BIORTHOGONAL-LIKE SEQUENCES AND GENERALIZED GABOR EXPANSIONS OF DISCRETE-TIME SIGNALS IN $l^2(Z)$

Youhong Lu & Joel M. Morris
Electrical Engineering Department, University of Maryland Baltimore County
5401 Wilken Ave., Catonsville, MD 21228–5398, USA

Abstract

In this paper, a biorthogonal-like sequence (BLS) theory and its application to the generalized Gabor expansion (equivalently, the generalized short-time Fourier transform/filterbank summation) are presented. A pair of BLSs are defined to be two sequences satisfying a biorthogonal-like condition (BLC). We show that two collections in a Hilbert space generated by a pair of BLSs in the joint time-frequency domain are complete, and either can be used as an analysis filter, and the other as a synthesis filter, for a generalized Gabor expansion of discrete-time signals. An efficient algorithm for computation of BLSs is addressed in detail, and examples are presented to illustrate our results.

I. Introduction

Given a discrete-time sequence $g(k)$, with $k \in Z$ the set of integers, and $g \in l^2(Z)$ the space of magnitude square-summable complex sequences, we construct a collection of joint time-frequency sequences $\{g_{mn}\}$ defined as

$$g_{mn}(k) = e^{i\frac{2\pi mk}{M}} g(k - nN) \qquad (1.1)$$

where M and N are positive integers with $M \geq N$, and m and n are integers such that $0 \leq m \leq M-1$ and $n \in Z$. From the STFT point of view, the collection $\{g_{mn}\}$ is obtained by translating and modulating a window $g(k)$, and from the filterbank point of view, the collection $\{g_{mn}\}$ corresponds to a set of uniform bandpass filters. Our framework will be based on the completeness of the collection $\{g_{mn}\}$ in $l^2(Z)$.

For any discrete-time signal $f(k) \in l^2(Z)$, we expect to represent the signal in terms of the collection $\{g_{mn}\}$, namely,

$$f(k) = \sum_m \sum_n C_{mn} g_{mn}(k) \qquad (1.2)$$

for some scalars C_{mn}. Equation (1.2) is called the Gabor expansion, $g(k)$ is called a synthesis filter, and $\{g_{mn}\}$ is a collection of synthesis sequences. The scalars C_{mn} are called the Gabor coefficients and are usually obtained via a collection of sequences generated by translating and modulating another single sequence $\gamma(k)$ such that

$$C_{mn} = \langle f, \gamma_{mn} \rangle = \sum_k f(k)\gamma^*_{mn}(k) \qquad (1.3)$$

where the sign $\langle \cdot, \cdot \rangle$ denotes the inner product and is defined as above. $\gamma(k)$ is called an analysis filter and $\{\gamma_{mn}\}$ is a collection of analysis sequences. We therefore have

$$f(k) = \sum_m \sum_n \langle f, \gamma_{mn} \rangle g_{mn}(k) \qquad (1.4)$$

It is not difficult to show that

$$f(k) = \sum_m \sum_n \langle f, g_{mn} \rangle \gamma_{mn}(k) \qquad (1.5)$$

Equations (1.4) and (1.5) are called the generalized Gabor expansion is an extention of the Gabor expansion proposed by Gabor ([1]). The $g(k)$ and $\gamma(k)$ are a pair of analysis and synthesis filters and defined as a pair of biorthogonal-like sequences (BLSs) (definition will be given in the next section).

There are several theories about the relationship between $\{g_{mn}\}$ and $\{\gamma_{mn}\}$. If $\{g_{mn}\}$ is a frame, then $\{\gamma_{mn}\}$ is the dual frame, and stability of the Gabor expansion is guaranteed. The computation of dual frames for discrete-time signals was presented in [2] that is based on the Gabor expansion for continuous-time signals ([3], [4]). We know that frame and dual frame pairs are in the subclass of the class of all analysis and synthesis pairs. In other words, $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ are not necessarily a frame and dual frame pair.

Knowing $\{g_{mn}\}$, we can also obtain the dual sequence $\{\gamma_{mn}\}$ via the " biorthogonal condition", that is, via a set of linear equations, if all sequences are defined to be periodic ([5], [6]). The definition of the biorthogonal function in [6] is not traditional, although it led to a method for the Gabor expansion and signal representation. The key idea is to imitate the discrete windowed Fourier transform since all sequences were

ONR GRANT # N00014-89-J1210

1058-6393/95 $4.00

assumed to be periodically extendible with the same period.

II. Biorthogonal-Like Sequences and generalized Gabor Expansion

Let $g(k) \in l^2(Z)$ be a sequence, and let M and N be two positive integers, then we rewrite the sequence defined in Eq.(1.1) with $M \geq N$ such that

$$g_{mn}(k) = e^{i\frac{2\pi mk}{M}} g(k - nN) \qquad (2.1)$$

where $0 \leq m \leq M-1$ and $n \in Z$. The integer M is called the frequency modulation parameter and the integer N is the time translation parameter. The collection $\{g_{mn}\}$ are joint time-frequency versions of $g(k)$ and is a basis for the Gabor expansion if the integers M and N satisfy certain conditions. We know that if $\{g_{mn}\}$ is complete, it is necessary that $\frac{N}{M} \leq 1$.

Suppose $g(k)$ is a low-pass filter, then it is clear that $\{g_{mn}\}$ is a bandpass filter for fixed m and n. Hence, the analysis of $\{g_{mn}\}$ can be done in terms of a filterbank structure. The filterbank structure was addressed in detail in [7]. In this paper, on the one hand, we study the properties of $\{g_{mn}\}$, such as completeness, by means of nonharmonic analysis. On the other hand, the filter design is not implemented in the frequency domain since the generalized Gabor expansion usually emphasizes the representation of a signal. For example, suppose $g(k)$ is not a lowpass filter, but is an optimal choice for representing some signal, then bandpass filter design techniques, summarized in [28], may not be better for an application of this kind. Moreover, the disjoint filterbank structure is not allowed in the generalized Gabor expansion since some finite energy sequences cannot be represented in terms of such filterbanks ([4], [18]).

Definition: Let $g(k)$ and $\gamma(k) \in l^2(Z)$ be two sequences, and $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ be defined by Eq. (2.1). We say g and γ are a pair of *biorthogonal-like sequences* (BLSs) if

$$< g_{m'n'}, \gamma_{m''n''} > = \frac{M}{N}\delta(m'-m'')\,\delta(n'-n'') \quad (2.2)$$

such that $\frac{M}{N} \leq 1$, *where* u', u'', v', *and* v'' *are integers*, and < , > indicates the inner product in the Hilbert space. Eq. (2.2) is called the *biorthogonal-like condition* (BLC) for discrete-time signals.

Remarks: (*a*) The BLC (Eq.(2.2)) is the traditional biorthogonal condition when M = N; and in the case $M \neq N$, if we scale $\gamma(k)$ by the factor N/M, then it is still the traditional biorthogonal condition. (*b*) When N < M, the oversampling case, there does not exist any sequences $g(k)$ and $\gamma(k) \in l^2(Z)$ satisfying Eq.(2.2) since both $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ are linearly dependent collections ([3]); when M = N, the critical-sampling case, both $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ satisfying Eq.(2.2) are unique and bases for $l^2(Z)$; and when N > M, the undersampling case, both $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ satisfying Eq.(2.2) are non unique and incomplete.

Remark (*b*) shows that in order for Eq.(2.2) to be well-defined, it is necessary to require that $\frac{M}{N} \leq 1$ since there may exist sequences $g(k)$ and $\gamma(k)$ satisfying Eq.(2.2) only when $\frac{M}{N} \leq 1$. Hence, we focus our attention on the case $\frac{M}{N} \leq 1$ when we study BLSs satisfying Eq.(2.2). It is well-known that in order for a signal to be represented in terms of $\{g_{mn}\}$ and $\{\gamma_{mn}\}$, it is necessary to require that $\frac{M}{N} \geq 1$. Hence, we focus our attention on the case $\frac{M}{N} \geq 1$ when we study signal representation. In this paper, we do not study $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ with $\frac{M}{N} \geq 1$ directly, but indirectly via the BLC (Eq.(2.2)) and $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ $\frac{M}{N} \leq 1$, i. e., we study BLSs.

The main point in the BLS theory is that a signal in $l^2(Z)$ can be represented in terms of $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ with $\frac{M}{N} \geq 1$ if and only if $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ with $\frac{M}{N} \leq 1$ are biorthogonal, i.e., satisfy Eq.(2.2). For example, if $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ are biorthogonal when $\frac{M}{N} = 1/2$, then $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ with $\frac{M}{N} = 2$ form a pair of analysis and synthesis sequences for the Gabor expansion. $R = \frac{M}{N}$ is defined to be the oversampling rate.

To be consistent, we modify the BLC. Switch M and N in Eq.(2.1), then a new, but equivalent, collection is created by

$$\gamma^{(uv)}(k) = e^{i\frac{2\pi vk}{N}} g(k - uM) \qquad (2.3)$$

where u and v are integers. Hence, the BLC Eq.(2.2) can be rewritten as

$$< g^{(u'v')}, \gamma^{(u''v'')} > = \frac{N}{M}\delta(u'-u'')\,\delta(v'-v'') \qquad (2.4)$$

where u', v', u'', and v'' are integers. Eq.(2.4) is equivalent to Eq.(2.2), but $\frac{N}{M} \leq 1$ is now our case of interest.

Theorem 1: Let sequences $\gamma(k)$ and $g(k)$ belong to $l^2(Z)$ and satisfy the BLC, Eq. (2.4), with $\frac{N}{M} \leq 1$, the pair $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ with $\frac{N}{M} \leq 1$ defined in Eq. (2.1), and let $f(k) \in l^2(Z)$ be an arbitrary sequence to be analyzed, then

$$(1)\ f(k) = \sum_m \sum_n < f, g_{mn} > \gamma_{mn}(k) \qquad (2.5)$$

$$(2)\ f(k) = \sum_m \sum_n < f, \gamma_{mn} > g_{mn}(k) \qquad (2.6)$$

The converse is also true: if Eqs.(2.5) and (2.6) hold, then sequences γ and g must be a pair of BLSs satisfying Eq.(2.4).

Remarks: Collections $\{g^{(uv)}\}$ and $\{\gamma^{(uv)}\}$ satisfying Eq.(2.4) with $\frac{N}{M} \le 1$ do not imply $\{g^{(uv)}\}$ and $\{\gamma^{(uv)}\}$ is complete although they are biorthogonal. The $\{g^{(uv)}\}$ and $\{\gamma^{(uv)}\}$ with $\frac{N}{M} \le 1$ are actually always incomplete. This theorem, however, shows that the corresponding $\{g_{mn}\}$ and $\{\gamma_{mn}\}$ with $\frac{N}{M} \le 1$ are complete, and representations Eq.(2.5) and (2.6) hold though they are not biorthogonal when $\frac{N}{M} \le 1$.

Theorem 2: Let function $g(k)$ and $\gamma(k)$ be in $l^2(Z)$, and $\{g^{(uv)}\}$ and $\{\gamma^{(uv)}\}$ generated by Eq.(2.3) for discrete-time sequences, then the following conditions are equivalent:

(a) $\{g_{mn}\}$ is a frame;

(b) $< g^{(u'v')}, \gamma^{(u''v'')} > = \frac{N}{M}\delta(u'-u'')\delta(v'-v'')$ (2.7)

(c) $< g, \gamma^{(uv)} > = \frac{N}{M}\delta(u)\,\delta(v)$ (2.8)

(d) $< \gamma, g^{(uv)} > = \frac{N}{M}\delta(u)\,\delta(v)$ (2.9)

(e) $\sum_i g(k-iN)\gamma^*(k-iN-uM) = \delta(u)/M$(2.10)

where u, u', u", i ∈ Z, and v, v', v" ∈ [0, 1, ..., N-1].

Theorem 3: Given $h(k)$ a sequence with length Q, and let $\{h_{mn}\}$ a collection defined by $h(k)$ and Eq. (2.1) with $M \ge Q$. Suppose $H(k) = \sum_n |h(k-nN)|^2 > 0$ for all k, and

$$g(k) = h(k)/G(k) \qquad (2.11)$$

where $G(k) = \sqrt{M \times h(k)}$, then:

(1) The collection of sequences $\{g_{mn}\}$ defined by Eqs.(2.1) and (3.6) is biorthogonal-like with itself, that is:

$$\sum_{k=0}^{Q-1} g(k)g^{(uv)}(k)^* = \frac{N}{M}\delta(u)\delta(v) \qquad (2.12)$$

where $\{g^{(uv)}\}$ is a collection of sequences defined by Eqs. (2.11) and (2.3);

(2) The $\{g_{mn}\}$ constitutes an orthonormal basis for $l^2(Z)$, that is:

$$\sum_k g_{m'n'}(k)g^*_{m''n''}(k) = \delta(m'-m'')\delta(n'-n'') \quad (2.13)$$

if $M = N = Q$; and

(3) If $M > N$, the collection $\{g_{mn}\}$ is linearly dependent.

This theorem shows that the collection $\{g_{mn}\}$ constructed above can never be an orthogonal basis for $l^2(Z)$ since the sequences are always linearly dependent; but the collection is complete in $l^2(Z)$, and any sequence $f(k) \in l^2(Z)$ can be expressed in terms of $\{g_{mn}\}$. If the frequency modulation parameter M is less than Q, which is the length of the window sequence $g(k)$, the above theorem is not valid and we have not found a closed-form solution for $\gamma(k)$. The more general solution will be studied in the next section, i.e., we will present a general algorithm for the computation of the BLSs.

III. Computation of BLSs

From section II, we showed that any signal sequence $f(k) \in l^2(Z)$ can be expressed in terms of two collections of BLSs. Given a collection of sequences, we can compute a collection of BLSs via the BLC (Eq.(2.6)). In order to numerically compute the BLS $\gamma(k)$, suppose the length of a given sequence $g(k)$ is finite, say Q_1, i.e., $g(k)$ is a FIR filter, and the length of the BLS $\gamma(k)$ is a finite Q_2. It is reasonable to seek FIR analysis and synthesis filters since the expansion always has numerical stability, and the sequence to be analyzed is completely determined by its expansion coefficients. Based on this assumption, we will numerically compute a BLS $\gamma(k)$ via Eq.(2.10) that is equivalent to the BLC Eq. (2.4). Notice that we do not set any other constraints beyond those given above. The length of the analyzed sequence is arbitrary and independent of the other parameters.

To simplify the discussion, we rewrite Eq.(2.10):

$$\sum_n g(k-nN)\gamma^*(k-nN-uM) = \frac{1}{M}\delta(u)$$

$$0 \le k \le N-1 \qquad (3.1)$$

Consider the range of u. It is not difficult to see that when u is an integer that does *not* satisfy

$$-\left\lfloor \frac{Q_2-1}{M} \right\rfloor \le u \le \left\lfloor \frac{Q_1-1}{M} \right\rfloor \qquad (3.2)$$

then Eq.(3.1) is automatically satisfied for all choices of $g(k)$ and $\gamma(k)$. Hence, we see that Eq.(3.1) stands for a finite number of linear equations. The important issue we must address first is how to choose the length Q_2 for a given length Q_1 such that the number of variables are greater than or equal to the number of equations. The set of equations in Eq.(3.1) is under-determined.

Theorem 4: Let $k_1 = \left\lfloor \frac{Q_1-1}{M} \right\rfloor$, $J = \lceil k_1 M/(R-1) \rceil$, and $L = \lfloor (k_1 M/(R-1) \rfloor + N$, where $R = M/N$, $\lceil n \rceil$ indicates the smallest integer greater than or equal to n, and $\lfloor n \rfloor$ indicates the largest integer less than or equal to n. Suppose Q_2 is an integer number between J and L such that $Q_2 = k_3 N$ for some integer k_3 satisfying $\frac{k_1 M}{M-N} \le k_3 \le 1 + \frac{k_1 M}{M-N}$, then Q_2 exists and satisfies

$$n1 = \{\left\lfloor \tfrac{Q_1-1}{M} \right\rfloor + \left\lfloor \tfrac{Q_2-1}{M} \right\rfloor + 1\}N = Q_2 \qquad (3.3)$$

Let Q_3 be any integer number such that $Q_3 > L$, then Q_3 exists and satisfies

$$n2 = \{\left\lfloor \tfrac{Q_1-1}{M} \right\rfloor + \left\lfloor \tfrac{Q_3-1}{M} \right\rfloor + 1\}N \le Q_3 \qquad (3.4)$$

Observe Eqs.(3.3) and (3.4), the number on the left side of the equations stands for the number of equations contained in Eq.(3.12), and the number on the right side of the equations stands for the number of variables contained in Eq.(3.1). Hence, this theorem is useful in the sense that we can find not only a length Q_2 for $\gamma(k)$ such that Eq.(3.3) is achieved and the number of equations and unknown variables are equal, but also a length Q_3 for $\gamma(k)$ such that the inequality (3.4) is satisfied and the number of equations is less than the number of variables.

We can now derive matrix algorithms based on Theorem 4 to compute the BLS. Note that to properly use Eq. (3.3) in Theorem 4, we assume the remainder of $(Q_1-1)/M$ is not less than N-1. If this is not satisfied, we can only use Eq. (3.4) in Theorem 4, and from the formulae expressing J and L, we see that if R = 1 (N = M, i.e., critical-sampling), the above Q_2 does not exist. This means that the above method is not applicable in the critical-sampling case, which we will consider separately later. So we assume the sampling rate R = M/N is greater than 1.

Let $g(k)$ be real, then it is clear that Eq.(3.1) can be written in matrix form:

$$\mathbf{H}\gamma = \frac{1}{M}\Delta \qquad (3.5)$$

where $\mathbf{H}$ is a $Q_2 \times Q_2$, or $n2 \times Q_3$, real matrix with Q_2, Q_3, and n1 defined in Theorem 4, γ is a $Q_2 \times 1$ variable matrix, i.e., $\gamma = [\gamma(0), \gamma(1), \gamma(2), \ldots, \gamma(Q_2 - 1)]^T$, and Δ is also a $Q_2 \times 1$ matrix with N ones and $Q_2 - 1$ zeros ($\Delta = [\ 1,\ 1,\ \ldots\ 1,\ 0,\ 0,\ \ \ldots,0]^T$). Next, we separately discuss the two cases: $\mathbf{H}$ is $Q_2 \times Q_2$ and $\mathbf{H}$ is $n2 \times Q_3$. If $\mathbf{H}$ is a $Q_2 \times Q_2$ matrix, then it is a square matrix. Suppose the rank of $\mathbf{H}$ is full or Q_2, then the solution to matrix Eq. (3.5) can be written directly from

$$\gamma = \frac{1}{M}\mathbf{H}^{-1}\Delta \qquad (3.6)$$

since in this case, the inverse of matrix $\mathbf{H}$ exists. The rank of $\mathbf{H}$ is usually full if the $g(k) \neq 0$ for all k. If the rank of $\mathbf{H}$ is not full ($< Q_2$), there are many solutions if the matrix Δ is in the range of $\mathbf{H}$.

If $\mathbf{H}$ is a $n2 \times Q_3$ matrix, suppose the rank of $\mathbf{H}$ is equal to the number of rows in $\mathbf{H}$, n2, then one solution of Eq.(3.5) is

$$\gamma = \frac{1}{M}\mathbf{H}^T(\mathbf{H}\mathbf{H}^T)^{-1}\Delta \qquad (3.7)$$

which corresponds to the minimum energy solution. Note that the solutions of Eqs.(3.6) and (3.7) are real. It shows that the BLS $\gamma(k)$ is real if the given sequence $g(k)$ is real.

The above solutions require that the rank of $\mathbf{H}$ is equal to the number of rows in the equations. If the rank of $\mathbf{H}$, denoted κ, is less than the number of rows, we always can find κ linearly independent equations so that we still can use Eq. (3.7) to compute $\gamma(k)$ if it exists.

Observe Eq.(3.1) again, we see that the set of equations can be partitioned into N disjoint subsets of equations. Equivalently, Eq.(3.5) can be partitioned into N disjoint submatrix equations. Hence, the algorithm is faster by a factor of N. We can compare our algorithm with algorithms in [5] and [6]. All algorithms used a matrix equation in which: (1) there are many zeros in the coefficient matrix, (2) the dimensions of the matrix are larger than the length of the given sequence, and (3) the relationship between the length and existence of filters were not addressed.

The following are examples on how to use Eqs.(3.6) and (3.7) to compute $\gamma(k)$ for the case $g(k) = ae^{-\frac{1}{2\sigma^2}(k-b)^2}$ for $k \in \{0, 1, 2, \ldots, 63\}$: (1). Figure 1 shows BLSs with different sampling parameters. From figure 2, we conclude that the normalized $\gamma(k)$ is more similar to the given $g(k)$ when we increase M and decrease N. This conclusion is similar to our previous work [2] where $g(k)$ and $\gamma(k)$ are a frame and dual frame, respectively.

Using Theorem 4 we can design a pair of analysis and synthesis filters such that the reconstruction is perfect. But we mentioned above that Theorem 4 cannot be used in the critical-sampling case since, in this case, integers J and L do not exist. To solve the critical-sampling case, let

$$G_k(z) = \sum_n g(k+nN)z^{-n}$$
$$\Gamma_k(z) = \sum_n \gamma^*(k+nN)z^{-n} \qquad (3.8)$$

Then in the critical-sampling case, the BLC can be written as

$$G_k(z^{-1})\Gamma_k(z) = \frac{1}{N}; \qquad (3.9)$$

We can see from Eq.(3.9) that the BLS $\gamma(k)$ must be an IIR filter if the $g(k)$ is a FIR filter, and can be obtained easily from Eq.(3.9) so that perfect reconstruction is possible in the critical-sampling case.

To demonstrate the generalized Gabor expansion for discrete-time signals, we used as example a frequency hop signal, i.e., $f(t) = \sin(2\pi\lambda t)$, $0 \le t \le t_1$, $f(t) = \sin(4\pi\lambda t)$, $0 \le t \le t_2$, $f(t) = \sin(6\pi\lambda t)$, $0 \le t \le t_3$, where λ is a constant. Its time-domain behavior and Gabor coefficients magnitude are shown in Fig. 3 with the same analysis and synthesis filters as in the first example. The Gabor coefficients magnitude plot exhibit several parallel lines in the JTF domain parallel to the time axis. The examples demonstrate some advantages of the generalized Gabor expansion when compared to the Fourier transform.

References

[1] D. Gabor, "Theory of Communication", *J. IEE* (London), vol. 93, No. III, Nov. 1946, 429-457.

[2] J. M. Morris and Y. Lu, "Discrete Gabor Expansion of Discrete-Time Signals in $l^2(\cdot)$ via Frame Theory", *Signal Processing*, vol. 40, No. 3, 1994.

[3] I. Daubechies, "The Wavelet Transform, Time-Frequency Localization and Signal Analysis", *IEEE Trans. Inform. Theory*, vol. 36, No. 5, Sept. 1990, pp. 961 - 1005.

[4] C. E. Heil and D. F. Walnut, "Continuous and Discrete Wavelet Transforms", *SIAM Review*, vol. 31, No. 4, December, 1989, pp. 628 - 666.

[5] S. Qian and D. Chen, "Discrete Gabor Transform", *IEEE Trans. SP*, vol. 41, No. 7, July 1993, pp. 2429 -2438.

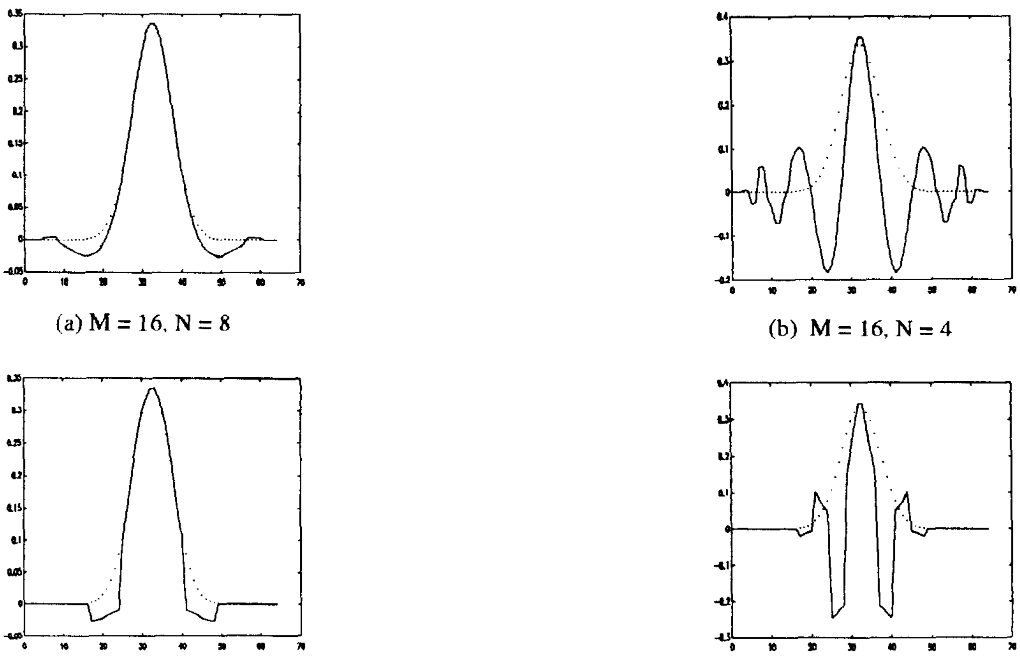
[6] J. Wexler and S. Raz, "Discrete Gabor Expansions", *Signal Processing*, vol. 21, No. 3, Nov. 1990, pp.207-221.

[7] M. R. Portnoff, "Time-Frequency Representation of Digital Signals and Systems Based on Short-Time Fourier Analysis", *IEEE Trans. ASSP*, vol. ASSP-28, No. 1, Feb., 1980, pp. 55-69.

[8] M. Zibulski and Y. Y. Zeevi,"Oversampling in Gabor Scheme", *IEEE Trans. SP.*, vol. 41, No. 8, Aug. 1993, pp. 2679 - 2683.

[9] J. M. Morris and Y. Lu, "On Discrete Gabor Expansion of Discrete-Time Signals via Frame Theory", 27th Conf. Inform. Sci. and Sys., The Johns Hopkins Univ., March, 1993.

[10] R. M. Young, *An Introduction to Nonharmonic Fourier Series*. Academic Press: New York, 1980.



(a) M = 16, N = 8

(b) M = 16, N = 4

(c) M = 8, N = 4

(d) M = 8, N = 2

Fig. 1: Biorthogonal-like function $\gamma(k)$ with length 64 and given $g(k) = ae^{-\frac{1}{2\sigma^2}(k-32.5)^2}$

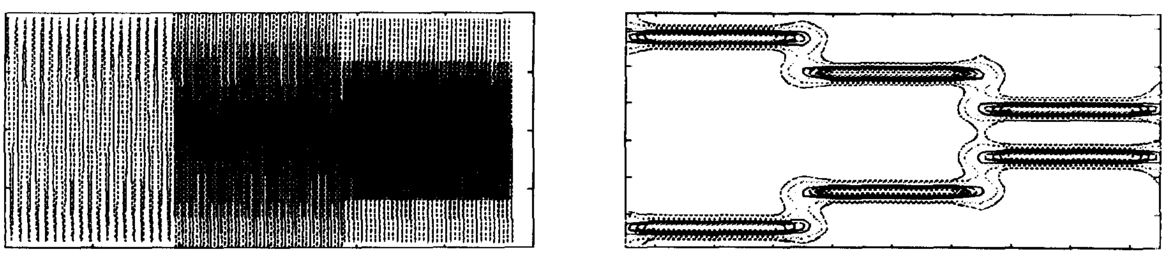


Fig. 2: A frequency hop signal and its Gabor coefficients magnitude.