Channel Capacity of Spatial Division Multiple Access Schemes

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Abstract

The radio frequency spectrum for wireless communications is a finite and scarce resource, there is a fundamental limit on the radio channels which can be made available to mobile communication. In this paper, we study the capacity limit of a wireless system with multiple receivers using an information theoretic model. We showed that the capacity region expands and varies between two boundaries depending on the relative positions of the users.

1 Introduction

A prime problem of current wireless communication systems is the contradiction between the finite spectrum available and the increasing demand for wireless communications. To mitigate this problem, various multiplexing schemes (*e.g.*, TDMA or CDMA) have been proposed to increase the channel capacity for the given spectrum. Most research is focused on the searching of a more efficient way of using the existing resource [1, 2, 3, 4, 5, 6, 7, 8]. Recently, it has been proposed that multiple receivers (or an antenna array) can be used at each cell site to increase the channel capacity via exploitation of the spatial diversity among different users. Such a system is referred as Spatial-Diversity-Multiple-Access (or SDMA). Significant progress has been made in incorporating the antenna array to the existing wireless communication system. However, one fundamental question remained unanswered is that the upper limit of the channel capacity of the SDMA system. In this paper, we present our studies on the channel capacity of a SDMA system from information theory point of view. Our objective is to reveal the upper bound of capacity increase from an antenna array. Although our results are derived based on an idealized model, they provide insights on how the exploitation of the spatial diversity affects the channel capacity.

1.1 Mathematical Model

We consider a single-cell network where mobiles (users) transmit to the cell site. We look at the mobile-to-cell-site link and model it as a Synchronized Discrete Time Multiple User Channel. Finally, Multiple receivers (array of antennas) are used to explore the spatial diversity among users.

1.2 Array of Receivers

Throughout this work, we assume that the sources are significantly far from the array (far-field sources) so that the wavefronts are planar. Incoming signals are narrowband[1], *e.g.*, the baseband signal does not vary over the length of the array except a phase shift. The Direction Of Arrival (DOA) is defined as the azimuth[2] angle (θ_k) in reference of the array. We assume that the DOAs are slowly varying and they are constant over many information bits. Given the above assumptions, for a single source $s_1(t)$, we can easily see that the data vector $\mathbf{x}(t)$ received from the antenna array is given by

$$x(t) = \underbrace{\mathbf{a}(\theta_1)s_1(t)}_{\text{directpath}} + \underbrace{\sum_{l=2}^{L_1} \alpha_l \mathbf{a}(\theta_l)s_1(t)}_{\text{multipath}} = \mathbf{a}_1 s_1(t),$$

where $L_1 - 1$ is the total number of multipath signals and α_l is the phase and amplitude difference between

[1] This narrow-band assumption is reasonable in cellular telephony since the carry frequency is usually 800 MHz while the bandwidth of the information transmission of each user is usually less than 1 MHz.

[2] To simplify our presentation, we assume that all the mobile units have the identical elevation angle.

the l^{th} multipath and direct path. $\mathbf{a}_1$ is defined as the *spatial signature* (SS) associated with source $s_1(t)$.

In the presence of co-channel users and noise, the data vectors $\mathbf{x}(\cdot)$ can be written as:

$$\mathbf{x}(t) = \sum_{k=1}^{K} \mathbf{a}_k s_k(t) + \mathbf{n}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t),$$
$$\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_K], \ \mathbf{s}(t) = [s_1(t), \ldots, s_K(t)]^T,$$

where $\mathbf{n}(t)$ are $\mathbf{s}(t)$ are uncorrelated and Gaussian distributed. Obviously, we can always normalize the σ_n^2 to unity without changing the signal-to-noise-ratio (SNR). Taking the expectation of the data vectors $\mathbf{x}(t)$, we obtain the following output covariance matrix

$$\mathbf{R}_x = E\{\mathbf{x}(t)\mathbf{x}^*(t)\} = \mathbf{A}\mathbf{R}_S\mathbf{A}^* + \mathbf{I}, \qquad (1)$$

where $\mathbf{R}_S = E\{\mathbf{s}(t)\mathbf{s}^*(t)\} = \mathrm{diag}\{p_1, \ldots, p_K\}$.

2 Capacity Regions of SDMA Systems

The problem under consideration is how fast we can *reliably* transmit the K signals to an array of multiple receivers in a standard Gaussian channel with certain SNR. Of course, the maximum achievable rate of reliable communications also depends on how the decoding is performed. In this paper, we only discuss two types of decoding methods, *i.e.*, (optimal) *joint decoding* and *independent decoding*. Joint decoding means that the decoding of all the signals is performed simultaneously, while independent decoding means that different signals are decoded independently and parallelly. Though joint decoding is optimal, it requires $O(2^K)$ computational complexity and can be too expensive to implement. Independent decoding may somehow be more practical since it only requires $O(K)$ complexity. The fundamental difference between joint and independent decoding is that the former treats all the $\{s_k(t)\}$ as signals whereas the latter treats $\{s_k(t)\}$, $k \neq i$ as noise when decoding $s_i(t)$.

2.1 Capacity for Joint Decoding

By multiple access information theory (see *e.g.*, [9], [7]), any achievable rate K-tuple $(R_1, \cdots, R_K)$ corresponding to the K users must satisfy:

$$R_k \le \max I(\mathbf{x}(t); s_k(t) | s_n(t), n \neq k);$$
$$R_{k_1} + R_{k_2} \le \max I(\mathbf{x}(t); s_{k_1}(t); s_{k_2}(t) | s_n(t), n \neq k_1, k_2);$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\sum_{k=1}^{K} R_k \le \max I(\mathbf{x}(t); s_1(t); \ldots; s_K(t)), \qquad (2)$$

with $k, K_1, K_2 = 1, \ldots, M$, $k_1 \neq k_2$ and $I(x; y|z)$ denotes the mutual information between x and y given z. For a Gaussian channel, it is not difficult to derive that

$$R_k \le L\frac{1}{2}\log\left(\det(\mathbf{I} + p_k \mathbf{a}_k \mathbf{a}_k^H)\right);$$
$$R_{k_1} + R_{k_2} \le \frac{1}{2}\log\left(\det(\mathbf{I} + p_{k_1}\mathbf{a}_{k_1}\mathbf{a}_{k_1}^H + p_{k_2}\mathbf{a}_{k_2}\mathbf{a}_{k_2}^H)\right);$$
$$\vdots \qquad \vdots$$
$$\sum_{k=1}^{K} R_k \le \frac{1}{2}\log\det(\mathbf{R}_x),. \qquad (3)$$

2.2 Capacity for Independent Decoding

In the case of independent decoding, the Multiple-Access-Channel (MAC) is an interference channel. The decoder for $s_k(t)$ considers the background noise as well as other sources as interference. The achievable rate satisfies:

$$R_k \le \max I(\mathbf{x}(t); s_k(t)) = \frac{1}{2}\log\left(\frac{\det(\mathbf{R}_x)}{\det(\mathbf{R}_x - p_k \mathbf{a}_k \mathbf{a}_k^H)}\right). \qquad (4)$$

3 Two-User Case

To simplify the discussion and provide more insights on how the spatial diversity of users affects the channel capacity, we first study the capacity region for the two-user case.

3.1 Capacity Region for Joint Decoding

From (3), it is seen for a Gaussian channel, any achievable rate pair, (R_1, R_2), must satisfy the following:

$$R_1 = \frac{1}{2}\log(\det(\mathbf{I} + p_1\mathbf{a}_1\mathbf{a}_1^*))$$
$$R_2 = \frac{1}{2}\log(\det(\mathbf{I} + p_2\mathbf{a}_2\mathbf{a}_2^*))$$
$$R_1 + R_2 = \frac{1}{2}\log(\det(\mathbf{I} + \mathbf{A}\mathbf{R}_s\mathbf{A}^*)). \qquad (5)$$

Since $\mathbf{I}+\mathbf{A}\mathbf{R}_s\mathbf{A}^*$, $\mathbf{I}+p_1\mathbf{a}_1\mathbf{a}_1^*$, and $\mathbf{I}+p_2\mathbf{a}_2\mathbf{a}_2^*$ are $M \times M$ matrices, their determinants can be difficult to evaluate. We alleviate this difficulty using the following easily proven lemma.

Lemma 1 *For any matrices* $\mathbf{D}_1$ *$(m \times n)$ and* $\mathbf{D}_2$ *$(n \times m)$,*

$$\det(\mathbf{I} + \mathbf{D}_1\mathbf{D}_2) = \det(\mathbf{I} + \mathbf{D}_2\mathbf{D}_1). \tag{6}$$

Apply Lemma 1 to (5), we obtain

$$R_1 \leq \frac{1}{2}\log(1 + Mp_1) \tag{7}$$

$$R_2 \leq \frac{1}{2}\log(1 + Mp_2) \tag{8}$$

$$R_1 + R_2 \leq \frac{1}{2}\log(1 + M(p_1 + p_2) + M^2 p_1 p_2 \sin^2 \angle(\mathbf{a}_1, \mathbf{a}_2)), \tag{9}$$

where $\angle(\mathbf{a}_1, \mathbf{a}_2)$ denotes the angle between the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

The region delimited by (7)-(9) is the capacity region for the two-user channel, as shown in Figure 1. By (9), the capacity region clearly depends on the physical locations of these two users.

Similarly, from (4), the capacity region for independent decoding is a rectangularly bounded by

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log\left(\frac{\det(\mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{I})}{\det(p_2\mathbf{a}_2\mathbf{a}_2^H + \mathbf{I})}\right) \\ &= \frac{1}{2}\log\left(1 + \frac{Mp_1 + M^2p_1p_2\sin^2\angle(\mathbf{a}_1, \mathbf{a}_2)}{1 + Mp_2}\right), \\ R_2 &\leq \frac{1}{2}\log\left(\frac{\det(\mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{I})}{\det(p_1\mathbf{a}_1\mathbf{a}_1^H + \mathbf{I})}\right) \\ &= \frac{1}{2}\log\left(1 + \frac{Mp_2 + M^2p_1p_2\sin^2\angle(\mathbf{a}_1, \mathbf{a}_2)}{1 + Mp_1}\right) \end{aligned} \tag{10}$$

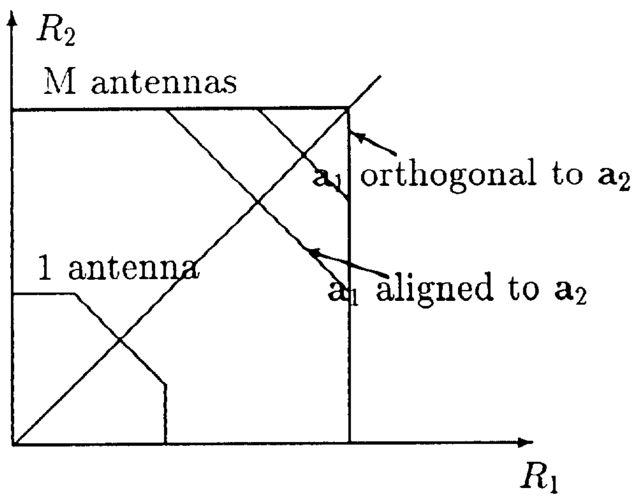

Figure 1: Capacity region for two users and an M-element array.

3.1.1 Outer Boundary

When the spatial locations of the users are such that their spatial signatures are orthogonal to each other, *i.e.*, $\angle(\mathbf{a}_1, \mathbf{a}_2) = \pi/2$, the capacity region is given by :

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log(1 + Mp_1),\ R_2 \leq \frac{1}{2}\log(1 + Mp_2) \\ R_1 + R_2 &\leq \frac{1}{2}\log(1 + Mp_1) + \frac{1}{2}\log(1 + Mp_2). \end{aligned} \tag{11}$$

And independent decoding gives:

$$\begin{aligned} R_1 = I(s_1(t); \mathbf{x}(t)) &= \frac{1}{2}\log(1 + Mp_1); \\ R_2 = I(s_2(t); \mathbf{x}(t)) &= \frac{1}{2}\log(1 + Mp_2). \end{aligned} \tag{12}$$

The capacity region is obviously optimal. It has the well known rectangular shape because user 1 and user 2 have orthogonal spatial signatures and therefore they do not interfere. One can separate them exactly by exploiting their spatial diversity. In this case, we note that independent decoding performs as well as join decoding.

3.1.2 Inner Boundary

When the spatial locations of the users are such that the spatial signatures are aligned to each other, then $|\cos^2(\underline{d_1}, \underline{d_2})| = 1$ and the capacity region is given by:

$$\begin{aligned} R_1 &\leq \frac{1}{2}\log(1 + Mp_1); \\ R_1 &\leq \frac{1}{2}\log(1 + Mp_2); \\ R_1 + R_2 &\leq \frac{1}{2}\log(1 + M(p_1 + p_2)). \end{aligned}$$

And independent decoding gives :

$$\begin{aligned} R_1 &= I(s_1(t); \mathbf{x}(t)) = \frac{1}{2}\log(1 + \frac{Mp_1}{1 + p_2}); \\ R_2 &= I(s_2(t); \mathbf{x}(t)) = \frac{1}{2}\log(1 + \frac{Mp_2}{1 + p_1}). \end{aligned}$$

The capacity region is obviously the smallest over all possible positions $(\mathbf{a}_1, \mathbf{a}_2)$. However, one gains a factor of M in SNR compared with the case without SDMA. With an (M, d) array of sensors, the receiver gets M replicas of the signal and adds them up coherently whereas the noise builds up incoherently.

4 The K-user Channel

Let T be a subset of $(1, \cdots, M)$ and T^c be the complement of T, $\mathbf{A}_T$ and $\mathbf{R}_{S,T}$ be the array manifold and signal covariances matrix corresponding to sources T. The covariance matrix of array output can be decomposed as

$$\mathbf{R}_x = \mathbf{I} + \mathbf{A}_T \mathbf{R}_{S,T} \mathbf{A}_T^H + \mathbf{A}_{T^c} \mathbf{R}_{S,T^c} \mathbf{A}_{T^c}^H$$

4.1 Capacity Region for Joint Decoding

With K users, any achievable rate K-tuple, $(R_1, \cdots, R_K)$, must satisfy:

$$\sum_{t \in T} R_t \leq \max I(\mathbf{x}; s_t,\, t \in T \,|\, s_u,\, u \in T^c)$$
$$= \frac{1}{2} \log(\det[\mathbf{I} + \mathbf{A}_T \mathbf{R}_{S,T} \mathbf{A}_T^H]) \qquad (13)$$

for all subsets T of $[1, .., M]$.
The region delimited by equation (13) is the capacity region for the K-Multiple Access Channel.
Proof:
The first inequality comes from standard information theory.
For the second part, we have:

1. $I(\mathbf{x}(t); s_t,\, t \in T \,|\, s_u,\, u \in T^c) = H(\mathbf{x}(t) |\, s_u,\, u \in T^c) - H(\mathbf{x}(t) |\, s_t, s_u)$,
2. $H(\mathbf{x}(t) |\, s_t, s_u) = H(\mathbf{n}) = \frac{1}{2} \log(2\pi e)^M$,
3. $H(\mathbf{x}(t) |\, s_u,\, u \in T^c) \leq \frac{1}{2} \log(2\pi e)^M \det(\mathbf{I} + \mathbf{A}_T \mathbf{R}_{S,T} \mathbf{A}_T^H)$

□

Definition 1 *We define the symmetric channel capacity to be*

$$R^S = \sum_{k=1}^{K} R_k \qquad (14)$$

4.1.1 Outer Boundary : The optimal spatial distribution of the users.

Theorem 1 *When the number of users, K, is less than the number of sensors, M, then the capacity region is maximal when users' spatial signatures are orthogonal to each other.*

Proof:
From (13), we need to prove that for given $\mathbf{R}_s$, $\det(\mathbf{I} + \mathbf{A}\mathbf{R}_s\mathbf{A}^H)$ is maximized when the columns of $\mathbf{A}$ are orthogonal. We shall prove the theorem by induction.

- The argument is obviously true for two users from
- Assume it is also true for $K - 1$ users.
- Use underscript K to denote variables for K users, By Lemma 1,

$$\begin{aligned}
&\det(\mathbf{I} + \mathbf{A}_K \mathbf{R}_{s,K} \mathbf{A}_K^H) \\
&= \det(\mathbf{I} + \mathbf{A}_K \mathbf{R}_{s,K}^{\frac{1}{2}} \mathbf{R}_{s,K}^{\frac{1}{2}} \mathbf{A}_K^H) \\
&= \det(\mathbf{I} + \mathbf{R}_{s,K}^{\frac{1}{2}} \mathbf{A}_K^H \mathbf{A}_K \mathbf{R}_{s,K}^{\frac{1}{2}}) \\
&= \det \begin{pmatrix} (\mathbf{I} + \mathbf{R}_{s,K}^{\frac{1}{2}} \mathbf{A}_{K-1}^H \mathbf{A}_{K-1} \mathbf{R}_{s,K}^{\frac{1}{2}} & \mathbf{c} \\ \mathbf{c}^H & 1 + K\sigma_K^2 \end{pmatrix} \\
&= \det\left(\mathbf{I} + \mathbf{A}_{K-1} \mathbf{R}_{s,K-1} \mathbf{A}_{K-1}^H\right) \cdot \det\Big(1 + K p_K \\
&\quad - \mathbf{c}^H (\mathbf{I} + \mathbf{R}_{s,K}^{\frac{1}{2}} \mathbf{A}_{K-1}^H \mathbf{A}_{K-1} \mathbf{R}_{s,K}^{\frac{1}{2}})^{-1} \mathbf{c}\Big),
\end{aligned} \qquad (15)$$

where

$$\mathbf{c} = \left(\sqrt{p_1 p_K} \mathbf{a}_1^H \mathbf{a}_K,\ \cdots,\ \sqrt{p_{K-1} p_K} \mathbf{a}_{K-1}^H \mathbf{a}_K\right)^H.$$

The first term is proportional to the channel capacity of $K - 1$ users and is maximized when the columns of $\mathbf{A}_{K-1}$ are orthogonal to each other. The second is maximized when $\mathbf{c} = 0$, which means all the columns of $\mathbf{A}_K$ are orthogonal to each other.

□

In this case, the capacity region for joint decoding and independent decoding are the same, both can be expressed as

$$\sum_{k=1}^{K} R_k = \frac{1}{2} \log \prod_{k=1}^{K} (1 + M p_k).$$

Theorem 2 *When the number of users, K, is strictly larger than the number of sensors, M, then the symmetric channel capacity is maximal when users' spatial signatures are such that rows of the array manifold matrix are orthogonal to each other.*

Proof:
Apply Lemma 1 and Theorem 1. □

4.1.2 Inner Boundary

Theorem 3 *The capacity region is minimal when users' spatial signatures are aligned with each other.*

The proof of this theorem is very similar to that of Theorem 1 and will not be presented.

5 The Gain with SDMA

5.1 *SNR* and Separation Ability

By replacing the omnidirectional antenna by an array with M sensors,

- one gains a factor of $O(M)$ in SNR :
When other users are silent, we look at the capacity of the i^{th} user ($S = (i)$).
For an omnidirectional antenna, we have:

$$R_i \leq \frac{1}{2}\log(1+p_i)$$

For a K sensor array, we have:

$$R_i \leq \frac{1}{2}\log(\det[I + p_i \mathbf{a}_i^H \mathbf{a}_i\ ]) = \frac{1}{2}\log(1+||\mathbf{a}_i||^2 p_i)$$

- one gains in separation ability :
As shown above, the capacity region varies depending upon the relative position (in space) of the mobiles. By doing adaptive spatial filtering one can take advantage of the region 'above' the inner boundary.

5.2 Independent Decoding against Joint Decoding

With K receivers, the gap between performances of independent decoding and joint decoding is reduced and asymptotically cancelled. Indeed with an infinite number of sensors, the receiver can almost completely separate the users. Users do not interfere anymore and the Multiple Access Channel is reduced to M independent point to point channels.

For example with 2 users : as $K \rightarrow \infty$ the separation ability increases. Unless X and Y are aligned, we have:

$$\begin{aligned} \cos^2(\mathbf{a}_X, \mathbf{a}_Y) = \frac{||\mathbf{a}_X^H \mathbf{a}_Y||^2}{K^2} &\rightarrow 0 \\ I(X;\underline{Q}) &\rightarrow I(X;\underline{Q}|Y) \\ I(Y;\underline{Q}) &\rightarrow I(Y;\underline{Q}|X) \end{aligned}$$

This suggests that, by the increasing of number of receivers, the capacity of independent decoding approaches the maximum capacity of joint decoding. Since the implementation of independent decoding is much less expensive than that of joint decoding, the SDMA scheme offers a computationally effective way to increase the channel capacity.

6 Conclusion

We have established in this paper the outer and inner boundaries of the capacity region for a MAC with multiple receivers. We showed that the increase of the channel capacity by exploiting the spatial diversity varies depending on the relative locations of different usres. We further proved that by increasing the number of receivers, the performance of the simple independent decoding is close to that of the optimal, but much costly joint decoding.

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