing to note that when operating under a noisy environment or when a large  $\mu_w$  has to be used in order to reduce the time lag, false peaks of  $\{w_i(k)\}$  might be located because of random fluctuations due to the noisy gradients of the stochastic algorithm. This phenomenon is similar to the so called "outliers" as described in [13]. In this case, the estimated delay  $\hat{D}_w(k)$  which is determined by  $\{w_i(k)\}$  may deviate significantly from the actual delay. As a result, the mean square delay error of  $\hat{D}_w(k)$  can become much larger than the theoretical value.

The computational load of updating  $w_i(k)$  is  $2P+1$  additions and  $4P+2$  multiplications. While  $2P+1$  additions and multiplications are required to compute  $e(k)$ . Furthermore, if interpolation of  $\{w_i(k)\}$  is also considered, then an extra  $20P$  additions and  $20P+10$  multiplications are needed assuming that the delay estimate has a resolution of 0.001.

### III. Performance Analysis of the ETDE

In ETDE, the filter coefficients of  $W(z)$  are replaced by  $\{\text{sinc}(i - \hat{D}_e(k))\}$  for  $-P \leq i \leq P$ , where  $\hat{D}_e(k)$  is the estimated delay. It is trivial from the convolution theorem that this filter will provide a time lag of  $\hat{D}_e(k)$  to the input signal  $x(k)$ . The major advantage of ETDE is that the time delay is now being adapted directly on a sample by sample basis. Consequently, this method requires no interpolation and false peaks will not occur as in LMSTDE. Similar to Widrow's LMS algorithm [12], the ETDE uses a stochastic gradient estimate which is obtained by differentiating the instantaneous error square,  $e^2(k)$ , with respect to  $\hat{D}_e(k)$ . The ETDE updating equation is given by

$$\begin{aligned}\hat{D}_e(k+1) &= \hat{D}_e(k) - \mu_e \frac{\partial e^2(k)}{\partial \hat{D}_e(k)} \\ &= \hat{D}_e(k) - 2\mu_e e(k) \sum_{i=-p}^p x(k-i) f(i - \hat{D}_e(k))\end{aligned}\quad (11)$$

where

$$e(k) = y(k) - \sum_{i=-p}^p \text{sinc}(i - \hat{D}_e(k)) x(k-i)$$

and  $f(v) = (\cos(\pi v) - \text{sinc}(v))/v$  and  $\mu_e$  is the convergence step size for the ETDE.

A *sinc* lookup table of dimensions  $K \times 2r+1$  together with a cosine mapping vector of length  $L$  are used to reduce the computations involved in calculating the *sinc* and  $f$  functions [9]. The lookup table is constructed with elements given by  $\text{sinc}(0.5i/(K-1)-j)$  for  $0 \leq i \leq K-1$  and  $-r \leq j \leq r$  while the mapping vector has elements with values  $\cos(\pi l/L)$  for  $0 \leq l \leq L-1$ . In this method, each iteration requires only  $4P+2$  additions and  $4P+3$  multiplications to calculate the delay estimate  $\hat{D}_e(k)$  while the computation for  $e(k)$  is the same as the LMSTDE. Overall speaking, the ETDE is less complicated than the LMSTDE since no interpolation is needed. However, a larger memory is required to store the lookup tables.

The expected value of the delay estimate,  $E\{\hat{D}_e(k)\}$ , has been shown [8] to have the following trajectory,

$$E\{\hat{D}_e(k+1)\} \approx E\{\hat{D}_e(k)\} - 2\mu_e \sigma_s^2 E\{f(\theta_0 + \theta_1 k - \hat{D}_e(k))\} \quad (12)$$

Assuming that  $\mu_e$  is chosen large enough so that  $\hat{D}_e(k)$  will not "fall out-of-lock" [13] at each iteration, we can approximate (12) by expanding the term  $f(\theta_0 + \theta_1 k - \hat{D}_e(k))$  using Taylor's series. By ignoring the higher order terms, we obtain

$$\begin{aligned}E\{\hat{D}_e(k+1)\} &\approx E\{\hat{D}_e(k)\} - 2\mu_e \sigma_s^2 E\{f(0)\} \\ &\quad + E\{f'(0)(\theta_0 + \theta_1 k - \hat{D}_e(k))\} \\ &= \left(1 - \frac{2}{3}\mu_e \sigma_s^2 \pi^2\right) E\{\hat{D}_e(k)\} \\ &\quad + \frac{2}{3}\mu_e \sigma_s^2 \pi^2 (\theta_0 + \theta_1 k)\end{aligned}\quad (13)$$

since  $f(0) = 0$  and  $f'(0) = -\pi^2/3$ .

A stability bound for  $\mu_e$  to ensure finite delay variance for static delay is given by [9]

$$0 < \mu_e < \frac{1}{\pi^2(\sigma_s^2 + \sigma_n^2/3)} \quad (14)$$

Solving (13) yields

$$\begin{aligned}E\{\hat{D}_e(k)\} &= \theta_0 + \theta_1 k - \frac{3\theta_1}{2\mu_e \pi^2 \sigma_s^2} \\ &\quad + \left(\hat{D}_e(0) - \theta_0 + \frac{3\theta_1}{2\mu_e \pi^2 \sigma_s^2}\right) \left(1 - \frac{2}{3}\mu_e \pi^2 \sigma_s^2\right)^k\end{aligned}\quad (15)$$

In steady state, the last term vanishes and  $E\{\hat{D}_e(k)\}$  lags  $D(k)$  by the third term which is directly proportional to  $\theta_1$  and inversely proportional to  $\mu_e$  and  $\sigma_s^2$ . Due to the similarity of (9) and (15), an estimate of  $\theta_1$  can also be acquired by fitting the trajectory of  $\hat{D}_e(k)$  onto a straight line.

Similar to (10), in this particular application, the mean square delay error of the ETDE,  $\epsilon_e$ , is given by

$$\epsilon_e = \frac{2\mu_e \sigma_s^2 (1 + SNR)}{SNR^2} + \left(\frac{3\theta_1}{2\mu_e \sigma_s^2 \pi^2}\right)^2 \quad (16)$$

## IV. Simulation Results

Simulation tests have been conducted to compare the performance of LMSTDE and ETDE. In our experiments, the sequences  $s(k)$ ,  $n_1(k)$  and  $n_2(k)$  are produced by a random number generator of Gaussian distribution with a white spectrum. The signal source has unity power and different SNRs are obtained by proper scaling of the random noise sequences. The parameters  $\theta_0$  and  $\theta_1$  are chosen to be 0.25 and 0.001 respectively. Different values of  $\mu_w$  and  $\mu_e$  have been used to evaluate their effects on the tracking capability of time-varying delays. The order of the filter,  $2P+1$ , is chosen to be 31 in both methods to ensure small enough truncation error. The delay estimate of each scheme has a resolution of 0.001. To fulfil this requirement, the size of the cosine and *sinc* tables are 1024 and 513 x 31 respectively in the ETDE while 10 bisections are incorporated in the LMSTDE. Simulation results provided are the averages of 100 independent runs.

Figure 2 shows the trajectories of the delay estimates at a SNR of 20 dB with  $\mu_w = \mu_e = 0.005$ . After the 200th iteration, the learning curves of the LMSTDE and the ETDE were found to be approximately parallel to the actual delay and their corresponding delay estimates lagged  $D(k)$  by roughly 0.11 and 0.06. Due to the approximations in deriving (9) and (15), there existed discrepancies between the measured and the theoretical time lags in both schemes.

The convergence dynamics of the LMSTDE and the ETDE at SNR = 0 dB for different values of step sizes are presented in Figure 3 and Figure 4 respectively. Under such a noisy environment, it can be observed in Figure 3 that the trajectory of  $\hat{D}_w(k)$  for  $\mu_w = 0.005$  fluctuated severely indicating a large delay variance. A single run of this experiment is depicted to illustrate that fluctuations were mainly due to mis-location of the peaks. It is also demonstrated for  $\mu_w = 0.0005$  and  $\mu_w = 0.00005$  that these

false peaks could be eliminated by choosing a smaller value of  $\mu_w$ . However, in these cases, the convergence speed decreased noticeably which in turn gave rise to a poorer tracking capability. On the other hand, as seen in Figure 4, ETDE performed much better than LMSTDE when  $\mu_e = 0.005$  and the delay was tracked with a time lag of approximately 0.07. In addition, the mean square delay error in this test was found to be 0.025 and it was close to the theoretical value as given by (16). When  $\mu_e$  was reduced to 0.00005, the algorithm failed to track the delay since the assumptions in (13) became invalid. Hence the lower bound of  $\mu_e$  in (14) should be modified to cope with nonstationary delays. Indeed, this modified lower bound can be obtained from (15) and it is given by  $3\theta_1/(2\pi^2\sigma_s^2 t_{\max})$  where  $t_{\max}$  is the maximum allowable time lag. In practice, only an approximate value of this bound can be found because both  $\theta_1$  and  $\sigma_s^2$  might not necessarily be available. Since the ETDE outperforms the LMSTDE for large step size operation, it has a greater potential to track fast moving delays.

Fitting the trajectory of the delay estimate onto a straight line by using the least square method, estimates of  $\theta_1$  can be determined by the slope of the line. Using the delay estimates from the 201th to the 500th iterations, their statistics for  $\mu_w = \mu_e = 0.005$  are tabulated in Table 1 together with the Cramer Rao Lower Bound (CRLB) of the Doppler [14]. It can be seen that accurate Doppler estimates were achieved in all cases. At a high SNR of 20 dB, the performance of the LMSTDE and the ETDE were similar. However, when SNR = 0 dB, the variance of  $\theta_1$  in the ETDE is much smaller than that of the LMSTDE and is comparable to the CRLB. In general, the ETDE is more efficient and effective for delay and Doppler estimation under noisy environment.

## V. Conclusions

The LMSTDE and the ETDE are two different adaptive algorithms for time delay estimation. The former adapts and interpolates the filter coefficients to determine the delay while the latter provides direct time delay measurements. The convergence dynamics the two methods have been analyzed and it is found that the ETDE gives a superior performance than the LMSTDE. Since false peaks may occur in the LMSTDE under noisy environment, its delay variance could be very large. In contrast, the ETDE always produces reliable delay and Doppler estimates as long as the step size is chosen large enough and the variance of the Doppler estimate is comparable to the CRLB.

### Acknowledgement

This work was partly supported by a research grant awarded by the Hong Kong Research Grant Council.

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		SNR = 20 dB	SNR = 0 dB
Mean	LMSTDE	$9.5327 \times 10^{-4}$	$9.7187 \times 10^{-4}$
	ETDE	$1.0108 \times 10^{-3}$	$1.0563 \times 10^{-3}$
Variance	LMSTDE	$7.7870 \times 10^{-9}$	$2.7161 \times 10^{-5}$
	ETDE	$3.9095 \times 10^{-9}$	$4.4154 \times 10^{-7}$
CRLB		$5.8653 \times 10^{-10}$	$8.7542 \times 10^{-8}$

Table 1 Statistics of Doppler estimate for the LMSTDE and the ETDE

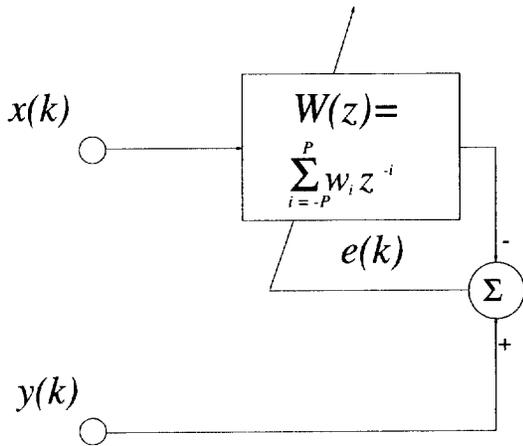


Figure 1 System block diagram for the LMSTDE

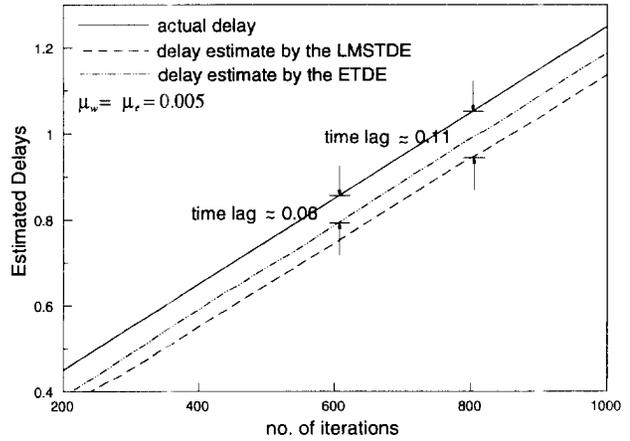


Figure 2 Delay estimates of the LMSTDE and the ETDE at  $SNR = 20$  dB

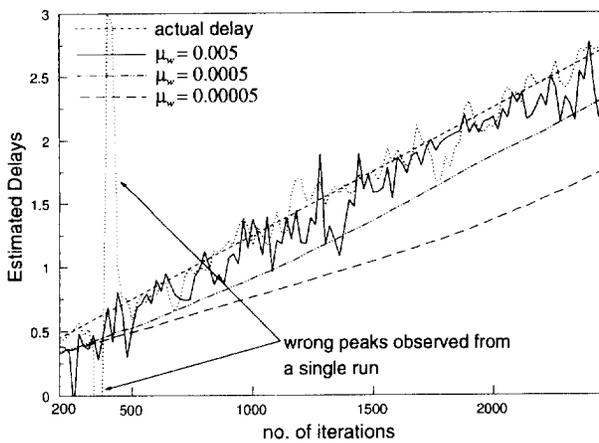


Figure 3 Delay estimates of the LMSTDE at  $SNR = 0$  dB for different  $\mu_w$

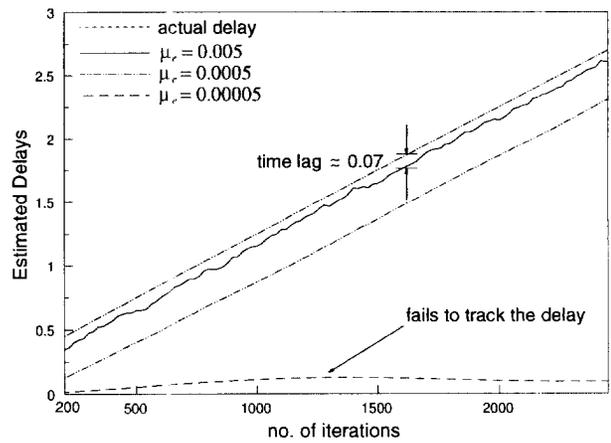


Figure 4 Delay estimates of the ETDE at  $SNR = 0$  dB for different  $\mu_e$