STATISTICALLY OPTIMAL SYNTHESIS BANKS FOR SUBBAND CODERS

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Abstract. In maximally decimated filter banks, the perfect reconstruction or biorthogonal solution is not necessarily the best choice when subband quantizers are present. Under suitable statistical assumptions, expressions for the best synthesis bank can be derived in terms of the analysis bank and other statistical quantities. In this paper we explore this topic for subband coders and the special case of transform coders. We highlight the statistical conditions under which the biorthogonal solution is still the best. Special cases where the optimal synthesis filter bank is the biorthogonal system followed by a scalar post filter are also considered.

1. INTRODUCTION

It is well-known [**1-3**] that an M-channel maximally decimated filter bank can be redrawn as in Fig. 1.1 where $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the polyphase matrices of the analysis and synthesis filter banks respectively. With $\mathbf{E}(z)$ taken to be a constant nonsingular matrix $\mathbf{T}$, the system reduces to a transform coder [**4**]. In the absence of the subband quantizers, the choice of the synthesis filters which results in perfect reconstruction [i.e., $\widehat{x}(n) = x(n)$] corresponds to the choice $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$. In this case the analysis/synthesis system forms a perfect reconstruction or biorthogonal set.

With subband quantizers present, what is the best choice of $\mathbf{R}(z)$ that minimizes the mean square value of the reconstruction error $x(n)-\widehat{x}(n)$? At least one issue is clear: the "best synthesis bank" must reduce to the traditional biorthogonal solution as the subband bit rate tends to infinity. Thus the synthesis bank should somehow depend on the subband bit rate.

Fig. 1.2 shows a simplified redrawing of the essentials of the problem. Here $\mathbf{x}(n)$ is the vector signal entering the polyphase matrix $\mathbf{E}(z)$, and is the blocked version [**1**] of the scalar input $x(n)$ as shown in Fig. 1.1. The vector of decimated subband signals is denoted as $\mathbf{p}(n)$, and $\mathbf{q}(n)$ is the vector of quantization errors. Finally $\widehat{\mathbf{x}}(n)$ is the reconstructed vector whose unblocked version will result in $\widehat{x}(n)$. Assuming [det $\mathbf{E}(z)$] is not identically zero for all z, $\mathbf{E}^{-1}(z)$ is well defined. We can always write

$$\mathbf{R}(z) = \mathbf{H}(z)\mathbf{E}^{-1}(z) \tag{1.1}$$

which gives us the structure of Fig. 1.3(a),(b). The output of $\mathbf{E}^{-1}(z)$ is $\mathbf{x}(n)$ if there is no quantization [$\mathbf{q}(n) = \mathbf{0}$]. In presence of quantization, the output of $\mathbf{E}^{-1}(z)$ has the noise component $\mathbf{w}(n)$, which is $\mathbf{q}(n)$ filtered through $\mathbf{E}^{-1}(z)$.

The mean square value of the reconstruction error is $E[|x(n) - \widehat{x}(n)|^2]$. Assuming that $x(n)$ is wide sense stationary (WSS), we know that the output $\widehat{x}(n)$ is cyclo WSS with period M [i.e., $(CWSS)_M$] rather than WSS [**11**]. So the quantity $E[|x(n)-\widehat{x}(n)|^2]$ has period M and we have to average this over M samples. This average value is equal to

$$E\Big[\Big(\mathbf{x}(n) - \widehat{\mathbf{x}}(n)\Big)^\dagger\Big(\mathbf{x}(n) - \widehat{\mathbf{x}}(n)\Big)\Big] \tag{1.2}$$

upto a scale factor of M (superscript † denotes transposed conjugate). So the problem reduces to finding the best $\mathbf{H}(z)$ that minimizes (1.2).

In Fig. 1.3(a), $\mathbf{H}(z)$ is an $M \times M$ transfer matrix whose input is *signal plus noise*. Our aim is to design it such that the output $\widehat{\mathbf{x}}(n)$ is as close to the signal component $\mathbf{x}(n)$ as possible in the mean square sense. The best $\mathbf{H}(z)$ is the familiar Wiener filter [**5**] for the vector process $\mathbf{x}(n) + \mathbf{w}(n)$ (assuming that $\mathbf{x}(n)$ and $\mathbf{w}(n)$ are jointly WSS). We say that $\mathbf{H}(z)$ is the *synthesis bank Wiener filter matrix.*

LPTV interpretation of the Wiener filter. It can be shown that inserting the LTI matrix $\mathbf{H}(z)$ is equivalent to inserting a scalar LPTV (linear periodically time varying) Wiener filter at the output $\widehat{x}(n)$ of the filter bank Fig. 1.3(b). This LPTV filter has period M. The above equivalence essentially follows from the relation between scalar LPTV filters and blocked filters (Chap. 10, [**1**]).

Aims of the Paper. We derive expressions for the Wiener filter matrix in terms of the joint statistics of appropriate signals. Special cases where subband noise sources are uncorrelated to the unquantized signal, and cases where they are uncorrelated to the *quantized signals* will be considered. We highlight the conditions under which the Wiener filter $\mathbf{H}(z) = \mathbf{I}$ (i.e., Wiener filtering is unncessary). Special cases where

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the Wiener filter matrix $\mathbf{H}(z)$ can be replaced with a scalar LTI Wiener filter $H(z)$ at the filter bank output will be considered in Sec. 6. The *transform coder case* (where $\mathbf{E}(z)$ is a constant) is considered in parallel during all developments; we will see that with optimal vector quantization in the subbands, the transform coder does not require a synthesis Wiener matrix (i.e., the *biorthogonal solution is the best*). Earlier papers analyzing subband errors include [6]–[8].

2. THE SYNTHESIS WIENER FILTER MATRIX

Consider Fig. 1.3(a) where a multi input multi output LTI system $\mathbf{H}(z)$ has input $\mathbf{y}(n) = \mathbf{x}(n) + \mathbf{w}(n)$. Here $\mathbf{y}(n)$ is a noisy input sequence with signal component $\mathbf{x}(n)$ and noise component $\mathbf{w}(n)$. Assume the $M \times 1$ sequences $\mathbf{x}(n)$ and $\mathbf{w}(n)$ are jointly wide sense stationary (WSS) random processes. So $\mathbf{y}(n)$ and the output $\widehat{\mathbf{x}}(n)$ are also WSS. Define the error sequence $\mathbf{e}(n) = \mathbf{x}(n) - \widehat{\mathbf{x}}(n)$ and the mean square error

$$\mathcal{E} = E[\mathbf{e}^{\dagger}(n)\mathbf{e}(n)] = \mathrm{Tr}\, E[\mathbf{e}(n)\mathbf{e}^{\dagger}(n)]. \qquad (2.1)$$

We would like to choose $\mathbf{H}(z)$ to be the Wiener filter, that is, the filter which minimizes $\mathcal{E}$. The solution is

$$\mathbf{H}(z) = \mathbf{S_{xy}}(z)\mathbf{S}^{-1}_{\mathbf{yy}}(z) \qquad \text{(Wiener filter)} \qquad (2.2a)$$

unless $[\det \mathbf{S_{yy}}(z)] \equiv 0$ for all z. Here $\mathbf{S_{xy}}(z)$ is the cross power-spectrum, which is the z-transform of the cross correlation $\mathbf{R_{xy}}(k) = E[\mathbf{x}(n)\mathbf{y}^{\dagger}(n-k)]$. Other equivalent forms for the Wiener filter are

$$\mathbf{H}(z) = [\mathbf{S_{xx}}(z) + \mathbf{S_{xw}}(z)]\mathbf{S}^{-1}_{\mathbf{yy}}(z) \qquad (2.2b)$$

$$\text{and} \quad \mathbf{H}(z) = [\mathbf{S_{yy}}(z) - \mathbf{S_{wy}}(z)]\mathbf{S}^{-1}_{\mathbf{yy}}(z). \qquad (2.2c)$$

The proof of this standard result follows from orthogonality principle (vector version). Note that

$$\widehat{\mathbf{x}}(n) = \sum_{m} \mathbf{h}(m)\mathbf{y}(n-m), \qquad (2.3)$$

where $\mathbf{h}(m)$ is the impulse response matrix, that is $\mathbf{H}(z) = \sum_m \mathbf{h}(m)z^{-m}$. Orthogonality principle says that the best filter (Wiener filter) should satisfy

$$E[\mathbf{e}(n)\mathbf{y}^{\dagger}(n-k)] = \mathbf{0}, \quad \text{for all } k. \qquad (2.4)$$

That is,

$$E[\mathbf{x}(n)\mathbf{y}^{\dagger}(n-k)] = E[\widehat{\mathbf{x}}(n)\mathbf{y}^{\dagger}(n-k)], \quad \text{for all } k.$$

The left hand side is $\mathbf{R_{xy}}(k)$. Substituting from Eq. (2.3) the equation simplifies to

$$\mathbf{R_{xy}}(k) = \sum_{m} \mathbf{h}(m)\mathbf{R_{yy}}(k-m), \qquad (2.5)$$

i.e., $\mathbf{S_{xy}}(z) = \mathbf{H}(z)\mathbf{S_{yy}}(z)$, from which the stated form of $\mathbf{H}(z)$ follows. The alternative forms in (2.2) follow using $\mathbf{S}_{(\mathbf{x}_1+\mathbf{x}_2),\mathbf{y}}(z) = \mathbf{S}_{\mathbf{x}_1,\mathbf{y}}(z) + \mathbf{S}_{\mathbf{x}_2,\mathbf{y}}(z)$ etc.

Case when Wiener Filter $= \mathbf{I}$

We see from Eq. (2.2c) that $\mathbf{H}(z) = \mathbf{I}$ if and only if $\mathbf{S_{wy}}(z) = \mathbf{0}$, or equivalently $\mathbf{R_{wy}}(k) = \mathbf{0}$. That is, $\mathbf{w}(n)$ and $\mathbf{y}(m)$ must be orthogonal for all choices of n, m. Thus the Wiener filter $\mathbf{H}(z) = \mathbf{I}$ if and only if the *noise random process* $\mathbf{w}(.)$ and the *noisy random process* $\mathbf{y}(.)$ are mutually orthogonal (equivalently uncorrelated, in the zero-mean case).

Memoryless Case

Suppose we are interested in finding a constant matrix $\mathbf{V}$ rather than a filter $\mathbf{H}(z)$ in Fig. 1.3. In this case, Eq. (2.4) has to hold only for $k = 0$, and Eq. (2.5) is replaced with $\mathbf{R_{xy}}(0) = \mathbf{V}\mathbf{R_{yy}}(0)$ so that the optimal solution is

$$\mathbf{V} = \mathbf{R_{xy}}(0)\mathbf{R}^{-1}_{\mathbf{yy}}(0). \qquad (2.6)$$

This assumes nonsingularity[2] of $\mathbf{R_{yy}}(0)$. We can write

$$\mathbf{R_{xy}}(0) = \mathbf{R_{xx}}(0) + \mathbf{R_{xw}}(0) = \mathbf{R_{yy}}(0) - \mathbf{R_{wy}}(0) \qquad (2.7)$$

using $\mathbf{y}(n) = \mathbf{x}(n) + \mathbf{w}(n)$. From this we can verify the following:

1. $\mathbf{V} = \mathbf{I}$ if and only if the noise $\mathbf{w}(n)$ and the noisy signal $\mathbf{y}(n)$ are orthogonal for each n. This is weaker than the requirement for $\mathbf{H}(z) = \mathbf{I}$ in Fig. 1.3.
2. $\mathbf{V} = \mathbf{R_{xx}}(0)[\mathbf{R_{yy}}(0)]^{-1} = \mathbf{R_{xx}}(0)[\mathbf{R_{xx}}(0) + \mathbf{R_{ww}}(0)]^{-1}$ whenever the noise $\mathbf{w}(n)$ and the noiseless signal $\mathbf{x}(n)$ are orthogonal for each n.

3. CASE WHEN SYNTHESIS WIENER MATRIX $= \mathbf{I}$

For the problem of optimal reconstruction under subband quantization, consider Fig. 1.3(a). If we wish to minimize $\mathbf{e}(n) = \mathbf{x}(n) - \widehat{\mathbf{x}}(n)$ in the mean square sense, the solution $\mathbf{H}(z)$ is given by any one of the three forms in Eq. (2.2). We now draw some conclusions from these expressions. We assume $\mathbf{x}(n)$ and the quantization noise vector $\mathbf{q}(n)$ are *jointly WSS,* so that Wiener filtering results can be used.

♠**Theorem 3.1.** The solution $\mathbf{H}(z)$ which minimizes the reconstruction error Eq. (2.1) (i.e., the Wiener filter) is given by $\mathbf{H}(z) = \mathbf{I}$ if and only if

$$E[\mathbf{u}(n)\mathbf{q}^{\dagger}(m)] = \mathbf{0}, \qquad \text{for all } m, n, \qquad (3.1)$$

that is, the quantized subband signal $\mathbf{u}(.)$ and the quantization noise $\mathbf{q}(.)$ are orthogonal random processes (equivalently, in the frequency domain $\mathbf{S_{uq}}(e^{j\omega}) = \mathbf{0}$ for all ω). ◇

Proof. From Sec. 2 we know that the Wiener filter $\mathbf{H}(z) = \mathbf{I}$ if and only if $\mathbf{S_{wy}}(z) = \mathbf{0}$. It only remains to make the connection between the pair $\mathbf{y}(.), \mathbf{w}(.)$ and

[2] If this is not true, we can find $\mathbf{u} \neq \mathbf{0}$ such that $\mathbf{u}^{\dagger}\mathbf{y}(n) = 0$ for all n and reduce the size of the vector process $\mathbf{y}(n)$.

the pair $\mathbf{u}(.), \mathbf{q}(.)$ For this, note that $\mathbf{x}(n), \mathbf{w}(n)$ and $\mathbf{y}(n)$ are the outputs of $\mathbf{E}^{-1}(z)$ in response to the specific inputs $\mathbf{p}(n)$, $\mathbf{q}(n)$, and $\mathbf{u}(n)$ respectively. Suppose $\mathbf{C}_1(z)$ and $\mathbf{C}_2(z)$ are two systems with jointly WSS inputs $\mathbf{x}_1(n)$ and $\mathbf{x}_2(n)$ and outputs $\mathbf{y}_1(n)$ and $\mathbf{y}_2(n)$. If $\mathbf{x}_1(m)$ and $\mathbf{x}_2(n)$ are mutually orthogonal (for all m, n), it can be verified that the two output processes are also orthogonal, and the converse holds when $\mathbf{C}_1(z)$ and $\mathbf{C}_2(z)$ are invertible. Thus, the quantities $\mathbf{y}(n)$ and $\mathbf{w}(m)$ are orthogonal for all n, m, (i.e., the Wiener filter $\mathbf{H}(z) = \mathbf{I}$) if and only if $\mathbf{q}(n)$ and $\mathbf{u}(m)$ are orthogonal for all (n, m) (i.e., the quantizer noise and the quantized signal are orthogonal random processes). $\nabla\nabla\nabla$

♠**Theorem 3.2. Memoryless case.** Consider now the transform coder case, where we use a constant matrix $\mathbf{T}$ instead of $\mathbf{E}(z)$, and use a memoryless Wiener matrix $\mathbf{V}$ rather than the general transfer function $\mathbf{H}(z)$ (Fig. 3.1). Then the solution $\mathbf{V}$ which minimizes the reconstruction error Eq. (2.1) is given by $\mathbf{V} = \mathbf{I}$ if and only if

$$E[\mathbf{u}(n)\mathbf{q}^\dagger(n)] = \mathbf{0}, \qquad \text{for all } n, \qquad (3.2)$$

that is, the quantized subband signal $\mathbf{u}(n)$ and the quantization noise $\mathbf{q}(n)$ are orthogonal (uncorrelated in the zero mean case) for each instant of time n. $\diamond$

This follows by a simple modification of the preceding proof. The requirement of Theorem 3.1 that the processes $\mathbf{u}(.)$ and $\mathbf{q}(.)$ be orthogonal is a rather strong one. This means that $E[\mathbf{u}(n)\mathbf{q}^\dagger(m)] = \mathbf{0}$ for all m, n. The ith element $u_i(n)$ of the vector $\mathbf{u}(n)$ is the ith subband signal (after decimation and quantization, see Fig. 1.3(b)). The kth element $q_k(m)$ of $\mathbf{q}(m)$ is the quantizer noise source for the kth subband. Thus the condition Eq. (3.1) means that every sample $u_i(n)$ of every subband signal is orthogonal to every sample $q_k(m)$ of every subband quantization noise source.

On the other hand the requirement of Theorem 3.2 that $\mathbf{u}(n)$ and $\mathbf{q}(n)$ be orthogonal for each n is much milder. It turns out that optimal vector quantization of subband signals achieves this property (Sec. 5, Sec. 6).

4. CLOSED-FORM EXPRESSIONS

When the synthesis Wiener matrix $\mathbf{H}(z)$ is not identity, it is of interest to find a closed form expression for it. Using $\mathbf{y}(n) = \mathbf{x}(n) + \mathbf{w}(n)$ the Wiener filter (2.2b) can be rewritten

$$\mathbf{H}(z) = \Big(\mathbf{S_{xx}}(z) + \mathbf{S_{xw}}(z)\Big) \times \Big(\mathbf{S_{xx}}(z) + \mathbf{S_{ww}}(z) + \mathbf{S_{wx}}(z) + \mathbf{S_{xw}}(z)\Big)^{-1}$$

Note that $\mathbf{w}(n)$ is the output of $\mathbf{E}^{-1}(z)$ in response to $\mathbf{q}(n)$. That is, $\mathbf{q}(n)$ would be the output of $\mathbf{E}(z)$ in response to $\mathbf{w}(n)$. This yields $\mathbf{S_{qx}}(z) = \mathbf{E}(z)\mathbf{S_{wx}}(z)$. Using this, and $\mathbf{S_{wx}}(z) = \widetilde{\mathbf{S}}_{\mathbf{xw}}(z)$ (where $\widetilde{\mathbf{F}}(z)$ denotes $\mathbf{F}^\dagger(1/z^*)$), we obtain the Wiener filter in terms of the joint statistics of the input $\mathbf{x}(n)$ and the subband quantization noise source $\mathbf{q}(n)$. With $\mathbf{G}(z) \stackrel{\Delta}{=} \mathbf{E}^{-1}(z)$, this is

$$\mathbf{H}(z) = \Big[\mathbf{S_{xx}}(z) + \mathbf{S_{xq}}(z)\widetilde{\mathbf{G}}(z)\Big] \times \Big(\mathbf{G}(z)\mathbf{S_{qq}}(z)\widetilde{\mathbf{G}}(z) + \mathbf{S_{xx}}(z) + \mathbf{G}(z)\mathbf{S_{qx}}(z) + \mathbf{S_{xq}}(z)\widetilde{\mathbf{G}}(z)\Big)^{-1}$$

Similarly, for the memoryless case where we use a constant matrix $\mathbf{V}$ rather than a filter $\mathbf{H}(z)$, we get, with $\mathbf{A} \stackrel{\Delta}{=} \mathbf{T}^{-1}$,

$$\mathbf{V} = \Big[\mathbf{R_{xx}}(0) + \mathbf{R_{xq}}(0)\mathbf{A}^\dagger\Big] \times \Big(\mathbf{A}\mathbf{R_{qq}}(0)\mathbf{A}^\dagger + \mathbf{R_{xx}}(0) + \mathbf{A}\mathbf{R_{qx}}(0) + \mathbf{R_{xq}}(0)\mathbf{A}^\dagger\Big)^{-1}$$

The preceding expressions do not make any assumptions as to whether the noise sources are correlated to each other and the signals, whether the noise sources are white, and so forth. The only assumption is that the input $\mathbf{x}(n)$ and the quantization noise vector $\mathbf{q}(n)$ are jointly WSS (so we can use Wiener filter theory).

5. ORTHOGONALITY IN OPTIMAL QUANTIZERS

Imagine that $\mathbf{x}$ is an M-component random vector to be quantized. Instead of quantizing each component separately in an optimal manner, the quantization can be performed by taking into account the dependancy between the components of $\mathbf{x}$. This is the idea behind vector quantization (VQ), and results in smaller mean square error [9]. The idea is to partition the M dimensional space of all vectors $\mathbf{x}$ in an appropriate manner, and assign codewords (M-component vectors) $\mathbf{c}_k$ in each of these regions R_k. The quantized value is

$$\mathbf{x}_q = \mathbf{c}_k \qquad \text{if } \mathbf{x} \text{ is in } R_k \qquad (5.1)$$

(we ignore details of border ambiguity). The set $\mathcal{C}$ of codewords $\mathbf{c}_k$ is the *codebook*. The design of the regions R_k and the codewords $\mathbf{c}_k$ is based on the probability density function pdf $f_{\mathbf{X}}(\mathbf{x})$ of the vector $\mathbf{x}$. The aim is to mininize the mean square error given by

$$\mathcal{E} = E[(\mathbf{x} - \mathbf{x}_q)^\dagger(\mathbf{x} - \mathbf{x}_q)] \qquad (5.2)$$

which can be rewritten as

$$\mathcal{E} = \sum_k \int_{R_k} f_{\mathbf{X}}(\mathbf{x})\Big(\mathbf{x} - \mathbf{c}_k\Big)^\dagger\Big(\mathbf{x} - \mathbf{c}_k\Big)d\mathbf{x} \qquad (5.3)$$

The quantizer optimized for a given pdf is often called the *pdf-quantizer* or the *Lloyd-Max quantizer* [9]. The central result that governs optimality of vector quantizers is the following.

♠**Theorem 5.1. Orthogonality in optimal quantizers.** Let $\mathbf{x}$ be a random vector (possibly complex) with probability density $f_{\mathbf{X}}(\mathbf{x})$ and let $\mathbf{x}_q$ be the optimally quantized value, in the sense that the mean square error $\mathcal{E}$ is minimized. Then the *quantized vector* $\mathbf{x}_q$ *is orthogonal to the quantization error* $\mathbf{e} = \mathbf{x} - \mathbf{x}_q$. That is, $E[\mathbf{e}\mathbf{x}_q^\dagger] = \mathbf{0}$. $\diamond$

This is proved in standard references, see for example [9]. In the zero mean case, this means that the error $\mathbf{e}$ is uncorrelated to the *quantized* value $\mathbf{x}_q$. This is unlike the popular notion in roundoff noise analysis [10], where the error is assume to be uncorrelated to the *unquantized* random variable $\mathbf{x}$. Since $\mathbf{x} = \mathbf{x}_q + \mathbf{e}$, we have

$$E[\mathbf{e}\mathbf{x}^\dagger] = E[\mathbf{e}\mathbf{e}^\dagger] + E[\mathbf{e}\mathbf{x}_q^\dagger] \qquad (5.4)$$

Thus the condition $E[\mathbf{e}\mathbf{x}_q^\dagger] = \mathbf{0}$ for optimal quantization, and the "popular notion" $E[\mathbf{e}\mathbf{x}^\dagger] = \mathbf{0}$ are approximately identical if the noise variance is relatively small.

6. QUANTIZER MODEL AND WIENER MATRIX

The exact nature of the Wiener filter $\mathbf{H}(z)$ in Fig. 1.3 depends on the model used for the subband quantizers, because that will affect the joint statistics of $\mathbf{x}(n)$ and $\mathbf{q}(n)$. We now consider several cases.

Case 1. Optimal VQ in the Subbands

If we quantize the subband vector $\mathbf{p}(n)$ using optimal VQ for each n, then the *error* vector $\mathbf{q}(n)$ is orthogonal to the *quantized* subband signal $\mathbf{u}(n)$. In the transform coding case [where $\mathbf{E}(z)$ and $\mathbf{H}(z)$ are constant matrices (Fig. 3.1)], we see then that the Wiener matrix $\mathbf{V} = \mathbf{I}$ (Theorem 3.2).

In the more general subband coder where $\mathbf{E}(z)$ and $\mathbf{H}(z)$ have memory (Fig. 1.3), the best solution is not necessarily $\mathbf{H}(z) = \mathbf{I}$ even with optimal vector quantizers in the subbands. The Wiener filter looks for deeper correlations, e.g., between $\mathbf{q}(n)$ and $\mathbf{u}(n+1)$ (Theorem 3.1). However, if all correlations between the quantized signal $\mathbf{u}(n)$ and the quantization noise $\mathbf{q}(m)$ can be neglected or made equal to zero (as in the case of a hypothetical vector quantizer which treats the set of all subband sequences as one infinite dimensional vector), then the Wiener solution $\mathbf{H}(z) = \mathbf{I}$.

Case 2. Traditional Subband Noise Model

Traditionally one assumes that the subband noise $\mathbf{q}(n)$ is uncorrelated to the *unquantized subband signal* $\mathbf{p}(m)$. Under zero mean conditions this implies that $\mathbf{q}(n)$ and $\mathbf{p}(m)$ are orthogonal for all m, n. That is $\mathbf{x}(.)$ and $\mathbf{w}(.)$ entering the Wiener filter are orthogonal processes (because linear filtering does not destroy orthogonality). Then the Wiener solution $\mathbf{H}(z)$ reduces to

$$\mathbf{H}(z) = \mathbf{S_{xx}}(z)[\mathbf{S_{xx}}(z) + \mathbf{S_{ww}}(z)]^{-1} \qquad (6.1)$$

Since $\mathbf{w}(n)$ is the output of $\mathbf{E}^{-1}(z)$ in response to $\mathbf{q}(n)$ we have $\mathbf{S_{ww}}(z) = \mathbf{G}(z)\mathbf{S_{qq}}(z)\widetilde{\mathbf{G}}(z)$ where $\mathbf{G}(z) = \mathbf{E}^{-1}(z)$. The Wiener solution becomes

$$\mathbf{H}(z) = \mathbf{S_{xx}}(z)\Big(\mathbf{S_{xx}}(z) + \mathbf{G}(z)\mathbf{S_{qq}}(z)\widetilde{\mathbf{G}}(z)\Big)^{-1}$$

Case 3. Case When Wiener Filter Is Scalar

Continuing with case 2, assume further that the components $q_i(n)$ of $\mathbf{q}(n)$ are zero-mean and white, and uncorrelated for different values of i, and that they have equal variance, i.e., $\sigma_{q_i}^2 = \sigma_q^2$. In this case $\mathbf{S_{qq}}(z) = \sigma_q^2\mathbf{I}$. Equality of $\sigma_{q_i}^2$ arises for example when $\mathbf{E}(z)$ is paraunitary [1], and bits have been allocated optimally in the subbands (not taking into account the presence of $\mathbf{H}(z)$ during this allocation; see [1]). In this case

$$\mathbf{H}(z) = \mathbf{S_{xx}}(z)\Big(\mathbf{S_{xx}}(z) + \sigma_q^2\mathbf{G}(z)\widetilde{\mathbf{G}}(z)\Big)^{-1} \qquad (6.2)$$

In the paraunitary case, $\mathbf{G}(z)\widetilde{\mathbf{G}}(z) = \mathbf{I}$ so that

$$\mathbf{H}(z) = \mathbf{S_{xx}}(z)\Big(\mathbf{S_{xx}}(z) + \sigma_q^2\mathbf{I}\Big)^{-1} \qquad (6.3)$$

Pseudocirculant property. If the input $x(n)$ to the filter bank is WSS, then the vector process $\mathbf{x}(n)$ is not only WSS but also has a pseudocirculant power spectrum $\mathbf{S_{xx}}(z)$ [11]. And since the sums, products and inverses of pseudocirculants are pseudocirculants, we see that the Wiener filter $\mathbf{H}(z)$ in this case is a pseudocirculant. That is, it is a blocked version [1] of a scalar filter $H(z)$! In this case we can replace the matrix Wiener filter $\mathbf{H}(z)$ of Fig. 1.3(b) with the scalar Wiener filter $H(z)$ at the output of the filter bank, as shown in Fig. 6.1. If the 0th row of the matrix $\mathbf{H}(z)$ is given by

$$\begin{bmatrix} H_{00}(z) & H_{01}(z) & \ldots & H_{0,M-1}(z) \end{bmatrix} \qquad (6.4)$$

then the scalar filter $H(z)$ is

$$H(z) = \sum_{i=0}^{M-1} z^{-i} H_{0,i}(z^M). \qquad (6.5)$$

This is not surprising, since the preceding noise model assumption implies that the reconstruction error $x(n) - \widehat{x}(n)$, which is usually $(CWSS)_M$, degenerates into a white WSS random process uncorrelated to the signal $x(n)$. So the scalar Wiener filter, which is usually an $LPTV$ system (see discussions in Sec. 1) degenerates into an LTI system

$$H(z) = S_{xx}(z)/(S_{xx}(z) + \sigma_q^2) \qquad (6.6)$$

where $S_{xx}(z)$ is the power spectrum of the input $x(n)$.

Summary of Case 3. Suppose the subband noise sources $q_i(n)$ are (a) zero-mean, white and uncorrelated to each other, (b) uncorrelated to the unquantized signals $p_j(n)$, (c) and have identical variances σ_q^2 (as in the case of paraunitary filter banks with optimal bit allocation). Then we can replace the matrix Wiener filter $\mathbf{H}(z)$ of Fig. 1.3 with the above scalar Wiener filter $H(z)$, at the output of the filter bank.

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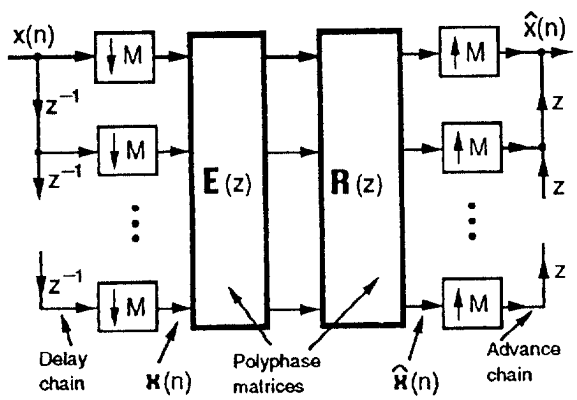

Fig. 1.1. The polyphase representation of a filter bank.

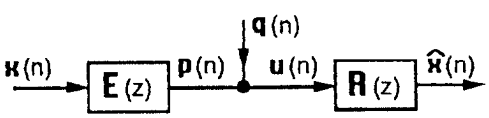

Fig. 1.2. Matrix model for a filter-bank with subband quantizers.

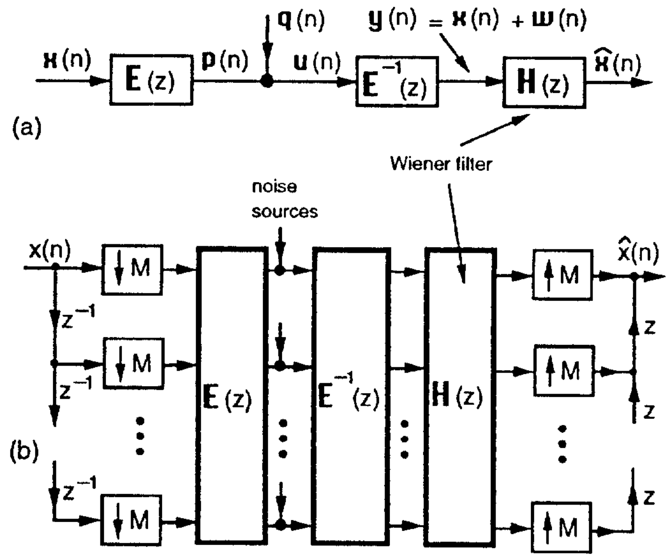

Fig. 1.3. Decomposing the synthesis bank into two parts. (a) Block diagram, and (b) internal details.

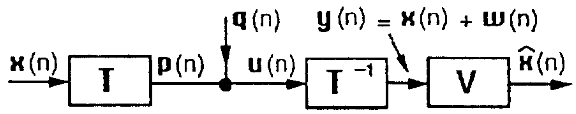

Fig. 3.1. The transform coder case.

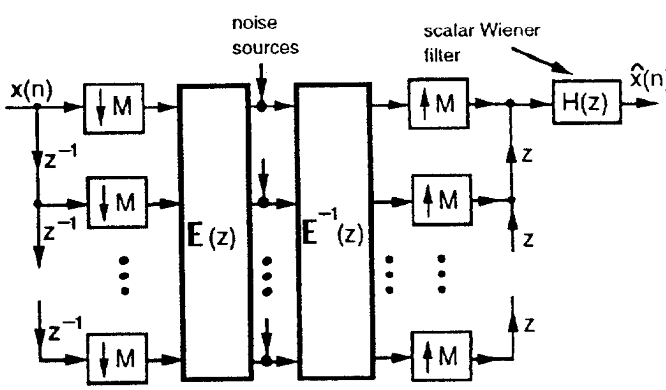

Fig. 6.1. The Wiener matrix reduced to a scalar LTI Wiener filter.