On Scale-Limited Extrapolation

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Abstract

We propose a scale-limited signal model based on wavelet representation and study the reconstructability of scale-limited signals via extrapolation in this research. In analogy with the band-limited case, we define a scale-limited time-concentrated operator, and examine various vector spaces associated with such an operator. It is proved that the scale-limited signal space can be decomposed into the direct sum of two subspaces and only the component in one subspace can be exactly reconstructed, where the reconstructable subspace can be interpreted as a space consisting of scale/time-limited signals.

1 Introduction

The band-limited signal model has been widely used in the past three decades [9], [10], and band-limited extrapolation has been extensively studied and applied in signal reconstruction [4], [8]. Possible applications include spectrum estimation, synthetic aperture radar (SAR) imaging, limited-angle tomography, beamforming and high resolution image restoration. The performance of an extrapolation algorithm is highly dependent on a proper modeling of the underlying signal. There are however signals which are not band-limited such as time-limited signals. Wavelet theory has recently attracted a lot of attention as a useful tool for signal modeling, and the multiresolution wavelet representation leads naturally to a scale-limited signal model.

The scale-limited model includes the band-limited model as a special case, since by choosing the wavelet basis to be the sinc functions, the scale-limited model is reduced to the band-limited one. To illustrate the additional modeling power of the scale-limited model, we may consider the following two examples. First, the cubic cardinal B-spline wavelet basis [2] spans a function space whose elements are second-order polynomials between knots and with continuous first-order derivate at knots. Many practical signals can be well approximated with such a function space. Second, time-localized wavelet bases such as the Haar and Daubeches wavelets are more suitable than the conventional Fourier basis in modeling signals with interesting transient information such as those arising from the electrocardiogram and radar applications.

There exist two fundamental questions in signal extrapolation, i.e. the reconstructability of a signal via extrapolation, and the sensitivity of the extrapolation process to noise. With respect to the band-limited case, these two questions have been examined thoroughly. The answer to the first question is positive. That is, a band-limited signal can be exactly reconstructed from its any segment when no noise exists. As to the second question, it is well known that the band-limited extrapolation process is an ill-posed problem. By adding a small amount of noise in observed data, the extrapolated solution may change dramatically. To overcome the ill-posedness of the extrapolation process, it is often to introduce a regularization technique.

Theory on band-limited signal modeling and extraplation has been well developed [10]. It can be dated back to the work [9] of Slepian in early 60's. In order to provide a meaningful explanation for band-limited signals, Slepian [9] constructed a complete set of band-limited functions by using the eigenfunctions of a time-concentrated band-limited operator, known as the prolate spheroidal wave functions (PSWFs). Papoulis [8] and Gerchberg [4] developed an iterative algorithm for band-limited signal extrapolation, and proved the convergence of the algorithm by using the PSWFs in the 70's. Many interesting problems can also be conveniently solved based on PSWFs. They include band-limited extrapolation for noisy data [14] and with unevenly sampled observations [1]. The discrete prolate spheroidal sequence (DPSS) has also been studied by researchers [5], [11].

In this research we attempt to answer the first fundamental question in the scale-limited context. In

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analogy with the band-limited case, we define a scale-limited time-concentrated operator, and examine various vector spaces associated with such an operator. It is proved that the scale-limited signal space can be decomposed into the direct sum of two subspaces and only the component in one subspace can be exactly reconstructed. We show that the reconstructable subspace can be well interpreted as a vector space consisting of scale/time-limited signals with a significant amount of energy in the observed interval. In contrast with the band-limited case where every band-limited signal can be exactly reconstructed via extrapolation regardless of the length and position of the observation interval, this result appears to be more consistent with our intuition. It is important to point out that the reconstructability of band-limited signals is fundamentally linked to the fact that every band-limited function is analytic. However, this assumption is too strong to hold in many practical situations. We feel that the scale-limited model provides not only a more general and but also more natural choice than the conventional band-limited one, and should receive more attention.

2 Scale-limited signals

The scale-limited signal model is based on multiresolution analysis and wavelet theory. Consider a sequence of successive approximation space $\mathcal{P}_j$ of $L^2(\mathbf{R})$ satisfying,

$$\cdots \subset \mathcal{P}_{-2} \subset \mathcal{P}_{-1} \subset \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \cdots,$$

with

$$\overline{\bigcup_j \mathcal{P}_j} = L^2(\mathbf{R}), \quad \bigcap_j \mathcal{P}_j = \{0\}.$$

Let $\phi(t)$ be the associated scaling function so that $\{\phi_{jk}(t)\}_{k\in\mathbf{Z}}$, where $\phi_{jk}(t) = 2^{j/2}\phi(2^j t - k)$, is an orthonormal basis of the wavelet subspace $\mathcal{P}_j$. The mother wavelet function corresponding to $\phi(t)$ is denoted by $\psi(t)$. Then, $\{\psi_{jk}(t) = 2^{j/2}\psi(2^j t - k),\ j,k \in \mathbf{Z}\}$ forms an orthonormal basis in $L^2(\mathbf{R})$. For any $f(t) \in L^2(\mathbf{R})$, we have

$$f(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t). \tag{1}$$

The projection $f_J(t)$ of $f(t)$ in $\mathcal{P}_J$ can be written as

$$f_J(t) = \sum_{k=-\infty}^{\infty} c_{J,k}\phi_{Jk}(t) = \sum_{j<J} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t). \tag{2}$$

We call $f_J(t)$ a scale-limited signal, since its wavelet coefficients are zero for $j \geq J$. The wavelet coefficients $b_{j,k}$, $j < J$, can be computed from coefficients $c_{J,k}$ by a fast recursive formula, and vice versa [6]. In practice, the recursion has to stop at some finite $J_0 < J$ so that (2) is modified to

$$f_{J_0,J}(t) = \sum_{k=-\infty}^{\infty} c_{J_0,k}\phi_{J_0k}(t) + \sum_{j=J_0}^{J-1} \sum_{k=-\infty}^{\infty} b_{j,k}\psi_{jk}(t), \tag{3}$$

known as the finite-scale wavelet transform.

3 Scale-limited time-concentrated operator and its properties

In analogy with the band limited time concentrated operator, we define the scale limited time concentrated operator H as an integral operator which maps $f(t) \in L^2(\mathbf{R})$ to $g(t) \in L^2[-T,T]$ via

$$\sum_{k=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(s)\phi_{Jk}(s)\, ds \right) \phi_{Jk}(t) = g(t), \tag{4}$$

for $t \in [-T,T]$. In words, this operator projects the function $f(t)$ into the wavelet subspace $\mathcal{P}_J$ and then truncates the projected function in the time domain. The signal extrapolation problem with a scale-limited model $\mathcal{P}_J$ can be formulated as the solution of $Hf = g$ for the projected $f_J(t)$ of $f(t)$ in $\mathcal{P}_J$ based on the observation $g(t)$. By reconstructability, we mean that $f_J(t)$ can be solved uniquely for any $g(t) \in L^2[-T,T]$. The operator H is clearly linear and bounded. To get more insight into this problem, it is important to examine various vector spaces associated with H and its adjoint H^*. It is easy to derive that HH^* defines an integral operator from $L^2[-T,T]$ to itself, i.e.

$$HH^*g(t) = \int_{-T}^{T} g(s)Q_J(s,t)ds,\ t \in [-T,T], \tag{5}$$

where

$$Q_J(s,t) \triangleq \sum_{k=-\infty}^{\infty} \phi_{Jk}(s)\phi_{Jk}(t), \quad (s,t) \in \mathbf{R}^2, \tag{6}$$

is the reproducing kernel for the reproducing kernel Hilbert space $\mathcal{P}_J$ [12], [13].

In what follows, we assume that $Q_J(s,t)$ is continuous in $[-T,T]^2$ and finite in $\mathbf{R}^2$. The eigenfunctions of HH^* can be examined by using the adjoint operator theory. For a real scaling function $\phi_{Jm}(t)$, the operator HH^* is self-adjoint and positive semidefinite so that all eigenvalues λ_k of HH^* are real and non-negative. We can arrange them in a descending order in terms of magnitude,

$$\infty > |\lambda_0| \geq |\lambda_1| \geq \cdots \geq 0,$$

and use $r_0(t)$, $r_1(t)$, $\cdots$ to denote their corresponding eigenfuctions, i.e.

$$HH^* r_k(t) = \lambda_k r_k(t), \qquad t \in [-T, T]. \qquad (7)$$

The compactness of HH^* can be proved in the following lemma.

Lemma 1 *The operator HH^* is compact.*

Proof: Let us define the kernel

$$Q_{J,M} = \sum_{|m| \leq M} \phi_{Jm}(s)\phi_{Jm}(t).$$

We know from wavelet theory that the scaling function is well concentrated in the time domain so that

$$||Q_J - Q_{J,M}|| = \sum_{|m|>M} \int_{-T}^{T}\int_{-T}^{T} \phi_{Jm}(s)\phi_{Jm}(t) dt\, ds \to 0,$$

as $M \to \infty$. Let $H_M H_M^*$ denote the integral operator with kernel $Q_{J,M}$. Then, we have

$$||HH^* - H_M H_M^*|| \leq ||Q_J - Q_{J,M}|| \to 0, \qquad \text{as } M \to \infty.$$

Since the rank of $H_M H_M^*$ is $(2M+1)^2$, it is compact. With the result in [7, pages 384, theorem 5.24.8], we conclude that HH^* is compact. □

By using the above lemma and spectrum theory [7], we claim that the set of functions $\{r_k(t)\}_{k\geq 0}$ is complete and forms an orthogonal basis in $L^2[-T, T]$.

Now, let us focus on the set of eigenfunctions with nonzero eigenvalues, i.e. $r_k(t)$ with $k \in \mathbf{K}$, where

$$\mathbf{K} = \{ \; 0, 1, \cdots K : \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_K > 0 \text{ and } \lambda_{K+1} = \lambda_{K+2} = \cdots = 0 \}.$$

By using (5) and (6), we can extend these eigenfunctions from $[-T, T]$ to $\mathbf{R}$ to define a new set of functions

$$\begin{aligned} \hat{r}_k(t) &= \frac{1}{\lambda_k} \int_{-T}^{T} r_k(s) Q_J(s,t) ds \\ &= \frac{1}{\lambda_k} \sum_n \left(\int_{-T}^{T} r_k(s)\phi_{Jn}(s) ds \right) \phi_{Jn}(t), \end{aligned} \qquad (8)$$

for $t \in \mathbf{R}$. Some properties of $\hat{r}_k \in L^2(\mathbf{R})$ were derived in [13] and summarized in Lemma 2.

Lemma 2 *The eigenfuction functions $\hat{r}_k(t)$, $k \in \mathbf{K}$, satisfy the following three properties.*

1. *For $k \in \mathbf{K}$, $\hat{r}_k(t) \in \mathcal{P}_J$, i.e. they are scale-limited.*
2. *The functions $\hat{r}_k(t)$, $k \in \mathbf{K}$, are orthonormal in $(-\infty, \infty)$ and orthogonal in $[-T, T]$, that is,*

$$\int_{-\infty}^{\infty} \hat{r}_k(t)\hat{r}_l(t) dt = \delta_{kl},$$

and

$$\int_{-T}^{T} \hat{r}_k(t)\hat{r}_l(t) dt = \lambda_k \delta_{kl}.$$

3. *For any $f(t) \in \mathcal{P}_J$ and $k \in \mathbf{K}$,*

$$\int_{-T}^{T} f(s) r_k(s) ds = \lambda_k \int_{-\infty}^{\infty} f(s)\hat{r}_k(s) ds.$$

4. *The corresponding eigenvalues λ_k with $k \in \mathbf{K}$ are real and $0 < \lambda_k \leq 1$.*

4 Main Result

Let us denote the space generated by the orthonormal basis $\hat{r}_k$, $k \in \mathbf{K}$, in $L^2(\mathbf{R})$ by

$$\mathcal{U}_J \triangleq \{ \text{ closed linear span of } \{\hat{r}_k\}_{k \in \mathbf{K}} \text{ in } L^2(\mathbf{R}) \}. \qquad (9)$$

It is clear from Property 1 of Lemma 2 that $\mathcal{U}_J$ is a linear subspace in $\mathcal{P}_J$. Also, by Properties 2 and 3, the eigenvalue λ_k can be interpreted as the energy contribution of $\hat{r}_k(t)$ in the time interval $[-T, T]$. However, unlike the band-limited case, the eigenfunctions $\hat{r}_k(t)$ are in general not complete in the wavelet subspace $\mathcal{P}_J$ so that $\mathcal{U}_J \neq \mathcal{P}_J$. This will be proved in the following main theorem. For convenience, we use $\mathcal{P}_T$ to denote a space consisting of all functions $f(t) \in L^2(\mathbf{R})$ with $f(t) = 0$ for $t \notin [-T, T]$. The orthogonal complements of $\mathcal{P}_J$ and $\mathcal{P}_T$ are $\mathcal{P}_J^\perp$ and $\mathcal{P}_T^\perp$. Clearly, $\mathcal{P}_T^\perp$ contains all functions $f(t) \in L^2(\mathbf{R})$ with $f(t) = 0$ for $t \in [-T, T]$.

Theorem 1 *For the $\mathcal{U}_J$ defined in (9), we have $\mathcal{P}_J = \mathcal{U}_J \oplus \mathcal{U}_J^\perp$ where $\mathcal{U}_J^\perp$ is the orthogonal complement of $\mathcal{U}_J$ in $\mathcal{P}_J$ and*

$$\mathcal{U}_J^\perp = \mathcal{P}_J \cap \mathcal{P}_T^\perp.$$

Proof: Given $J > 0$, any $f(t) \in \mathcal{P}_J$ can be decomposed as $f(t) = f_1(t) + f_2(t)$, where $f_1(t)$ is the projection of $f(t)$ onto $\mathcal{U}_J$, and $f_2(t)$ is the projection of $f(t)$ onto $\mathcal{U}_J^\perp$. Therefore, we have

$$f_1(t) = \sum_{n \in K} a_n \hat{r}_n(t), \qquad (10)$$

where

$$a_n = \langle f, \hat{r}_n \rangle = \int_{-\infty}^{\infty} f(t)\hat{r}_n(t)dt.$$

We want to prove that $f_2(t) \in \mathcal{P}_J$ and $f_2(t) \in \mathcal{P}_T^{\perp}$. First, since $f(t) \in \mathcal{P}_J$, it follows that $f_2(t) \in \mathcal{P}_J$. Next, we will show that $f_2(t) = 0$ for $t \in [-T,T]$. This is equivalent to proving

$$f(t) = f_1(t) = \sum_{n\in K} a_n \hat{r}_n(t), \qquad t \in [-T,T]. \qquad (11)$$

Recall that $\{r_k(t)\}$ with all $k \geq 0$ forms an orthogonal basis of $L^2[-T,T]$. To prove (11), we need to show that

$$\langle f, r_k \rangle_T = \langle \sum_{n\in\mathbf{K}} a_n \hat{r}_n, r_k \rangle_T \qquad \text{for } k \geq 0, \qquad (12)$$

where the notation $\langle a,b \rangle_T = \int_{-T}^{T} a(t)b(t)dt$ is used. We first consider the case $k \in \mathbf{K}$. To verify the equality in (12), we have $\langle r_n, r_k \rangle_T = \lambda_k \delta_{nk}$ from Property 2 in Lemma 2. By using Property 3 in Lemma 2 and (10), it is easy to see that (12) holds for $k \in \mathbf{K}$. For the case $k \notin \mathbf{K}$, we know

$$\langle \sum_{n\in\mathbf{K}} a_n \hat{r}_n, r_k \rangle_T = \langle \sum_{n\in\mathbf{K}} a_n r_n, r_k \rangle_T = 0, \qquad \text{for } k \notin \mathbf{K},$$

since the functions $\{r_k(t)\}$ with $k \geq 0$ form an orthogonal basis in $[-T,T]$. Furthermore, we have

$$\begin{aligned}
\langle f(t), r_k(t) \rangle_T &= \int_{-T}^{T} f(t) r_k(t) dt \\
&\overset{(1)}{=} \int_{-T}^{T} \left(\int_{-\infty}^{\infty} f(s) Q_J(s,t) ds \right) r_k(t) dt \\
&\overset{(2)}{=} \int_{-\infty}^{\infty} f(s) \left(\int_{-T}^{T} r_k(t) Q_J(s,t) dt \right) ds \\
&= 0.
\end{aligned}$$

In the above derivation, equality (1) is based on the fact $f(t) \in \mathcal{P}_J$ and equality (2) is due to that any $r_k(t)$ with $k \notin \mathbf{K}$ is in the null space of the integral operator HH^* defined by (5). □

By using the orthogonal projection, it follows that the coefficients a_n for $n \in \mathbf{K}$ in (10) can be written as

$$a_n = \langle f, \hat{r}_n \rangle = \frac{1}{\lambda_n} \langle f, r_n \rangle_T = \frac{1}{\lambda_n} \int_{-T}^{T} f(t) r_n(t) dt.$$

The coefficients a_n, $n \in \mathbf{K}$, can be completely determined with the knowledge of the segment of $f(t)$ over the interval $[-T,T]$. It means that $f_1(t)$ can be uniquely determined from $f(t)$ on $[-T,T]$. In the band-limited case, since $f_2(t) = 0$ for $t \in [-T,T]$ as proved above, we have $f_2(t) = 0$ for $t \in R$ by using the analytic property of the function $f_2(t)$. This is however not true for general wavelet bases. As discussed above, a scale-limited signal $f(t) \in \mathcal{P}_J$ can be written as $f(t) = f_1(t) + f_2(t)$, where $f_1(t)$ and $f_2(t)$ are the projections of $f(t)$ onto $\mathcal{U}_J$ and $\mathcal{U}_J^{\perp}$, respectively. The component $f_1(t)$ can be uniquely determined from the values of $f(t)$ in $[-T,T]$ while the component $f_2(t)$ cannot since $f(t)$ contains no information of $f_2(t)$ in $[-T,T]$.

In the context of signal extrapolation, the value of a signal $f(t)$ is observed in $[-T,T]$. We first assume that the function $f(t)$ has a certain finest resolution (or scale) J. It is clear to see that, for a certain scale $J > 0$, we can determined only a finite portion of the signal $f(t)$. Now we assume that the observation of $f(t)$ inside the interval $[-T,T]$ can uniquely determine the value of $f(t)$ up to $[-\Pi,\Pi]$ in the time domain. Hence, we have the following signal model for the extrapolation problem. We represent $f(t)$ with the wavelet basis ϕ_{jk} in (2). Usually, the function $\phi(t)$ is well localized in time with a center around 0 (time) and distributed in frequency interval $[-\xi_0,\xi_0]$ with a center 0 (frequency). By using the scaling property, $\phi_{jk}(t)$ is localized around $2^{-J}k$ in time and distributed in frequency interval $[-2^J\xi_0, 2^J\xi_0]$ with a center 0 (frequency). Then, we may interpret the wavelet coefficient $c_{J,k} =< f, \phi_{J,k} >$ as the "information content" of f near $2^{-J}k$ in time and $[-2^J\xi_0, 2^J\xi_0]$ in frequency. This concept is illustrated in Fig. 1(a) and (b), where the dot (j,k) in (a) denotes the time and scale indices of the wavelet coefficient $c_{J,k}$ while the dots in (b) denote the influence of the wavelet coefficient $c_{J,k}$ in the time and frequency domains. Now, suppose that the energy of $f(t)$ is well concentrated in rectangle regions as shown in Fig. 1(b), i.e.

$$\mathcal{D} = [-\Pi,\Pi] \times [(-2^J\xi_0, 2^J\xi_0)]. \qquad (13)$$

Then, from [3], only the wavelet coefficients $c_{J,k}$, for $k \in \mathcal{K}$, where

$$\mathcal{K} = \{k : 2^{-J}|k| \leq \Pi + t_\epsilon\} \qquad (14)$$

are needed for a good approximation of $f(t)$ whose energy is concentrated in $\mathcal{D}$ given by (13). As in the Fig 1(b), $\mathcal{K}$ contains the index inside the dashed rectangle. Thus, the space

$$\mathcal{P}_{J,\mathcal{K}} = \{f(t) : \; f(t) = \sum_{k\in\mathcal{K}} c_{J,k}\phi_{J,k}(t)\} \qquad (15)$$

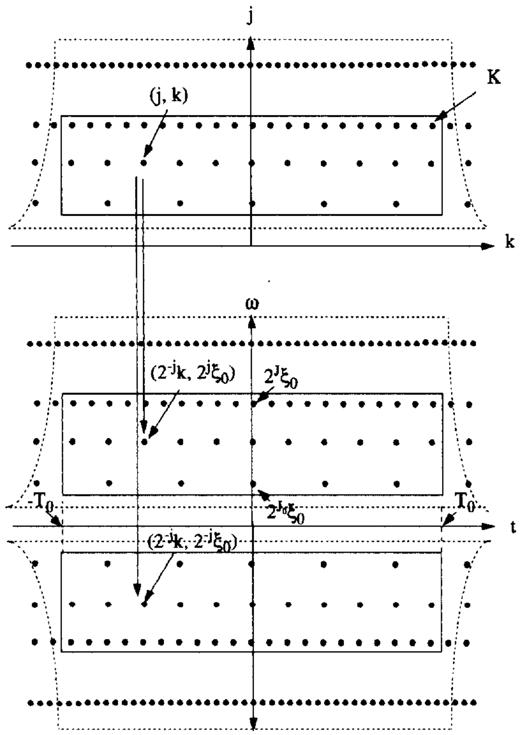

Figure 1: Scale/time-limited signal model: (a) the signal space in the wavelet domain and (b) the corresponding signal space in the time-frequency domain.

provides a good signal model. As a consequence, the wavelet coefficients $c_{J,k}$, $k \in \mathcal{K}$, can be uniquely determined from the value of $f(t)$ in $[-T, T]$. Therefore, we have $\mathcal{P}_{J,\mathcal{K}} = \mathcal{U}_J$.

Given T, the Π value will depend on the regularity of the signal and the scale parameters J. The mathematical relationship between these parameters is still open. However, by considering the compact support wavelet, we can simply estimate the Π value from the result in Theorem 1. Let $f_J(t_1)$ be the projection of $f(t_1)$ onto $\mathcal{P}_J$, which is the convolution of $f(t_1)$ with $\phi(2^J t - t_1)$. It is obvious that if the projection $f_J(t_1)$ does not intersect with the observation interval $[-T, T]$, it cannot be determined from the observations in $[-T, T]$. Hence, the extrapolation interval $[-\Pi, \Pi]$ can be determined by shifting the scaling function $\phi_J(t)$ along the time axis.

5 Conclusion

Instead of using the traditional Fourier-based technique, the scale-limited signal model based on the wavelet representation is investigated for signal extrapolaton. We proved properties of scale-limited extrapolaton by examining the eigenfunctions of the scale-limited time-concentrated operator. Based on the results, we can prove the convergence of the generalized PG extrapolation algorithm and develop a regularization solution for noisy data extrapolation. This will be detailed in a separate roport under preparation.

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