ON PERFECT COSINE-MODULATED FILTER BANKS

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Abstract

In this paper, we present a design technique for perfect-reconstruction cosine-modulated filter banks. The design of analysis and synthesis filter banks in this case is much more efficient in terms of computation compared with previous work ([1], [2], [3], [8]). In particular, a design technique for perfect-reconstruction of FIR analysis and synthesis filter banks with arbitrary length is presented.

The existence of synthesis/analysis filter banks for given analysis/synthesis filter banks are addressed via frame theory ([9]) and biorthogonal-like functions (BLFs) ([6], [7]), which have advantages over traditional short-time Fourier transform (STFT) and filter bank summations (FBS) ([7]). A set of Gaussian analysis or synthesis filter banks is possible in our filter bank system.

1. Introduction

Multirate analysis and synthesis filter systems are useful in signal analysis and representation (coding). There are many techniques for this kind of system design, in which the system has the perfect reconstruction (PR) property. For example, the halfband filter based technique, the power complementary based technique, the laped latice based technique, and the paraunitary based technique. The most efficient techniques for implementation of this system, we believe, is the tree-structured filter bank systems.

Cosine-modulated analysis and synthesis filter systems were studied by many researchers because their design is simpler and more realizable than for a general filter bank systems ([1], [3], [4], [8]). This paper also addresses the cosine-modulated filter bank system by using a different approach from the current and traditional one.

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Most past and current design techniques set a fixed relationship between analysis and synthesis filter banks: for example, $g_m(k) = \gamma_m(N-k-1)$, where $g_m(k)$ and $\gamma_m(k)$ are m^{th} band analysis and synthesis filters, respectively, and N is the length of filters, so that the system design problem becomes a problem of the design of a set of analysis or synthesis filters. This design usually required one to solve a non-linear equation, leading to the need for non-linear optimization methods. There are two disadvasntages in non-linear optimization: (1) one usually faces local minima problem, and (2) the convergence and convergence rate are highly related to the initial conditions. Because of these, it may be difficult to achieve the global minimum. In other words, it may be difficult to achieve a desired design error.

In some applications, on the other hand, the performance of a filter bank system may mainly depends on the set of analysis filter banks, and in other applications, the performance of a filter bank system may mainly depends on the set of synthesis filter banks, but not both. For example, in the analysis of a signal, the requirements on the analysis filters are such that the features (information) of the signal should be easily and completely described at the output of the analysis filter banks, and the signal and the output of analysis filter banks should be one-to-one and onto (bijective); and, in the signal representation, the requirements on the synthesis filters are such that they match the localization of a signal.

A set of fixed analysis or synthesis filters may not achieve this goal for most applications. To achieve this goal, one should have freedom in the choice of analysis or synthesis filters. It is not necessary to require filters be good bandpass filters. For example, the Gaussian filter was proved to be good in many applications; we, therefore, should consider a set of analysis filters to be Gaussian.

We see that different applications may have completely different requirements on the set of analysis filters or synthesis filters, and the traditional filter bank design technique, including some biorthogonal filter

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banks, may not meet these requirements. Moreover, it has no reason to set requirements on both analysis and synthesis filter banks. Hence, the performance of the synthesis or analysis filters, such as passband and stopband ripples and cutoff frequency for the desired analysis or synthesis filters may not be important as long as the system has PR, or the reconstruction error achieves a desired value.

In this work, we mainly discuss the design of the set of synthesis filters for a given desired set of analysis filters. It is equivalent to the design of the set of analysis filters for a given desired set of synthesis filters since the filter bank system still holds if we exchange the set of analysis filters with the set of synthesis filters in the system based on biorthogonal-like sequence concept ([6]).

In Section 2, we will derive the PR condition in the time domain for a cosine-modulated filter bank system shown in Figure 1, where M is the number of bands and R is the down and up sampling rate with $R \le M$. This system is the same as in [1]; In Section 3, we present a time-domain design technique based on a biorthogonal-like sequence concept and its associated design existence condition on the given prototype filter. In Section 4, we will give the theoretical background of the design technique in Section 3 via frame theory ([7]). Section 5 provides some examples that illustrate results derived in this paper.

2. Perfect Reconstruction Condition (PRC)

The filter bank system discussed in this paper is shown in Figure 1. Suppose that we are given a baseband prototype filter $g(k)$ of finite energy (or say, $g(k) \in l^2(Z)$, where $l^2(Z)$ is a sequence space such that $\sum_k |f(k)|^2 < \infty$ for any sequence $f(k) \in l^2(Z)$), then we can construct M-band cosine-modulated analysis filter banks of the form

$$g_m(k) = \sqrt{2}\, g(k) \cos\left(\frac{(2m+1)}{2M} k\pi + \theta_m\right), \quad m = 0, 1, 2, \ldots, M-1 \qquad (2.1)$$

where $\theta_m = \frac{\pi}{4}(-1)^m$. This set of analysis filters is the same as the one in [3]. There are some different forms for the analysis filters in [1] and [8], but they do not have essential differences. The $g(k)$ is necessarily a symmetrical filter for cosine-modulated filter banks, i.e., $g(k) = g(N-k-1)$, where N is the length of $g(k)$ and is assumed to be $2LM+1$ with integer $L \ge 1$.

Suppose $x(k)$ is the input to the analysis filter banks, then the outputs of the analysis banks are $x_m(j)$, and can be expressed in terms of an inner product:

$$x_m(j) = \langle x, g_m \rangle = \sum_k x(j-k) g_m(k) \qquad (2.2)$$

In the filter bank structure of the analysis part, the output of the analysis filter banks will be down-sampled by a factor of R with $R \le M$, i.e., the output of analysis part is $a_{mn} = x_m(nR)$.

In the synthesis part of the filter bank structure, the down-sampled signal a_{mn} is to be up-sampled by a factor of R, then passed through M synthesis filter banks of the form

$$\gamma_m(k) = \sqrt{2}\, \gamma(k) \cos\left(\frac{(2m+1)}{2M} k\pi - \theta_m\right), \quad m = 0, 1, \ldots, M-1 \qquad (2.3)$$

where $\gamma(k)$ is a prototype filter for the synthesis filter banks. The output of the synthesis filter banks are

$$\hat{x}_m(k) = \langle a_{mn}, \gamma_m \rangle = \sum_n a_{mn} \gamma_m(k-nR) \qquad (2.4)$$

Adding the M outputs of the synthesis filter banks, one has

$$\hat{x}(k) = \sum_{m=0}^{M-1} \hat{x}_m(k) = \sum_{m=0}^{M-1} \sum_n a_{mn} \gamma_m(k-nR) \qquad (2.5)$$

The above procedure defines a well-known filter banks structure for signal decomposition and reconstruction based on filter bank system theory, and can be illustrated using a very general structure (Figure 1). If the filter banks are PR, then $\hat{x}(k)$ should be equal to $x(k)$ except for some delay. In this case, the analysis and synthesis filter banks should satisfy a perfect-reconstruction condition (PRC); that is, the overall system transfer function should be unity, possibly with some delay ([4], [5]). It is not difficult to show that the filter bank system shown in Figure 1 achieve PR if and only if

$$2\sum_m \sum_n g_m(nR-l)\gamma_m(k-nR) = \delta(k-l-\Delta) \qquad (2.6)$$

where Δ is an integer and indicates a pure delay. In this paper, $\Delta = N$.

Consider the maximal-decimated filter bank system first, i.e., $R = M$, then Eq.(2.6) is equivalent to

$$\sum_{mn} g(k-nM-2mM+2LM)\gamma(k-nM)\delta(k-l-2mM) + \sum_{mn}(-1)^{(m+n)} g(k+nM-2mM-M)\gamma(k-nM)\delta(k+l-2m-M) = \frac{1}{M}\delta(k-l-\Delta) \qquad (2.7)$$

the above equation is satisfied if

$$\sum_n g(k-nM-2mM)\gamma(k-nM) = \frac{1}{M}\delta(m) \qquad (2.8)$$

and

$$\sum_n (-1)^n g(k+nM-2mM-M)\gamma(k-nM) = 0 \qquad (2.9)$$

The Eqs.(2.8) and (2.9) define the PRC for the system shown in Figure 1 with maximal-decimated sampling rate. Suppose analysis and synthesis filter banks are orthogonal, then Eq.(2.9) is automatically satisfied for any choice of filter $g(k)$, and Eq.(2.8) becomes

$$\sum_{n} g(k-nM)g(k-nM-2mM) = \frac{1}{M}\delta(m) \qquad (2.10)$$

which is the important formulation that was used by [3] and [8] in their system design. Since Eq.(2.10) is a non-linear difference equation, one has to use non-linear optimization in the design. Notice that both sides of Eq.(2.10) are periodic functions with period M. Simple manipulations on Eq.(2.10) shows

$$< g,\ g^{(u,v)}> \ = \ \delta(u)\delta(v), \qquad (2.11)$$

$$\text{with} \quad g^{(uv)}(k) \ = \ e^{i\frac{2\pi v k}{M}}\, g(k - 2uM) \qquad (2.12)$$

Eq.(2.11) shows that $\{g^{(u,v)}\}$ is a collection of orthogonal sequences (orthonormal if $g(k)$ has unit energy). It was proved in [7], however, that $\{g^{(u,v)}\}$ is not a basis for $l^2(Z)$.

There are several papers reporting design techniques for the cosine-modulated filter banks in the time domain ([1], [2]); they require one to solve a matrix equation developed for general filter bank system design and, hence, costs too much for the cosine-modulated filter bank system. Moreover, non-linear optimization is still needed in the design. The techniques in [1] and [2] are good in the sense that they are available for general filter bank system design, but since cosine-modulated filter banks are a special case of general filter banks, the algorithm for filter design may be simplified.

For the case R < M, we can still derive a PRC using the similar steps. Since the derivation needs many steps, we do not discuss it here. In the next section, we will present a completely different approach, from the current one, for filter bank design that avoids non-linear optimization, and takes neglectable time in the design since a recursive method is used.

3. A Recursive Design Technique

In this section, we suppose that filter $g(k) = g(N-k-1)$ is a given desired prototype filter for the set of analysis filter banks, i.e., $g(k)$ is fixed. For a maximal-decimated filter bank system, the prototype filter for the synthesis filter banks is usually a IIR filter ([10]) if it exists and may be difficult to design. This section, however, will address a recursive arbitrary length FIR synthesis filter design technique and its associated convergence condition based on biorthogonal-like sequences. An important feature of this technique is that we can quickly design any length synthesis filter for a given range of error, and when the length of the synthesis filter approaches infinity, the design error approaches zero.

Let $g(k)$ and $\gamma(k) \in l^2(Z)$ be two sequences. We say that $g(k)$ and $\gamma(k)$ are a pair of *biorthogonal-like sequences* (BLSs) ([6]) if

$$< \gamma,\ g^{(u,v)}> \ = \ \delta(u)\delta(v), \qquad (3.1)$$

$$\text{with} \quad g^{(uv)}(k) \ = \ e^{i\frac{2\pi v k}{M}}\, g(k - 2uM) \qquad (3.2)$$

where u is a integer and $0 \le v \le M-1$. Equation (3.2) is called the *biorthogonal-like condition* (BLC).

Theorem 1: Equation (3.1) admits a solution $\gamma(k)$ of finite energy if and only if the collection $\{g^{(uv)}\}$ is an independent set; if it has a solution, it has many solutions; and the space spanned by linear combinations of $\{g^{(uv)}\}$ is a strict subspace of the finite energy sequence space.

Theorem 2: The minimun energy solution to Eq.(3.1) is

$$\gamma(k) = \sum_{v=0}^{M} \sum_{u} a_{uv} g^{(uv)}(k) \qquad (3.3)$$

where $a_{uv} = \sum_{i=0}^{v-1} \sum_{j=0}^{u-1} a_{ij} < g^{(ij)}, g^{(uv)} >$. and $a_{00} = 1$. (3.4)

Theorem 1 presents an existence condition on $g(k)$ for a solution to Eq.(3.1), and Theorem 2 supplies an important solution and its associated solution method. These results will be used later.

Consider PRC, Eqs.(2.8) and (2.9) again. Using simple manipulation on the first PRC, we have the equivalent PRC as follows:

$$< \gamma,\ g^{(u,v)}> \ = \ \delta(u)\delta(v), \qquad (3.5)$$

where $u \in Z$, $0 \le v \le M-1$, and $g^{(u,v)}(k)$ is defined in Eq.(3.2). We see that the prototype filters of both analysis and synthesis filter banks should satisfy the BLC.

Theorem 3: The minimum-energy solution obtained via Eqs.(3.3) and (3.5) to Eq.(3.5) is automatically satisfied by the second PRC, Eq.(2.9).

Using Theorems 1, 2, 3, and PRC, we can draw several important conclusions for the cosine-modulated filter bank system:

1. Given a filter $g(k)$, if Eq.(3.5) has a solution, then there is the filter $\gamma(k)$ such that $g(k)$ and $\gamma(k)$ are prototype filters for cosine-modulated analysis and/or

synthesis filter banks, respectively, and the filter bank system achieve PR.

2. The filter design can be achieved simply by a fast recursive formula. For any given allowed design error, we can always design a FIR filter with arbitrary length.

4. Theoretical Background for our Design

In this section, we give the theoretical basis for the design technique presented in Section 3 via frame theory for discrete-time sequences, i.e., the relationship between the input signal to the filter bank system and the output of the analysis filter bank, the relationship between the input and output of the synthesis filter bank, and the relationship between the analysis and synthesis filters.

Let $\{g_n\}$ be a collection of sequences, then it is a *frame* if there exist constants A and B called frame bounds with $\infty > B \geq A > 0$ such that ([7])

$$A\,\|f\|^2 \leq \sum_n |\langle f, g_n\rangle|^2 \leq B\|f\|^2, \forall f \in l^2(Z) \quad (4.1)$$

(*i*) If $A \approx B$, the frame is said to be *snug*;

(*ii*) If $A = B$, the frame is *tight*, and

$$\sum_n |\langle f, g_n\rangle|^2 = A\|f\|^2, \qquad \forall f \in l^2(Z);$$

(*iii*) If $A = B = 1$ and $\| g \| = 1$, then $\{g_n\}$ is an orthonormal basis for $l^2(Z)$; and

(*iv*) If the elements of $\{g_n\}$ are linearly independent, then the frame is said to be *exact*.

If $\{g_n\}$ is a frame, then we are guaranteed that there exists the collection $\{\gamma_n\}$ called the dual frame such that for any $f \in l^2(Z)$, we have

$$f(k) = \sum_n \langle f, g_n\rangle \gamma_n(k) = \sum_n \langle f, \gamma_n\rangle g_n(k) \quad (4.2)$$

The convergence and stability of the above representation is guaranteed. Moreover, the isomorphicity between f and $\langle f, g_n\rangle$, and between f and $\langle f, \gamma_n\rangle$ is a very important property that can be used in signal processing since we can view $\langle f, g_n\rangle$ or $\langle f, \gamma_n\rangle$ as the output of a analysis system. The computation of dual frames for discrete-time signals was presented in [10] and summarized as follows

Define a frame operator such that

$$S(\cdot) = \sum_n \langle \cdot, g_n\rangle g_n \quad (4.3)$$

then it is shown that S is a positive definite operator if $\{g_n\}$ is a frame. It implies that the inverse operator S^{-1} exists. The frame theory proves that $\gamma_n = S^{-1}g_n$ for the $\gamma(k) \in l^2(Z)$. The $\gamma(k)$ can be obtained via series approximation, i.e.,

$$\gamma_n(k) = \frac{2}{A+B}\sum_{m=0}^{\infty}\left(I - \frac{2}{A+B}S\right)^m g(k) \quad (4.4)$$

where constants A and B are lower and upper frame bounds, respectively.

Now let $g_{mn}(k) = g_m(nM-k)$ and $\gamma_{mn}(k) = \gamma_m(k-nM)$, where $g_m(k)$ and $\gamma_m(k)$ are cosine-modulated filter banks, respectively, then since the operators in $g_{mn}(k)$ are linear with respect to both m and n, it is not difficult to prove that $\gamma = S^{-1}g$. Using the frame operator, we can derive

$$\sum_m\sum_n g(k\text{-}nM)g(k-nM-2mM)\gamma(k-2mM) = \frac{1}{M}\delta(m) \quad (4.5)$$

We can prove that the solution to this equation is unique and equal to the minimum-energy solution to the BLC, Eq.(3.1). Hence, in this section, we use frame theory to conclude that the filter bank system designed in Section 3: (1) is convergent and stable, (2) the input and output of the analysis filter banks have one-to-one correspondence, and (3) the system achieves PR.

5. Design Examples

In this section, we present two examples to demonstrate results derived in this paper. In the first example, a set of cosine-modulated Gaussian filters is used in the analysis filter bank, and in the second example, a set of cosine-modulated filters chosen from [1] is used as the analysis filter bank.

Example 1: In this example, a truncated Gaussian sequence is used as the prototype filter in the analysis filter bank. The Gaussian filter and computed prototype filter for the synthesis filter bank are shown in Figure 2. Figure 3 shows the amplitude responses of the analysis filters, and Figure 4 shows the amplitude responses of the designed synthesis filters. The length of the filter is 257, and there are 8 banks in the system. This pair of filters are important since they are useful in image processing.

Example 2: In this example, the prototype filter (dotted curve in Figure 3(a)) is chosen from [1], where it was used in the pseudo filter bank system. Using our technique, we design a prototype filter shown in Figure 3(a). We see that the two filters have little difference, but because of the difference, the system achieves PR. In this system, the length of the filter is 49, and M = 8.

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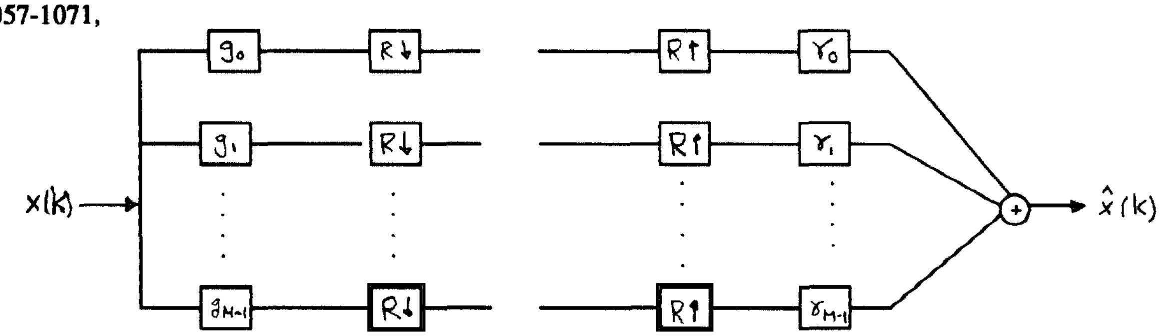


Fig. 1: M-band filter banks

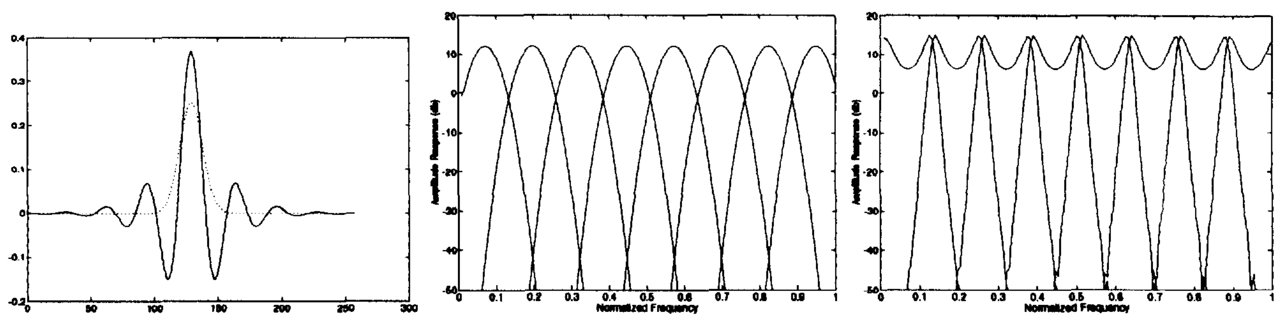
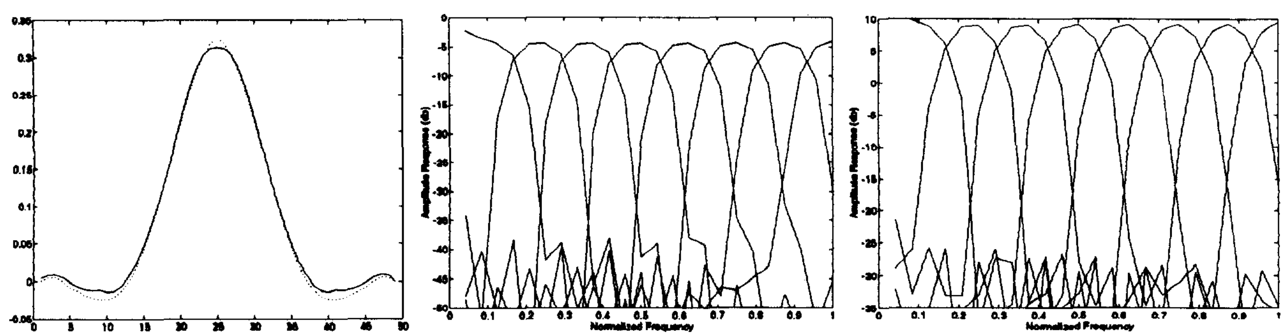

(a) Filters (b) Analysis (c) Synthesis

Figure 2: Analysis and Synthesis filters, and Their Frequency Response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