Least Squares Filters for Approximating Perfect Reconstruction Filter Banks*

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Abstract

This paper presents a least squares (LS) design methodology for approximating perfect reconstruction filter banks. A filter bank can be represented as a multi-input multi-output (MIMO) LTI system whose transfer function is described by the filter bank polyphase matrix. Using this MIMO system representation we frame a general least squares filter design problem. Then given an arbitrary set of rational analysis filters we find the causal synthesis filters for a filter bank which achieves the best causal LS approximation to perfect reconstruction.

1 Introduction

In this paper we develop a least squares (LS) design methodology for solving the following problem. Given an arbitrary set of analysis filters and some desired reconstruction delay find the causal synthesis filters resulting in the best possible approximation to a PRFB. The best approximation minimizes the squared error between the response of an ideal PRFB and the response achievable with causal synthesis filters. As we will demonstrate this LS approximation problem for PRFBs is a special case of a more general LS design problem for multi-input multi-output (MIMO) linear systems.

Many methods have been proposed for designing causal PRFBs. Most of these techniques produce FIR QMF filter pairs or equivalent M channel QMF designs. All of these design techniques for subband filters are subject to the severe constraints imposed by aliasing cancellation and the required absence or minimization of phase and amplitude distortion. The LS design methodology developed in this paper attempts to relieve some of these severe design constraints during the selection of the analysis filters, allowing the designer to choose from a much broader class of subband filters. The penalty for choosing arbitrary analysis filters is that in general the corresponding synthesis filters allowing for perfect reconstruction are non causal. Any attempts to approximate a PRFB with causal synthesis filters will in general result in some aliasing distortion as well as phase and amplitude distortion. However once a set of subband analysis filters is selected this LS design technique produces the causal synthesis filters which simultaneously minimize the combined aliasing, phase and amplitude distortion for a given reconstruction delay.

To begin we translate the PRFB approximation problem into a MIMO LS design problem in sections 2 and 3. This representation for filter banks as MIMO systems is a direct consequence of the general unifying theory for PRFBs developed primarily by Vaidyanathan in [2] [3] [4], and Vetterli in [5] [6] and is based on the polyphase representation. Once framed in the multivariate context we recognize the PRFB approximation problem as a classical LS equalization problem for a MIMO LTI system. In section 3.1 we outline a computational algorithm for finding the LS solution. These algorithms are applied in an actual design example in section 4.

2 Multivariate System Analysis for Filter Banks

In this section we consider the multi-input multi-output (MIMO) representation for filter banks. This MIMO system representation arises naturally from the filter bank polyphase representation. Consider the filter bank polyphase representation depicted in figure 1. $\mathbf{H}(z)$ is the type I polyphase matrix representing and the analysis filters. $\mathbf{G}(z)$ is the type II polyphase matrix representing and the synthesis filters. The first operation performed on the input sequence is a serial to parallel conversion. The delay elements and decimators convert the input scalar sequence u_k into a sequence of vectors $\mathbf{u}_k$. The resulting vector sequence $\mathbf{u}_k$ is filtered using the MIMO LTI filter $\mathbf{H}(z)$ to produce $\{\mathbf{x}_k\}$. Subsequently $\{\mathbf{x}_k\}$ is filtered by the synthesis filter $\mathbf{G}^T(z)$ to produce the output vector sequence $\{\mathbf{y}_k\}$.

$$\mathbf{y}(z) = \mathbf{G}^T(z)\mathbf{x}(z) = \mathbf{G}^T(z)\mathbf{H}(z)\mathbf{u}(z)$$

The output vector sequence $\{\mathbf{y}_k\}$ is then converted back to a scalar sequence $\{y_k\}$ by the zero-filling and

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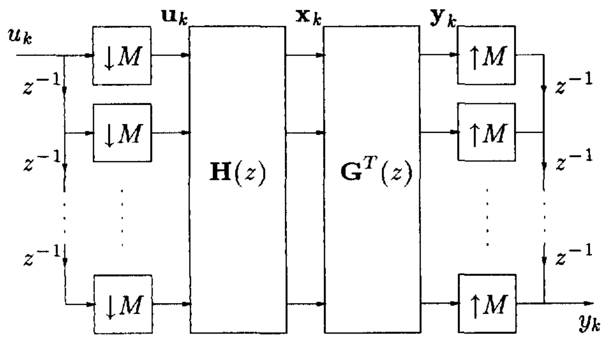

Figure 1: Multirate filter bank / MIMO filter

the delay operators which perform a parallel to serial conversion.

3 PRFB Approximation and Multivariate LS Equalization

We now consider the necessary and sufficient conditions for perfect reconstruction (PR) in terms of the polyphase product matrix $\mathbf{F}(z) = \mathbf{G}^T(z)\mathbf{H}(z)$. A filter bank achieves perfect reconstruction iff:

$$\mathbf{G}^T(z)\mathbf{H}(z) \;=\; \begin{bmatrix} 0 & z^{-m_o}\mathbf{I}_{M-r} \\ z^{-m_o-1}\mathbf{I}_r & 0 \end{bmatrix} \quad (1)$$

This condition for perfect reconstruction was originally derived by Vaidyanathan in [3]. A filter bank whose polyphase product matrix satisfies condition 1 achieves perfect reconstruction with a delay of $L = N + (M-1)$ where $N = m_oM + r$. Then the output $y(z)$ is precisely a delayed version of the input, $y(z) = z^{-L}u(z)$. Given an arbitrary set of analysis filters with polyphase matrix $\mathbf{H}(z)$, perfect reconstruction is obtained when the synthesis filter polyphase matrix $\mathbf{G}^T(z)$ satisfies $\mathbf{G}^T(z) = \mathbf{C}^N(z)\mathbf{H}^{-1}(z)$, where $\mathbf{C}^N(z)$ is defined as the pseudocirculant matrix on the RHS of equation 1. The inverse polyphase matrix $\mathbf{H}^{-1}(z)$ is in general non-causal. Therefore most PRFB design techniques constrain the analysis filters so that the inverse polyphase matrix $\mathbf{H}^{-1}(z)$ is causal.

In this paper we pursue an altogether different strategy. Given an arbitrary set of analysis filters, we find the causal synthesis filters which result in the best possible LS approximation to PR for a given reconstruction delay. The down-side of this approach is that aliasing, amplitude and phase distortions are introduced. The main advantage is that the designer can select the analysis filters from a much broader class of rational filters. The resulting PR approximation error can in principle be made arbitrarily small by allowing sufficient reconstruction delay.

To solve this approximation problem we first consider the deterministic least squares design problem

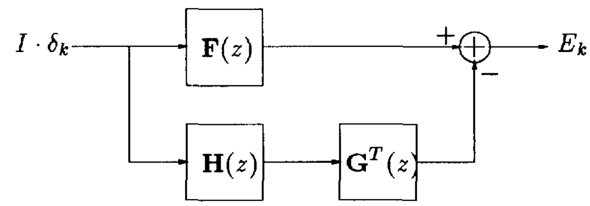

Figure 2: MIMO least squares design problem.

for causal MIMO LTI filters

Given: $\mathbf{H}(z), \mathbf{F}(z)$ rational
Find: $\mathbf{G}^T(z)$ rational and causal which minimizes

$$V(\mathbf{G}^T) = \|\mathbf{F} - \mathbf{G}^T * \mathbf{H}\|^2 = \sum \mathbf{Tr}[E_k^T E_k]$$
$$E_k = F_k - \sum G_{k-j}^T H_j \quad (2)$$

This represents the LS equalization or deconvolution problem for MIMO LTI systems depicted in figure 2. $\mathbf{F}(z)$ is the desired channel transfer function and $\mathbf{H}(z)$ is the actual or measured channel response. The design goal is to find a causal MIMO filter $\mathbf{G}^T(z)$ which minimizes the the total energy in the error sequence $\{E_k\}$.

Given the MIMO system representation for filter banks, the PRFB approximation problem can be framed as a special case of the MIMO LS design problem 2. Perfect reconstruction requires the polyphase product matrix $\mathbf{F}(z)$ to equal $\mathbf{C}^N(z)$. This condition is necessary and sufficient for perfect reconstruction with a delay of $L = N+(M-1)$. Let the actual or measured channel response equal the analysis filter polyphase matrix $\mathbf{H}(z)$. Then solving 2 we obtain the synthesis filter polyphase matrix $\mathbf{G}^T(z)$. This polyphase matrix represents the causal synthesis filters which minimize the energy in the PR approximation error. Given this identification of PRFB approximation as a MIMO LS problem we solve for the optimum causal synthesis filters using the solution to the multivariate LS problem. The LS solution $\mathbf{G}^T(z)$ is

$$\begin{aligned} \mathbf{G}^T(z) &= [\mathbf{F}(z)\mathbf{U}^T(z^{-1})]_+\mathbf{H}_o^{-1}(z) \quad (3) \\ \mathbf{H}(z) &= \mathbf{H}_o(z)\mathbf{U}(z) \\ \mathbf{U}(z)\mathbf{U}^T(z^{-1}) &= \mathbf{I} \;\; \text{Paraunitary} \end{aligned}$$

where the operator $[\cdot]_+$ denotes the causal part of its argument. The matrix $\mathbf{H}_o(z)$ is the minimum phase equivalent of $\mathbf{H}(z)$ where $\mathbf{H}_o(z)$ and its inverse $\mathbf{H}_o^{-1}(z)$ are both causal. $\mathbf{H}(z)$ is the product of its minimum phase equivalent $\mathbf{H}_o(z)$ and a paraunitary matrix $\mathbf{U}(z)$. Thus the causal filter $\mathbf{H}_o^{-1}(z)$ is the MIMO whitening or innovations filter for $\mathbf{H}(z)$. The most challenging aspect of computing the solution $\mathbf{G}^T(z)$ is to find the minimum phase equivalent of the filter $\mathbf{H}(z)$. This task is accomplished by performing a matrix spectral factorization on the product $\mathbf{H}(z)\mathbf{H}^T(z^{-1})$ which is positive definite on the unit circle. This problem is addressed in the following section.

3.1 Algorithm for Least Squares Design

In this section we outline an algorithm for computing the optimal (LS) synthesis filters from a set of rational analysis filters and a desired reconstruction delay. To begin we are given M rational analysis filters $h_k(z) = \frac{b_k(z)}{a_k(z)}$ for $k = 0$ to $M-1$ where $a_k(z)$ and $b_k(z)$ are FIR polynomials.

Qualitatively these M analysis filters are $1/M$ band rational filters whose pass bands are partitioned to cover all frequencies on the unit circle. In more precise terms the only requirement imposed on the M analysis filters for this design process is that the symmetric polyphase matrix product, $\mathbf{H}(z)\mathbf{H}^T(z^{-1})$, is strictly positive definite on the unit circle.

$$\mathbf{H}(e^{j\theta})\mathbf{H}^T(e^{-j\theta}) > 0 \quad \text{for } \forall\, \theta \tag{4}$$

This requirement is a sufficient condition for the matrix spectral factorization which is performed during the LS design process.

Before computing the minimum phase equivalent of the analysis filter polyphase matrix we must first convert the rational analysis filters into their polyphase components. Given a rational filter $h_k(z)$ find the polyphase components $h_{k,n}(z)$ satisfying:

$$h_k(z) = \sum_{n=0}^{M-1} z^{-n} h_{k,n}(z^M) = \frac{b_k(z)}{a_k(z)} \tag{5}$$

Each $h_{k,n}(z)$ is simply the z-transform of a shifted and subsampled version of the impulse response of $h_k(z)$. Thus each polyphase component $h_{k,n}(z)$ is a delayed and aliased version of $h_k(z)$.

$$h_{k,n}(z^M) = \frac{1}{M}\sum_{m=0}^{M-1} \frac{z^n W_M^{mn} b_k(zW_M^m)}{a_k(zW_M^m)}$$

where $W_M = e^{-j2\pi/M}$. Expressing each polyphase component as a rational polynomial we have $h_{k,n}(z) = \frac{b_{k,n}(z)}{a_{k,n}(z)}$ where

$$a_{k,n}(z^M) = \prod_{l=0}^{M-1} a_k(zW_M^l) \tag{6}$$

$$b_{k,n}(z^M) = \sum_{m=0}^{M-1} z^n W_M^{mn} b_k(zW_M^m) \prod_{\substack{l=0 \\ l\neq m}}^{M-1} a_k(zW_M^l) \tag{7}$$

Equations 6 and 7 allow us to compute the numerator and denominator polynomials for each polyphase component. Additionally every polyphase component $h_{k,n}(z)$ has the same denominator polynomial. This fact helps reduce computations but more importantly it leads to a polyphase matrix representation which is conveniently factored. To demonstrate this, suppose we have found the numerator and denominator polynomials, $b_{k,n}(z)$ and $a_{k,n}(z)$, for each polyphase component $h_{k,n}(z)$ in each of the M analysis filters. Then the analysis filter polyphase matrix is

$$\mathbf{H}(z) = \begin{bmatrix} h_{0,0}(z) & h_{0,1}(z) & \cdots & h_{0,M-1}(z) \\ h_{1,0}(z) & h_{1,1}(z) & \cdots & h_{1,M-1}(z) \\ \vdots & \vdots & & \vdots \\ h_{M-1,0}(z) & h_{M-1,1}(z) & \cdots & h_{M-1,M-1}(z) \end{bmatrix} \tag{8}$$

Since every element of a given row vector has the same denominator polynomial we can factor out all the denominator polynomials into a diagonal matrix $\mathbf{A}^{-1}(z)$ as follows:

$$\mathbf{H}(z) = \mathbf{A}^{-1}(z)\mathbf{B}(z) \tag{9}$$

$$\mathbf{A}(z) = \text{diag}\{d_0(z), d_1(z), \cdots d_{M-1}(z)\} \tag{10}$$

$$d_k(z) = a_{k,n}(z) = \prod_{l=0}^{M-1} a_k(z^{1/M} W_M^l) \tag{11}$$

$$[\mathbf{B}(z)]_{k,n} = b_{k,n}(z) \tag{12}$$

After converting the analysis filters into polyphase form we need to find the minimum phase equivalent of the polyphase matrix $\mathbf{H}(z)$. The problem is

$$\begin{array}{ll} \text{Given:} & \mathbf{H}(z) \text{ rational} \\ \text{Find:} & \mathbf{H}_o(z) \text{ satisfying} \\ & \mathbf{H}(z) = \mathbf{H}_o(z)\mathbf{U}(z) \\ & \mathbf{U}(z)\mathbf{U}^T(z^{-1}) = \mathbf{I} \;\; \text{paraunitary} \\ & \mathbf{H}_o(z) \;\text{ and }\; \mathbf{H}_o^{-1}(z) \;\;\text{causal} \end{array} \tag{13}$$

This is solved by performing a multivariate spectral factorization on the rational product matrix $\mathbf{S}(z) = \mathbf{H}(z)\mathbf{H}^T(z^{-1})$. A matrix spectral factorization algorithm takes a matrix which is positive definite on the unit circle and symmetrically factors it into minimum phase components.

$$\begin{array}{ll} \text{Given:} & \mathbf{S}(z) \text{ rational} \\ & \text{where } \mathbf{S}(e^{j\theta}) > 0 \;\; \text{for } \forall\, \theta \\ \text{Find:} & \mathbf{\Gamma}(z) \text{ rational satisfying} \\ & \mathbf{S}(z) = \mathbf{\Gamma}(z)\mathbf{\Gamma}^T(z^{-1}) \\ & \mathbf{\Gamma}(z) \;\text{ and }\; \mathbf{\Gamma}^{-1}(z) \;\;\text{causal} \end{array} \tag{14}$$

Solving this problem with $\mathbf{S}(z) = \mathbf{H}(z)\mathbf{H}^T(z^{-1})$ we define the minimum phase equivalent of $\mathbf{H}(z)$ as follows:

$$\mathbf{H}_o(z) \equiv \mathbf{\Gamma}(z)$$

$$\mathbf{U}(z) \equiv \mathbf{H}_o^{-1}(z)\mathbf{H}(z)$$

Then $\mathbf{H}(z) = \mathbf{H}_o(z)\mathbf{U}(z)$, where $\mathbf{U}(z)$ is paraunitary and $\mathbf{H}_o(z)$ is minimum phase.

The matrix spectral factorization problem 14 is central to the solution of the multivariate Wiener filtering problem, multivariate prediction, and linear optimal control theory. Many computational algorithms have been developed for solving it. Of these techniques the Newton-Raphson procedure first proposed by Wilson [7] and further refined by Jezek and Kucera [1] is used for this PRFB approximation problem.

This procedure computes successive iterations $\mathbf{\Gamma}_k(z)$ which converge to the actual minimum phase solution $\mathbf{\Gamma}(z)$. This particular algorithm operates on a strictly FIR polynomial matrix $\mathbf{S}(z) = \mathbf{\Gamma}(z)\mathbf{\Gamma}^T(z^{-1})$. Conveniently the computational procedure for finding the polyphase matrix components (equations 9-12 and 6-7) results in the factored form $\mathbf{H}(z) = \mathbf{A}^{-1}(z)\mathbf{B}(z)$ where $\mathbf{A}(z)$ and $\mathbf{B}(z)$ are both FIR polynomial matrices. From this factored form it is a straight forward procedure to again factor $\mathbf{H}(z)$ into a new polynomial matrix $\mathbf{B}(z)$ and a single monic denominator polynomial $a(z)$.

$$\mathbf{H}(z) = \frac{1}{a(z)}\mathbf{B}(z) \tag{15}$$

$$a(z) = \det(\mathbf{A}(z)) = \prod_{k=0}^{M-1}\prod_{l=0}^{M-1} a_k(z^{1/M}W_M^l)$$

$$[\mathbf{B}(z)]_{k,n} = b_{n,k}(z)\prod_{\substack{m=0\\m\neq k}}^{M-1}\prod_{l=0}^{M-1} a_m(z^{l/M}W_M^l) \tag{16}$$

Exploiting this new factorization 15 we perform a one dimensional spectral factorization on the FIR polynomial $a(z)$ to obtain its minimum phase equivalent $a_o(z)$. Then we find the minimum phase equivalent of the polynomial matrix $\mathbf{B}(z)$ (equation 16) using Wilson's algorithm [7]. The minimum phase equivalent of $\mathbf{H}(z)$ can then be written

$$\mathbf{H}_o(z) = \frac{1}{a_o(z)}\mathbf{B}_o(z) \tag{17}$$

Next we compute the paraunitary matrix $\mathbf{U}(z)$ satisfying $\mathbf{H}(z) = \mathbf{H}_o(z)\mathbf{U}(z)$, by finding the causal inverse $\mathbf{H}_o^{-1}(z)$. Let $\mathbf{B}_o^{(\text{adj})}(z)$ be the matrix of cofactors (the adjugate matrix) of $\mathbf{B}_o(z)$.

$$\mathbf{B}_o(z)\mathbf{B}_o^{(\text{adj})}(z) = \det(\mathbf{B}_o(z))\cdot\mathbf{I}$$

Let $\beta_o(z) = \det(\mathbf{B}_o(z))$ then $\mathbf{U}(z)$ is computed as follows:

$$\mathbf{U}(z) = \frac{a_o(z)}{\beta_o(z)}\mathbf{B}_o^{(\text{adj})}(z)\mathbf{H}(z) = \mathbf{H}_o^{-1}(z)\mathbf{H}(z)$$

Finally we apply these results to the solution of the LS approximation problem for PRFBs. Substituting $\mathbf{C}^N(z)$ for $\mathbf{F}(z)$ we can compute the least squares synthesis filter polyphase matrix $\mathbf{G}^T(z)$ using equation 3.

3.2 Performance Analysis

This LS filter design process produces filter banks which in general exhibit aliasing, magnitude and phase distortion. Thus we want an expression for the filter bank output $y(z)$ which is a function of the input alias components and the analysis and synthesis filters. Such an expression has been developed in [4] and [3].

$$y(z) = \frac{1}{M}\mathbf{g}^T(z)\mathbf{H}_A^T(z)\mathbf{u}_A(z) \tag{18}$$

$\mathbf{g}(z)$ is a column vector whose elements are the synthesis filters. $\mathbf{H}_A(z)$ is the alias component (AC) matrix for the analysis filters and $\mathbf{u}_A^T(z) = [u(z), u(zW_M), \cdots u(zW_M^{M-1})]$ is the vector of the alias components of the input. Next we define the aliasing gain $\mathbf{T}(z)$, such that $y(z) = \mathbf{u}_A^T(z)\mathbf{T}(z)$ where $\mathbf{T}(z) = \frac{1}{M}\mathbf{H}_A(z)\mathbf{g}(z)$. The components of the aliasing gain vector, $\mathbf{T}(z)$, represent the equivalent filters applied to each input alias component:

$$y(z) = \sum_{k=0}^{M-1} t_k(z)u(zW_M^k)$$

$$\mathbf{T}^T(z) = [t_0(z), t_1(z), \cdots t_{M-1}(z)]$$

Perfect reconstruction occurs when $\mathbf{T}(z) = [z^{-L}, 0, \cdots 0]$. Next we would like to express $\mathbf{T}(z)$ in terms of $\mathbf{H}(z)$ and $\mathbf{G}^T(z)$. By definition of the polyphase matrix, $g(z) = \mathbf{G}^T(z^M)\mathbf{J}\psi(z)$. The polyphase matrix $\mathbf{H}(z)$ is related to the alias component (AC) matrix $\mathbf{H}_A(z)$ by the following expression $\mathbf{H}_A(z) = \mathbf{W}^H\mathbf{D}(z)\mathbf{H}^T(z^M)$, where $\mathbf{W}^H$ is the inverse DFT matrix and $\mathbf{D}(z) = \text{diag}\{1, z^{-1}, \cdots z^{-(M-1)}\}$. The aliasing gain becomes

$$\mathbf{T}(z) = \frac{1}{M}\mathbf{W}^H\mathbf{D}(z)\mathbf{H}^T(z^M)\mathbf{G}(z^M)\mathbf{J}\psi(z) \tag{19}$$

In the following LS design example we plot the magnitude and group delay of the aliasing gain components computed using expression 19.

4 Design Example 6^{th} order Chebyshev

For this design example we approximate a three channel PRFB. The analysis filters selected for this task are are all type II Chebyshev filters. All three analysis filters, lowpass, bandpass, and highpass, were designed so that their 3dB points where at the normalized frequencies 1/3 and 2/3 respectively. The lowpass and highpass filters are actually 5^{th} order type II Chebyshev filters while the bandpass filter is a 6^{th} order type II Chebyshev. All three filters were designed with a maximum stopband ripple 20db below the passband. The synthesis filter magnitudes are plotted in figure 3. Following the LS design procedures outlined in the preceding sections, several different sets of optimal (LS) synthesis filters were designed by varying the reconstruction delay. The magnitudes of the LS synthesis filters with a reconstruction delay of 44 are plotted in figure 3. The resulting sequence of filter banks exhibits an increasingly better approximation to perfect reconstruction as the reconstruction delay is increased. The magnitudes of the resulting aliasing gain components, $\mathbf{T}(e^{j\theta}) = [t_0(e^{j\theta}), t_1(e^{j\theta}), t_2(e^{j\theta})]$, are plotted in figure 5 for reconstruction delays of 8, 12, 16 and 24. Clearly as the delay increases the magnitude of the zeroth term, $t_0(e^{j\theta})$, is converging to a constant value of one. Concurrently the other aliasing gain terms are fairly rapidly converging to zero. In concert with the magnitude response, the group delay of the $t_0(e^{j\theta})$ term is also converging to a constant which equals the reconstruction delay. This is evidenced in figure 6. Note that the group delays plotted in figure 6 for the $t_0(z)$ term, are normalized to a delay of 1 and the resulting deviations from a value of 1 represent a percentage error from the nominal delay.

This is precisely the convergence behavior we would expect. Increasing the reconstruction delay allows the causal filter bank to better approximate a pure delay element.

5 Conclusion

In this paper we have successfully implemented a least squares filter design methodology for approximating perfect reconstruction filter banks. This methodology allows the designer to select subband filters from a broad class of rational filters.

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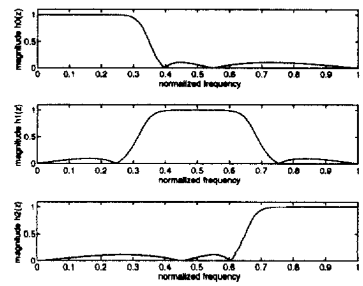

Figure 3: Chebyshev analysis filters - Magnitude

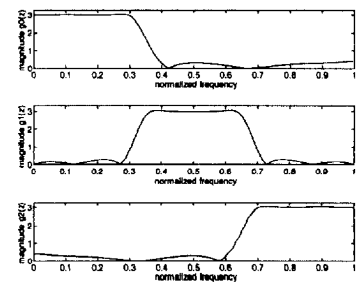

Figure 4: LS synthesis filters (delay = 44) - Magnitude

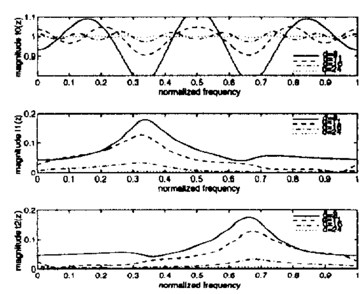

Figure 5: Aliasing Gain $\mathbf{T}(e^{j\theta})$ - Magnitude

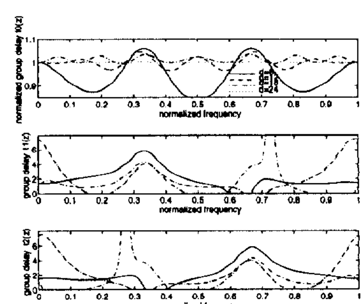

Figure 6: Aliasing Gain $\mathbf{T}(e^{j\theta})$ - Group delay