An Accurate Multirate Sampling Technique and Its Application to Time-Scale Analysis

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Abstract

An accurate multirate sampling technique is presented, which constructs a sequence $x(ns\Delta t)$ from a sampled signal $x(n\Delta t)$ for any positive s, where $x(n\Delta t)$ is assumed to be sampled with the sampling theorem being satisfied. It is proven that both $x(n\Delta t)$ and $x(ns\Delta t)$ can be characterized by a sequence $\{c_m\}$ which can be calculated from the DFT of $x(n\Delta t)$. The theory behind the accurate sampling rate conversion is then used to develop a parallel and recursive structure for time-scale analysis. It is indicated that the new approach not only accurately realizes the time-scale analysis, but also possesses a parallel and recursive structure appropriate for efficient hardware implementation.

I. Introduction

Multirate sampling is to construct a sampled sequence $x(ns\Delta t)$ of a continuous time signal $x(t)$ from another sampled sequence $x(n\Delta t)$, where $x(t)$ is assumed to be band-limited, and $x(n\Delta t)$ was sampled with the sampling theorem being satisfied. Multirate sampling is a crucial step in speech analysis, bandwidth compression, radar and sonar signal processing [2],[5]. In time-scale analysis, for example, the basic problem involved is to compute

$$y(s,n) = \sqrt{|s|}\sum_k w[(k-n)s\Delta t]r(k\Delta t), \qquad (1)$$

where n stands for time and s for scale. In radar signal processing, $w(-t)$ is the transmitted signal, and $r(t)$ is the received signal reflected from a moving object. The time n in $y(s,n)$ contains the distance information of the object, and the scale s contains the velocity information of the moving object.

In problem (1), $w(n\Delta t)$ is supposed to be known, $w(ns\Delta t)$ needs to be constructed from $w(n\Delta t)$ for a series of s before Equation (1) can be computed , where the s is called sampling rate conversion ratio. The available method[2],[5] to change the sampling rate for $s = \frac{N}{M}$ is first to increase the sampling rate by filling $M-1$ zeros between every samples of $w(n\Delta t)$ and passing the output through a low pass filter, then pick up every N samples from the output of the low pass filter to produce the $w(ns\Delta t)$. Because of the non-ideal low pass filter, the $w(ns\Delta t)$ can not be accurately constructed, especially at the header and tail of $w(ns\Delta t)$.

This paper presents a novel method that accurately constructs the $w(ns\Delta t)$ from $w(n\Delta t)$ for any positive s under the assumption that $w(t)$ is band-limited and $w(n\Delta t)$ was sampled with the sampling theorem being satisfied. It is indicated that $w(n\Delta t)$ and $w(ns\Delta t)$ can be characterized by a sequence $\{c_m\}$ which can be calculated from the DFT of $w(n\Delta t)$. The theory behind the accurate sampling rate conversion is then used to develop a parallel and recursive structure to implement Equation (1) for any $s = \frac{N}{M}$ with N and M being integers. It is indicated that this structure is suitable for real-time processing and appropriate for efficient hardware implementation.

The organization of this paper is as follows. The theory of the accurate sampling rate conversion is given in Section II. A parallel and recursive structure to implement Equation (1) is described in Section III. Finally some conclusions are given in Section IV.

II. The Theory of The Accurate Sampling Rate Conversion

B. Both $x(n\Delta t)$ and $x(ns\Delta t)$ can be characterized by a sequence $\{c_m\}$

Suppose $x(t)$ is a continuous function of a continuous variable t, and finite in duration, then this $x(t)$ can be represented by

$$x(t) = \begin{cases} x(t), & 0 \le t < T, \\ 0, & otherwise \end{cases} \qquad (2)$$

By Fourier series decomposition, $x(t)$ can be represented by

$$x(t) = \sum_{m=-\infty}^{\infty} c_m e^{jm\frac{2\pi}{T}t}, \quad 0 \le t < T \qquad (3)$$

where

$$c_m = \frac{1}{T}\int_0^T x(t)e^{-jm\frac{2\pi}{T}t}dt. \quad (4)$$

Suppose $x'(t) = x(st)$ for any positive s, then $x'(t)$ is also a continuous function of the continuous variable t, and spreads over $0 \le t < \frac{T}{s}$. When $s > 1$, $x'(t)$ is a compressed version of $x(t)$, and a dilated version of $x(t)$ when $0 < s < 1$. From Equation (3), $x'(t)$ can be represented by

$$x'(t) = \sum_{m=-\infty}^{\infty} c_m e^{jm\frac{2\pi}{T}st}, \quad 0 \le t < \frac{T}{s}. \quad (5)$$

Now suppose both $x(t)$ and $x'(t)$ are sampled at every Δt, then

$$x(n\Delta t) = \sum_{m=-\infty}^{\infty} c_m e^{jm2\pi\frac{\Delta t}{T}n}, \quad 0 \le n \le \frac{T}{\Delta t} - 1, \quad (6)$$

and

$$x'(n\Delta t) = \sum_{m=-\infty}^{\infty} c_m e^{jm2\pi\frac{s\Delta t}{T}n} \quad (7)$$

$$0 \le n \le [\frac{T}{s\Delta t}] - 1$$

where $\frac{T}{\Delta t}$ is assumed to be an integer, and $[\frac{T}{s\Delta t}]$ denotes the largest integer smaller than $\frac{T}{s\Delta t}$ when $\frac{T}{s\Delta t}$ is not an integer, and equals to $\frac{T}{s\Delta t}$ when $\frac{T}{s\Delta t}$ is an integer.

It is interesting to notice that $x'(n\Delta t)$ can be considered as the result by discretizing the $x(t)$ with a sampling interval $s\Delta t$ because $x'(n\Delta t) = x(ns\Delta t)$ and $x(t)$ is a function of a continuous variable t. Therefore s is called sampling rate conversion ratio. Suppose $\frac{T}{\Delta t} = N$ and $\frac{T}{s\Delta t} = M$, then the sampling rate conversion ratio $s = \frac{N}{M}$.

We conclude above discussion that sequences obtained from discretizing a continuous function $x(t)$ of a continuous variable t with different sampling rates can be characterized by or constructed from a same complex sequence $\{c_m\}$ which is independent of the sampling rate. Equations (6) and (7) give the relation between $x(n\Delta t)$ and $x(ns\Delta t)$ for any conversion ratio s. This relation indicates that if c_m can be determined somehow from $x(n\Delta t)$ then $x(ns\Delta t)$ can be accurately constructed from $x(n\Delta t)$. The next section will discuss how to determine the $\{c_m\}$ from $x(n\Delta t)$.

B. $\{c_m\}$ can be obtained from the DFT of $x(n\Delta t)$

Suppose $X(f)$ is the Fourier transform of $x(t)$, then

$$X(f) = \int_0^T x(t)e^{-j2\pi ft}dt. \quad (8)$$

Comparing Equation (8) and (4) yields

$$c_m = \frac{1}{T}X(\frac{m}{T}). \quad (9)$$

Suppose $x(t)$ is band-limited with high cutoff frequency f_c, that is

$$X(f) = \begin{cases} X(f), & |f| < f_c \\ 0, & otherwise \end{cases} \quad (10)$$

then $c_m = 0$ for those m's such that $|\frac{m}{T}| > f_c$, or $|m| > Tf_c$. Let Ω be the set of m such that $|m| \le Tf_c$. It is obvious that Ω is independent of sampling rate Δt or $s\Delta t$.

Suppose $\bar{X}(f)$ is the Fourier transform of $x(n\Delta t)$, and $\frac{T}{\Delta t} = N$ with N being an integer, then

$$\bar{X}(f) = \sum_{n=0}^{N-1} x(n\Delta t)e^{-j2\pi fn\Delta t}. \quad (11)$$

It is well known that $\bar{X}(f)$ is related to $X(f)$ by

$$\bar{X}(f) = \frac{1}{\Delta t}\sum_{k=\infty}^{\infty} X(f - \frac{k}{\Delta t}), \quad (12)$$

and if $\Delta t \le \frac{1}{2f_c}$, the $X(f)$ can be recovered completely from the first period of $\bar{X}(f)$ by

$$X(f) = \Delta t\bar{X}(f), \quad |f| < f_c \le \frac{1}{2\Delta t} \quad (13)$$

Therefore from Equations (9) and (11), the following is resulted

$$c_m = \frac{\Delta t}{T}\bar{X}(\frac{m}{T})$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n\Delta t)e^{-j\frac{2\pi}{N}mn} \quad (14)$$

where $\sum_{n=0}^{N-1} x(n\Delta t)e^{-j\frac{2\pi}{N}mn}$ is the DFT $X_D(m)$ of the sequence $x(n\Delta t)$. Therefore

$$c_m = \frac{1}{N}X_D(m), \quad m \in \Omega \quad (15)$$

Equation (15) indicates that the c_m in Equation (6) and (7) can be obtained from the DFT of the sequence $x(n\Delta t)$. $x(ns\Delta t)$ can thereby be accurately constructed from $x(n\Delta t)$ for any positive s if $x(n\Delta t)$ was sampled with sampling theorem being satisfied.

Next section will develop a parallel and recursive structure to implement Equation (1) for time-scale analysis by using the results obtained in this section.

III. A Recursive and Parallel Structure for Time-Scale Analysis

Now let us return to the basic problem in time-scale analysis which is to implement Equation (1). Suppose $w(t)$ is a band-limited function of a continuous variable t defined on [0,T), a sampled signal $w(n\Delta t)$ is available where Δt satisfies the sampling theorem, and $\frac{T}{\Delta t} = N$ with N being an integer, then from the discussion in Section II, the following is obtained

$$w(ns\Delta t) = \frac{1}{N} \sum_{m \in \Omega} W_m e^{j2\pi \frac{s\Delta t}{T} nm} \qquad (16)$$

$$n = 0, 1, ..., [\frac{T}{s\Delta t}] - 1$$

where W_m can be computed from $w(n\Delta t)$ by

$$W_m = \sum_{n=0}^{N-1} w(n\Delta t) e^{-j\frac{2\pi}{N}nm}, \quad m \in \Omega \qquad (17)$$

and Ω is the set of m such that $m < \frac{N}{2}$ and $W_m \neq 0$.

The following discussion assumes $\frac{T}{s\Delta t} = M$ with M being an integer, or the sample rate conversion ratio $s_M = \frac{N}{M}$, then

$$w(ns_M\Delta t) = \frac{1}{N} \sum_{m \in \Omega} W_m e^{j\frac{2\pi}{M}nm}. \qquad (18)$$

$$n = 0, 1, ..., M - 1.$$

In Equation (18), let

$$u_m(s_M, n) = W_m e^{j\frac{2\pi}{M}nm} \qquad (19)$$

where $n = 0, 1, ..., M - 1$, and $m \in \Omega$.

then the Z transform $U_m(s_M, Z)$ of $u_m(s_M, n)$ can be proven[1] to be

$$U_m(s_M, Z) = W_m \frac{1 - Z^M}{1 - e^{j\frac{2\pi}{M}m}Z}, \quad m \in \Omega. \qquad (20)$$

Let $R(Z)$ and $Y(s_M, Z)$ be the Z transforms of $r(n\Delta t)$ and $y(s_M, n)$ in Equation (1) respectively, then

$$Y(s_M, Z) = \frac{1}{\sqrt{NM}} \sum_{m \in \Omega} U_m(s_M, Z) R(Z) \qquad (21)$$

Let

$$Y_m(s_M, Z) = U_m(s_M, Z) R(Z), \quad m \in \Omega \qquad (22)$$

then from Equation (20)

$$y_m(s_M, n) = r_m(n) + y_m(s_M, n-1) e^{j\frac{2\pi}{M}m} \qquad (23)$$

where $y_m(s_M, n)$ is the time-domain representation of $Y_m(s_M, Z)$, and

$$r_m(n) = W_m\{r(n\Delta t) - r[(n - M)\Delta t]\} \qquad (24)$$

Finally, from Equation (21) the time-scale distribution $y(s_M, n)$ will be

$$y(s_M, n) = \frac{1}{\sqrt{NM}} \sum_{m \in \Omega} y_m(s_M, n) \qquad (25)$$

Equations (23)-(25) represent a parallel and recursive structure to implement Equation (1) for $s_M = \frac{N}{M}$ with M being any integer. It is indicated that the computation required to implement Equation (1) for $s = \frac{N}{M}$ with different M are identical, although the convolutional kernel $w(ns\Delta t)$ increases its length M ($M > N$, dilation) or decreases its length M ($M < N$, compression). The computation is determined by the number of the independent harmonic components in $w(n\Delta t)$. To obtain scales $s_m = \frac{N}{M}$ with different N, all we need to do is to increase the length N of $w(n\Delta t)$ by appending certain number of zeros at the tail of $w(n\Delta t)$. Therefore we can realize Equation (1) for any $s = \frac{N}{M}$ with N and M being integers.

If $w(t)$ is a real-valued signal, the complex-value calculation in Equations (23)-(25) can be avoided by using the conjugate symmetry of W_m. This will reduce the computational complexity. If $w(t)$ is symmetric, then $W_m = W_{-m}$, and more computational complexity can be reduced. It is also important to point out that the computational structures for $s_M = \frac{N}{M}$ with different M are identical, therefore the algorithm developed here for time-scale analysis is proper for efficient hardware implementation.

Summary and Discussion

The first part of this paper presents a method for sampling rate conversion, which constructs a sampled sequence of a band-limited signal $x(t)$ from a sampled sequence of the same signal at any different sampling rate. By analyzing Equation (6)-(7) and (15), it is interesting to notice that if both $\frac{T}{\Delta t}$ and $\frac{T}{s\Delta t}$ are integers, then the DFT of $x(n\Delta t)$ and $x(ns\Delta t)$ can be easily obtained from each other by inserting or kicking off some zeros at high frequencies except for a scaling factor. It is known[3] that appending zeros at the tail of a time sequence before taking its DFT is equivalent to increasing the frequency-domain sampling rate of its DFT. This paper reveals a similar process that inserting zeros into the DFT of a time sequence at the center (where DFT is defined only on positive frequencies) is equivalent to increasing the time-domain sampling rate.

The second part of this paper presents a parallel and recursive structure to implement the time-scale analysis (1). For a fixed scale s_M, this structure, in fact, represents an efficient time-domain realization of a frequency-domain filter. This structure will be useful in multirate filter banks because of the frequency domain relationship of $w(n\Delta t)$ and $w(ns\Delta t)$ discussed in the first part. This relationship suggests that the convolution of $w(ns\Delta t)$ with $r(n\Delta t)$ can be equally efficiently realized by the structure for any $s = \frac{N}{M}$ with N and M being integers.

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