Generalized Sampling Theorems for Wavelet Subspaces

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Abstract

In this article, a method for creating sampling theorems in wavelet subspaces is given. Several sufficient conditions are provided for the sampling theorems to hold. The notion of continuous multiresolution analysis is introduced and the associated sampling theorems are extended to different (scale) wavelet subspaces.

1. Introduction

The classical Whittaker-Shannon sampling theorem [4] states: if the function f is bandlimited (in the sense of Paley-Wiener) to $[-2\pi W, 2\pi W]$, then

$$f(t)=\sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right)\frac{\sin\pi(2Wt-n)}{\pi(2Wt-n)}$$

Many extensions of this classical sampling theorem have been proposed. The possibility of extension to the wavelet subspaces was first recognized by Gilbert Walter [5].

Walter's result utilized the notion of a Multiresolution analysis ([1] and [3]).

Definition. The sequence $\{V_j\}$ is said to form a **(discrete) multiresolution analysis (MRA)** of $L^2(\mathrm{R})$ if the following five conditions are satisfied.
(1°) V_j are closed and nested i.e. $\cdots\subset V_{-1}\subset V_0\subset V_1\subset\cdots$;
(2°) $\mathrm{Clos}_{L^2}(\bigcup_{j\in Z}V_j)=L^2(\mathrm{R})$;
(3°) $\bigcap V_{j\in Z}=\{0\}$;
(4°) $f(t)\in V_j \Leftrightarrow f(2t)\in V_{j+1},\ j\in Z$;
(5°) There is a function ϕ, such that $\{\phi(t-k)\}$ is a Riesz basis of V_0 with Riesz bounds A and B.

A basis $\{y_n\}$ in a Hilbert space is a **Riesz basis** if it is equivalent to an orthonormal basis, that is, there exists a bounded invertible operator T transforms $\{y_n\}$ to an orthonormal basis $\{x_n\}$.

Here ϕ is called a **scaling function**, and it generates a (discrete) multiresolution analysis (MRA) V_j, $j\in Z$. If, in addition, $A=B=1$, then $\{\phi(t-k)\}$ becomes an orthonormal (o.n.) basis of V_0, and ϕ is called an **orthonormal scaling function.**

Walter's result. If ϕ is a real continuous function such that
(i) $\phi(t)=O(|t|^{-1-\epsilon})$ as $t\to\pm\infty$, where ϵ is some small positive number.
(ii) $\hat{\phi}^*(\omega)=\sum_n\phi(n)e^{-i\omega n}\neq 0$, for all $\omega\in R$.
(iii) ϕ is an o.n. scaling function generating the multiresolution system $\{V_m\}$, $m\in Z$. Then for $f\in V_0$,

$$f(t)=\sum_{n=-\infty}^{+\infty} f(n)S(t-n)$$

where

$$\hat{S}(\omega)=\hat{\phi}(\omega)/\hat{\phi}^*(\omega)$$

with $\hat{\phi}(\omega)$ being the Fourier transform of ϕ, and the convergence is uniform, i.e.

$$\lim_{N\to\infty}\|f(t)-\sum_{n=-N}^{N} f(n)S(t-n)\|_\infty=0$$

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It should be noted that if the multiresolution system $\{V_m\}$ is not chosen carefully, the above result might not hold. Since any function of L^p is actually an equivalence class of functions. If a sampling theorem holds for one representative of the class, it might not hold for another one. For example, if a function f satisfies

$$f(t)=\sum_n f(n)S(t-n)$$

then using the function f_e

$$f_e(x)=\begin{cases} f(x) & x\neq 0 \\ f(0)-1 & x=0 \end{cases}$$

in the same equivalence class as f. It follows that

$$f_e(x)\neq\sum_n f_e(n)S(x-n)$$

$$=\sum_n f(n)S(x-n)-S(x)$$

Hence the sampling theorem does not hold for f_e. Thus an appropriate representative of the equivalence class is needed for the sampling theorem to hold.

2. Sampling theorems in V_0

As mentioned in section 1, if the representative of the equivalence class is not chosen appropriately, the sampling theorem will not hold. Accordingly, conditions are given below which ensure appropriate membership from the equivalence class. Among the most important properties in V_0 is that $\{\phi(t-k)\}$, $k\epsilon Z$ forms a basis of this subspace of $L^2(R)$. Thus for any $f\epsilon V_0$, there exists a unique sequence $\{a_k\}$ in l^2 such that

$$f(t)=\sum_k a_k\phi(t-k) \quad in\ L^2(R).$$

In this section, the function f will always denote some member of V_0 and $\{a_k\}$ denotes the l^2 sequence associated with f by the above equation.

THEOREM 1. If $\{\phi(t-n)\}$ forms a Riesz basis of $V_0 \subset L^2(R)$ and
(i) $\{\phi(n)\}\epsilon l^1$
(ii) $\hat{\phi}^*(\omega)=\sum_n\phi(n)e^{-i\omega n}\neq 0$, for all $\omega\epsilon R$;
(iii) $f\epsilon V_0$ and $f(n)= {}_k a_k\phi(n-k)$
then

$$f(t)=\sum_{n=-\infty}^{+\infty} f(n)S(t-n)$$

in L^2, where $S(t)$ is defined by

$$\hat{S}(\omega)=\hat{\phi}(\omega)/\hat{\phi}^*(\omega)$$

and $\{S(t-n)\}$ also forms a Riesz basis of V_0.

A formal presentation of the above theorem is given below. For a detailed proof, see [2]. Assume that translates of a function ϕ, that is $\{\phi(x-n)\}$ form a basis for V_0. Then for any f in V_0, it is true that

$$f(t)=\sum_n a_n\phi(t-n).$$

The objective is to show that

$$f(t)=\sum_n f(n)S(t-n)$$

where S(t) has Fourier transform

$$\hat{S}(\omega)=\hat{\phi}(\omega)/\hat{\phi}^*(\omega).$$

The Discrete Fourier transform (DFT) $\hat{\phi}^*(w)$ of $\{\phi(n)\}$ is given by

$$\hat{\phi}^*(\omega)=\sum_n\phi(n)e^{-i\omega n}.$$

Instead of showing that

$$f(t)=\sum_n f(n)S(t-n),$$

the Fourier transform of this equation can be shown to hold:

$$\hat{f}(\omega)=\sum_n f(n)e^{-i\omega n}\hat{S}(\omega).$$

The last equation can be written as

$$\hat{f}(\omega)=\hat{f}^*(\omega)\hat{\phi}(\omega)/\hat{\phi}^*(\omega)$$

or

$$\hat{f}(\omega)\hat{\phi}^*(\omega)=\hat{f}^*(\omega)\hat{\phi}(\omega).$$

Since

$$f(t)=\sum_k a_k\phi(t-k),$$

let

$$f(n)=\sum_k a_k\phi(n-k)$$

and taking the Fourier transform of f(t) gives

$$\hat{f}(\omega)=\sum_n a_n e^{-i\omega n}\hat{\phi}(\omega)$$

Substitution of these equations into

$$\hat{f}(\omega)\hat{\phi}^*(\omega)=\hat{f}^*(\omega)\hat{\phi}(\omega)$$

gives

$$\sum_n a_n e^{-i\omega n}\sum_n \phi(n)e^{-i\omega n}\hat{\phi}(\omega)$$

$$=\sum_n\sum_k a_k\phi(n-k)e^{-i\omega n}\hat{\phi}(\omega)$$

or

$$\sum_n a_n e^{-i\omega n}\sum_n \phi(n)e^{-i\omega n}$$

$$=\sum_n\sum_k a_k\phi(n-k)e^{-i\omega n}.$$

The left hand side of the final equation is the product of the DFT of a and ϕ, on the right appears the DFT of the convolution of a and ϕ. The equality follows by interchanging the order of summation on the right side:

$$\sum_n\sum_k a_k\phi(n-k)e^{-i\omega n}$$

$$=\sum_k\sum_n a_k\phi(n-k)e^{-i\omega n}$$

$$=\sum_k a_k e^{-i\omega k}\sum_k \phi(k)e^{-i\omega k}.$$

Many known scaling functions satisfy conditions (i) and (ii) in the above theorem. However, the "proper choice condition" $f(n)=\sum_k a_k\phi(n-k)$, which is needed for the sampling theorem to be valid in the sense of L^2 is usually very hard to check. On the other hand, this condition can be relaxed if the choice of functions in V_0 is restricted, and ϕ and S are defined "properly". The following theorem and it's corollaries show the condition which makes the "proper choice condition" holds and the sampling theorem is valid both in the L^2 sense and L^∞ sense (uniform convergence).

THEOREM 2. If ϕ is a basis of V_0 such that

(i) $\{\phi(n)\}\in l^1$, $\hat{\phi}(\omega)\in L^1$;
(ii) $\hat{\phi}^*(\omega)=\sum_n\phi(n)e^{-i\omega n}\neq 0$, for all $\omega\in R$;
(iii) $f\in V_0$, $\{a_n\}\in l^1$;
Define

$$\hat{S}(\omega)=\hat{\phi}(\omega)/\hat{\phi}^*(\omega)$$

and suppose $\phi(t)$, $S(t)$ and $f(t)$ are defined using the inverse Fourier transform formula

$$g(t)=\mathcal{F}^{-1}(g)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{g}(\omega)\,e^{i\omega t}d\omega$$

Then

$$f(t)=\sum_{n=-\infty}^{+\infty} f(n)\,S(t-n)$$

where the convergence is both in $L^2(R)$ and $L^\infty(R)$.

COROLLARY 1. If ϕ is a real scaling function such that
(i) $\{\phi(n)\}\epsilon l^1$, $\hat{\phi}(\omega)\epsilon L^1$;
(ii) $\hat{\phi}^*(\omega)=\sum_n\phi(n)e^{-i\omega n}\neq 0$, for all $\omega\epsilon R$;
(iii) $f\epsilon V_0$, $\{a_n\}\epsilon l^1$;
Define

$$\hat{S}(\omega)=\hat{\phi}(\omega)/\hat{\phi}^*(\omega)$$

and suppose $\phi(t)$, $S(t)$ and $f(t)$ are all continuous and in $L^1(R)$. Then

$$f(t)=\sum_{n=-\infty}^{+\infty} f(n)\,S(t-n)$$

where the convergence is both in $L^2(R)$ and $L^\infty(R)$.

COROLLARY 2. If ϕ is a continuous o.n. scaling function such that
(i) $\{\phi(n)\}\epsilon l^1$, $\phi(t)=O(t^{-1-\epsilon})$ as $t\to\pm\infty$ and $\hat{\phi}(\omega)\epsilon L^1$;
(ii) $\hat{\phi}^*(\omega)=\sum_n\phi(n)e^{-i\omega n}\neq 0$, for all $\omega\epsilon R$;
Define

$$\hat{S}(\omega)=\hat{\phi}(\omega)/\hat{\phi}^*(\omega)$$

and suppose $S(t)$ and $f(t)$ are all continuous and all in $L^1(R)$. Then for $f\epsilon V_0$,

$$f(t)=\sum_{n=-\infty}^{+\infty} f(n)\,S(t-n)$$

where the convergence is both in $L^2(R)$ and $L^\infty(R)$.

Theorem 2 and its corollaries can be proved similarly as theorem 1.

3. Continuous Multiresolution Analysis and Sampling Theorems in U_α

Using sinc type function

$$\phi(t)=\frac{\sin\pi t}{\pi t}$$

[2] and [4] shows that it is the scaling function with V_0 being functions bandlimited to $[-\pi,\pi]$. Correspondingly, V_n is the subspace of $L^2(R)$ of functions bandlimited to $[-2^n\pi,2^n\pi]$. In order to cover the "scale gap" in the discrete MRA, new spaces will now be "filled in".

First an identification is made between V_n with U_{2^n}. Use the fact that $x\mapsto 2^x$ is a one-to-one and onto mapping between R and $(0,+\infty)$. A continuous scaling scheme, henceforth called continuous Multiresolution analysis can be found by utilizing U_α with α nonnegative.

Definition. A collection of subspaces U_α, $\alpha\epsilon R^+$ (R^+ denotes the set of all positive real numbers) of $L^2(R)$ is said to form a **continuous multiresolution analysis** if
(i) $\{U_{2^n}\}$ forms a discrete MRA
(ii) $f(x)\epsilon U_1 \Leftrightarrow f(\alpha x)\epsilon U_\alpha$, $\alpha\epsilon R$;

The function ϕ which generates discrete MRA $\{U_{2^n}\}$ will be called the scaling function which generates the continuous multiresolution analysis U_α, $\alpha\epsilon R^+$. Clearly, every discrete MRA uniquely determines a continuous MRA.

It is not hard to prove $\{\phi(\alpha t-k)\}$ is a Riesz basis of U_α [2]. For any $f\epsilon V_\alpha$, there exists $\{a_k\}$ such that

$$f(t)=\sum_k a_k\phi(\alpha t-k)\ \ in\ L^2(R).$$

Hence in the space U_α, the "proper choice condition" becomes

$$f(n/\alpha)=\sum_k a_k\phi(n-k) .$$

The following is the extension of theorem 1 for the space U_α.

THEOREM 3. If ϕ is real scaling function such that
(i) $\{\phi(n)\}\in l^1$
(ii) $\hat{\phi}^*(\omega)=\sum_n\phi(n)e^{-i\omega n}\neq 0$, for all $\omega\in R$;
(iii) $f\in U_\alpha$ and $f(n/\alpha)=\sum_k a_k\phi(n-k)$
then

$$f(t)=\sum_{n=-\infty}^{+\infty} f(n/\alpha)S(\alpha t-n)$$

in L^2, where $S(t)$ is defined as the inverse Fourier transform of

$$\hat{S}(\omega)=\hat{\phi}(\omega)/\hat{\phi}^*(\omega) .$$

Extension of theorem 2 and its corollaries can be formulated similarly.

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