Constructing MRAs from Desired Wavelet Functions

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Abstract

This paper develops a technique for constructing an orthonormal wavelet that is optimized to a desired signal in the least squares sense, and whose associated scaling function generates an orthonormal multiresolution analysis (MRA). The key development in this paper is a recursive equation for finding the scaling function from a given wavelet, whose closed form solution gives constraints on the wavelet that guarantee an orthonormal scaling function and multiresolution analysis. The matching algorithm uses Lagrangian multipliers to minimize the mean square error between the desired and optimum wavelet power spectra.

1 Introduction

Many of the design techniques for orthonormal wavelets begin by designing a scaling function, $\phi(x)$, or a generating sequence, h_k in such a way that they satisfy the conditions for orthonormality. The wavelet is derived from $\phi(x)$ or h_k by way of the following 2-scale relations

$$\phi(x) = 2\sum_{k=-\infty}^{\infty} h_k \phi(2x-k) \qquad (1)$$

$$\psi(x) = 2\sum_{k=-\infty}^{\infty} (-1)^k h_{1-k} \phi(2x-k) \qquad (2)$$

In a wavelet decomposition, a signal is represented by the sum of a series of detail functions at different scales or resolution.

$$s(x) = \sum_{j=-\infty}^{\infty} D_j(x) \qquad (3)$$

where the detail functions, $D_j(x)$, are found by projecting $s(x)$ onto the wavelet subspace, W_j.

$$D_j(x) = \sum_{k=-\infty}^{\infty} d_k^j 2^{j/2} \psi(2^j x - k) \qquad (4)$$

$$d_k^j = \langle s(x), \psi_{j,k} \rangle \qquad (5)$$

where $\langle \cdot, \cdot \rangle$ is the L^2 inner product and $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$ is the orthonormal basis of W_j. For signals, $s(x)$, with significant high frequency content, like speech signals, or image texture, it would be worthwhile to have an orthonormal wavelet that matches the high frequency band pass of a signal. Current design techniques do not lend themselves to matching the wavelet itself to a signal, only its scaling function. This paper develops a technique for designing symmetric wavelets in such a way as to minimize the norm of the error between the power spectrum of the wavelet and that of the desired signal and to guarantee that the associated scaling function will generate an orthonormal multiresolution analysis. In Section 2 of this paper we give a quick review of multiresolution decompositions. In Section 3 we derive a recursive equation for deriving $|\Phi(\omega)|$ from $|\Psi(\omega)|$. In Section 4 we develop the constraints on $|\Psi(\omega)|$ that guarantee orthonormality. Section 5 contains the constrained optimization algorithm for matching wavelets to a desired signal spectrum and Section 6 shows two examples.

2 Multiresolution Decomposition

Mallat [1] showed that the discrete wavelet transform can be used to generate an orthonormal multiresolution decomposition of a discrete signal consisting of a series of detail functions and a residual low resolution approximation of the original signal. The decomposition is done by convolving the original sequence with a pair of quadrature mirror filters, h (low pass) and g (high pass). In order to perfectly reconstruct

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the original signal from the detail functions and the residual approximation, the following must be true of the Fourier spectrum magnitudes of h and g [1, 3, 4].

$$|H(\omega)|^2 + |G(\omega)|^2 = 1 \tag{6}$$

Cancellation of any aliasing is guaranteed by setting $g_k = (-1)^k h_{1-k}$ [4]. The filters, h and g, are related to the mother wavelet, $\psi(x)$, and the scaling function, $\phi(x)$, by their 2-scale relations [1, 2],

$$\psi(x) = 2\sum_k g_k \phi(2x-k) \tag{7}$$

$$\phi(x) = 2\sum_k h_k \phi(2x-k) \tag{8}$$

given in the frequency domain by

$$\Psi(\omega) = G(\frac{\omega}{2})\Phi(\frac{\omega}{2}) \qquad \Phi(\omega) = H(\frac{\omega}{2})\Phi(\frac{\omega}{2}) \tag{9}$$

3 Finding ϕ from ψ

A recursive equation for finding $|\Phi(\omega)|$ from $|\Psi(\omega)|$ can be found using (6) and (9).

$$\begin{aligned} |\Phi(2\omega)|^2 + |\Psi(2\omega)|^2 &= |H(\omega)\Phi(\omega)|^2 + \\ &\qquad |G(\omega)\Phi(\omega)|^2 \\ &= \left[|H(\omega)|^2 + |G(\omega)|^2\right] |\Phi(\omega)|^2 \\ &= |\Phi(\omega)|^2 \end{aligned} \tag{10}$$

Substituting for $\omega = \pi n, \quad n \in \mathbb{Z}$ gives

$$|\Phi(\pi n)|^2 = |\Phi(2\pi n)|^2 + |\Psi(2\pi n)|^2 \tag{11}$$

Since we are seeking to construct an orthonormal multiresolution analysis, $\phi(x)$ must be orthonormal, and it's Poisson summation must be equal to 1 everywhere.

$$\sum_{m=-\infty}^{\infty} |\Phi(\omega + 2\pi m)|^2 = 1 \tag{12}$$

If $\Phi(\omega)$ is normalized such that $\Phi(0) = 1$, then from the Poisson summation

$$|\Phi(2\pi n)| = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} \tag{13}$$

and Equation (11) can be rewritten as

$$|\Phi(\pi n)| = \begin{cases} 1 & \text{for } n = 0 \\ |\Psi(2\pi n)| & \text{for } n \neq 0 \end{cases} \tag{14}$$

Therefore, at integer multiples of π, Φ can be computed directly from values of Ψ. Substituting for $\omega = \pi n/2$ in (10) gives

$$\left|\Phi\left(\frac{\pi n}{2}\right)\right|^2 = |\Phi(\pi n)|^2 + |\Psi(\pi n)|^2 \qquad \text{for } n \neq 0 \tag{15}$$

At integer multiples of $\pi/2$, Φ can be computed from values of Ψ and the previously calculated values of Φ. Repeated substitution leads to the following closed form solution.

$$\left|\Phi\left(\frac{\pi n}{2^\ell}\right)\right|^2 = \sum_{p=0}^{\ell} \left|\Psi\left(\frac{2\pi n}{2^p}\right)\right|^2 \qquad \text{for } n \neq 0 \tag{16}$$

4 Guaranteeing Orthonormality

A multiresolution analysis is orthonormal if the following conditions on $\phi(x)$ and $\psi(x)$ are true [2]:

$$\langle \phi_{j,k}, \phi_{j,m} \rangle = \delta_{k,m} \tag{17}$$

$$\langle \phi_{j,k}, \psi_{j,m} \rangle = 0 \tag{18}$$

$$\langle \psi_{j,k}, \psi_{\ell,m} \rangle = \delta_{j\ell} \cdot \delta_{k,m} \tag{19}$$

If $\Phi(0) = 1$ and $g_k = (-1)^k h_{1-k}$, as assumed previously, then it can be shown that if condition (17) is true, conditions (18) and (19) must also be true [6]. Therefore, an orthonormal multiresolution analysis is guaranteed when the scaling function is orthonormal, thereby satisfying (12). Let $\Delta\omega = \pi/2^\ell$ in (16). Then,

$$|\Phi(n\Delta\omega)|^2 = \sum_{p=0}^{\ell} \left|\Psi\left(\frac{2^{\ell+1} n\Delta\omega}{2^p}\right)\right|^2 \quad \text{for } n \neq 0 \tag{20}$$

Setting $n\Delta\omega = n\Delta\omega + 2\pi m$ and summing over m gives

$$\begin{aligned} &\sum_{m=-\infty}^{\infty} |\Phi(n\Delta\omega + 2\pi m)|^2 \\ &= \sum_{m=-\infty}^{\infty} \sum_{p=0}^{\ell} \left|\Psi\left(\frac{2^{\ell+1}}{2^p}(n\Delta\omega + 2\pi m)\right)\right|^2 \end{aligned} \tag{21}$$

The left side of (21) is the Poisson summation sampled at $\Delta\omega$ and must be equal to 1 everywhere if $\phi(x)$ is orthonormal. Therefore, substituting back in for $\Delta\omega$ gives a necessary condition on Ψ that will guarantee an orthonormal multiresolution analysis.

$$\sum_{m=-\infty}^{\infty} \sum_{p=0}^{\ell} \left|\Psi\left(\frac{2\pi}{2^p}\left(n + 2^{\ell+1} m\right)\right)\right|^2 = 1 \tag{22}$$

A wavelet whose spectrum satisfies the condition in (22) will by (21) guarantee that the Poisson summation for $\Phi(\omega)$ is equal to 1 everywhere. Therefore, (22) is necessary and sufficient to guarantee that $\phi(x)$ generates an orthonormal multiresolution analysis.

5 Matching Wavelets

Finding the matched wavelet is done numerically with discrete Ψ. We will assume that the resultant wavelet is real and therefore has a symmetric frequency spectrum. Assume the scaling function derived from the wavelet in (16) has a minimum sample spacing of $\Delta\omega_{min} = \pi/2^M$. The minimum sample spacing required of Ψ is $2\Delta\omega_{min} = \pi/2^{M-1}$. Now let's assume that $\Psi(\omega)$ is bandlimited to $\pi K_L < |\omega| < \pi K_U$, where $K_L, K_U \in \Re$, then the argument of (22) is limited to

$$\pi K_L < \left|\frac{2\pi}{2^p}(n + 2^{\ell+1}m)\right| < \pi K_U \quad (23)$$

Let $Y(k) = |\Psi(2k\Delta\omega_{min})|^2, \quad k \in \mathbb{Z}$, then condition (22) and (23) become

$$\sum_{m=-\infty}^{\infty}\sum_{p=0}^{\ell} Y\left(\frac{2^M}{2^p}(n + 2^{\ell+1}m)\right) = 1 \quad (24)$$

$$2^{M-1}K_L < \left|\frac{2^M}{2^p}(n + 2^{\ell+1}m)\right| < 2^{M-1}K_U \quad (25)$$

$$0 \le \ell \le M$$

Assuming that $|\Psi(\omega)| = |\Psi(-\omega)|$, the conditions in (24) generate a set of L linear equality constraints in $Y(k)$ of the form

$$\sum_{i=1}^{L}\alpha_{ik}Y(k) = 1 \quad (26)$$

where $k = \lceil 2^{M-1}K_L\rceil, \ldots, \lfloor 2^{M-1}K_U\rfloor$ since n and m are integers and we are matching only one side of a symmetric spectrum. Let the desired signal spectrum, sampled at $2\Delta\omega_{min}$, be given as $S(k)$ and let $W(k) = |S(k)|^2$ be its power spectrum. Then the objective function, E, to be minimized is defined as the mean square error between Y and W, normalized by the energy in W.

$$E = \frac{\sum_k (W(k) - Y(k))^2}{\sum_k W(k)^2} \quad (27)$$

Equations (27) and (26) can be written in vector notation as

$$E = \frac{(W - Y)^T(W - Y)}{W^T W} \quad (28)$$

$$AY = \underline{1} \quad (29)$$

where $A = \{\alpha_{ik}\}$ and $\underline{1}$ is a vector of 1's with length L. It is important to note that A is a function of K_L and K_U only. Once the bandlimits are set, and A is derived, it can be used for matching wavelets to any desired signal within the same bandlimits! The objective function is chosen to be the mean square error between the power spectra of Ψ and S, so that the minimization problem is linear and has a closed form solution using Lagrangian multipliers. The Lagrangian is given as

$$L = \frac{(W - Y)^T(W - Y)}{W^T W} + \lambda(AY - \underline{1}) \quad (30)$$

and the objective function is minimized by setting $\nabla L = 0$ and solving for Y [5].

$$\nabla L = \frac{2}{W^T W}(W - Y) + A^T\lambda = 0 \quad (31)$$

$$AY = \underline{1} \quad (32)$$

which gives

$$Y = A^T(AA^T)^{-1}(\underline{1} - AW) + W \quad (33)$$

Since $Y(k) = |\Psi(k\Delta\omega)|^2$, then we must also include the inequality constraints, $Y(k) \ge 0$. If the solution in (33) has a negative value, then it can be set to 0 with an additional equality constraint in A.

From the error, E, derived from (33),

$$E = \frac{(\underline{1} - AW)^T(AA^T)^{-1}(\underline{1} - AW)}{W^T W} \quad (34)$$

we see that the error in the match is a function of the deviation of AW from $\underline{1}$. If the desired signal already satisfies the constraints for an orthonormal MRA, then the deviation from $\underline{1}$ is 0, $E = 0$, and from (33), $Y = W$. As W moves away from the constraints, the error in the match, E, increases. It can also be seen from both (33) and (34) that any scale factor on W would affect both the solution, Y, and the error in the match, E. Let the input spectrum be normalized by a constant, a. The solution in (33) and the error in (34) become

$$Y(a) = A^T(AA^T)^{-1}(\underline{1} - \frac{1}{a}AW) + \frac{1}{a}W \quad (35)$$

$$E(a) = \frac{(\underline{1} - \frac{1}{a}AW)^T(AA^T)^{-1}(\underline{1} - \frac{1}{a}AW)}{\frac{1}{a^2}W^T W} \quad (36)$$

Setting $dE(a)/da = 0$ and solving for a gives the value of the normalizing factor on W that will produce the minimum error, E.

$$a = \frac{\underline{1}^T(AA^T)^{-1}AW}{\underline{1}^T(AA^T)^{-1}\underline{1}} \quad (37)$$

6 Examples

6.1 Known Wavelet

In the first example, the desired signal is Meyer's wavelet defined in the frequency domain as

$$|\Psi_m(\omega)| = \begin{cases} 0 & |\gamma| \leq \frac{2\pi}{3} \text{ or } |\gamma| \geq \frac{8\pi}{3} \\ sin\frac{\pi}{2}v(\frac{3|\gamma|}{2\pi} - 1) & \frac{2\pi}{3} \leq |\gamma| \leq \frac{4\pi}{3} \\ cos\frac{\pi}{2}v(\frac{3|\gamma|}{4\pi} - 1) & \frac{4\pi}{3} \leq |\gamma| \leq \frac{8\pi}{3} \end{cases} \tag{38}$$

where

$$v(\gamma) = \begin{cases} 0 & \gamma \leq 0 \\ \gamma & 0 \leq \gamma \leq 1 \\ 1 & \gamma \geq 1 \end{cases} \tag{39}$$

The bandlimits, $\{K_L, K_U\}$, in (23) for Meyer's wavelet are $\{2/3, 8/3\}$. Let $\Delta\omega_{min} = \pi/2^4$, then $Y(k)$ in (26) is non-zero for $k = 6, \ldots, 21$. The constraint matrix, $A = \{\alpha_{ik}\}$, derived from (24) is given in Figure 1.

$$A = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

Figure 1: A for $\{K_L, K_U\} = \{2/3, 8/3\}$

Calculate $W(k)$ using the following expression:

$$W(k) = |\Psi_m(k\Delta\omega_{min})|^2 \tag{40}$$

The matched wavelet spectrum is found from (35) where the normalizing factor, a, in this case is 1 since AW in (37) is $\underline{1}$. Since the desired spectrum is known to be orthonormal, the solution should match the desired spectrum exactly, which it does (Figure 2). The scaling function (Figure 3) is found using the closed form equation (16).

6.2 Arbitrary Signal

The next example will use the same bandlimits as in the example above, so the constraint matrix, A, remains unchanged. The desired signal spectrum is a truncated gaussian and is shown in Figure 4 along with its Poisson summation. It is clear from the Poisson summation that the desired signal is not orthonormal. $W(k)$ is again calculated as the squared magnitude of the desired spectrum and $Y(k)$ is found using Equation (35) where $a = 1.2924$. The optimization procedure operates on the non-zero portion of the positive frequency axis, $k = 6, \ldots, 21$. The results are shown in Figure 5. The matched wavelet spectrum is found by taking the square root of each element in $Y(k)$ and reflecting it onto the negative frequency axis. It is shown along with its Poisson summation in Figure 6, and is clearly orthonormal. The scaling function, found from (16), is shown in Figure 7 along with its Poisson summation. It too is orthonormal, as expected. The time or space domain wavelets and scaling function are shown in Figures 8 and 9.

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[6] A. H. Tewfik, D. Sinha, P. Jorgensen, "On the Optimal Choice of a Wavelet for Signal Representation," *IEEE Transactions on Information Theory*, v. 38, no. 2, March 1992.

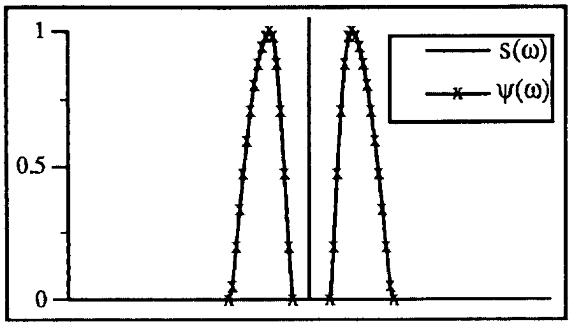

Figure 2: Desired and Matched Wavelet Spectra

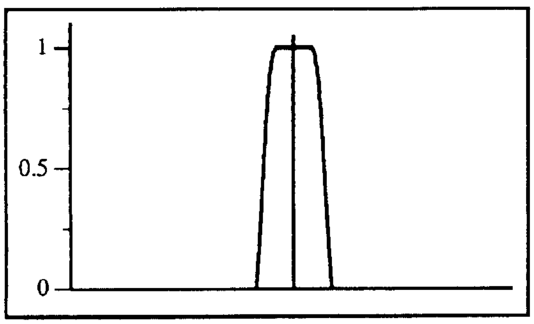

Figure 3: Meyer's Scaling Function, $\Phi(\omega)$

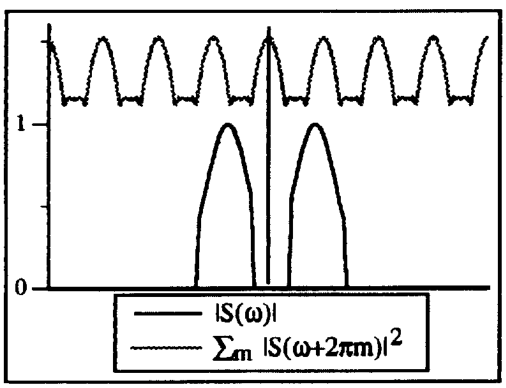

Figure 4: Desired Signal Spectrum

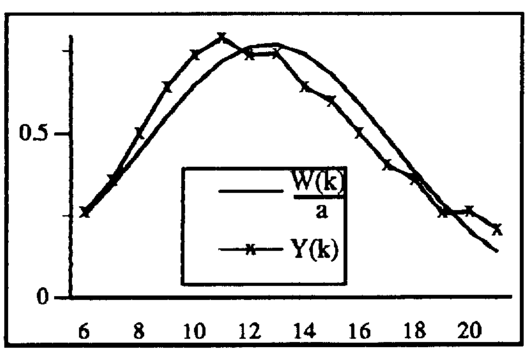

Figure 5: Optimization Results-Arbitrary Signal

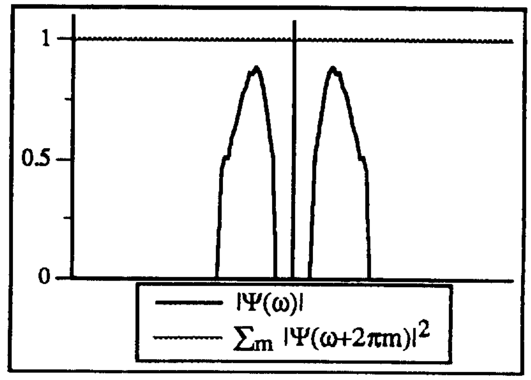

Figure 6: Matched Wavelet Spectrum

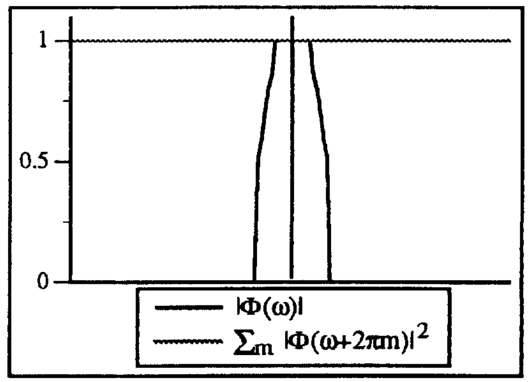

Figure 7: Resulting Scaling Function Spectrum

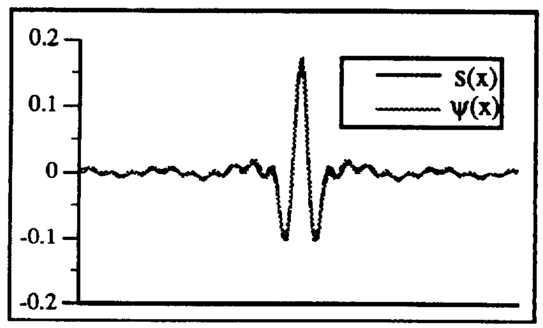

Figure 8: Desired and Matched Wavelets

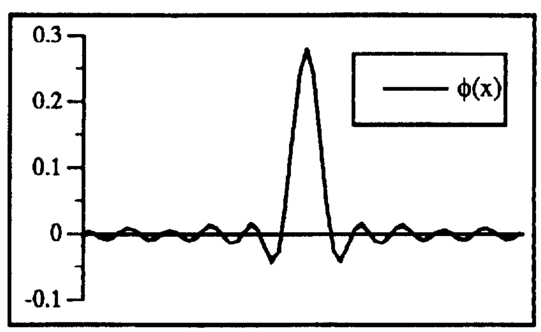

Figure 9: Resulting Scaling Function