OVERSAMPLED GABOR EXPANSION INTO ONE-SIDED EXPONENTIAL FUNCTIONS

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Abstract

In this paper we consider the Gabor representation which uses a one-sided exponential window for detection and analysis of transient signals. Earlier results on the critically sampled case are extended to the more practically useful oversampled case. For oversampling by an integer factor we derive an explicit analytical expression for the dual window (dual frame) function required for computing the Gabor representation. Based on this expression we develop an efficient procedure for computing the Gabor coefficients. Finally, we demonstrate the performance of the method by numerical examples.

1. Introduction

This paper considers the Gabor representation for detection and analysis of transient signals [1]. Suppose we are given a continuous time signal $\{y(t), -\infty \leq t \leq \infty\}$ and a window function $g(t)$. The Gabor representation of $y(t)$ using the window function $g(t)$ is

$$y(t) = \sum_{m,n=-\infty}^{\infty} C_{mn} g_{mn}(t) \quad (1)$$

where

$$g_{mn}(t) = g(t - na) \exp[j2\pi mbt] \quad (2)$$

$a > 0$, $b > 0$ and $ab \leq 1$. The condition $ab \leq 1$ is necessary for the existence of the representation. $ab = 1$ is the case of critical sampling while $ab < 1$ is the oversampled case where $1/ab$ is the oversampling factor [2, 3]. In the case of critical sampling the representation is not always stable; for example when $g(x)$ is a Gaussian function [4, 5]. The importance of oversampling in the Gabor scheme was recognized and demonstrated by several authors [7, 8]. In this paper we focus therefore on the oversampled case.

As in [6, 10] we propose to use a one-sided exponential function as the Gabor window function $g(t)$. This function represents quite well the jump discontinuity and the gradual decay characterizing many physical transient phenomena. Also, we show that for the case of integer oversampling the dual window (frame) has finite support and therefore is most suitable for Gabor analysis of continuous data streams.

In this paper we derive an analytical expression for the dual frame associated with the one sided exponential window function for the oversampled case. Based on this expression we develop an efficient procedure for computing the Gabor coefficients. Then we demonstrate the performance of the method by numerical examples.

2. Mathematical Background

In this section we review the definition of the Zak transform [9] and the theory of frames [11] which are very useful tools in problems of Gabor representation.

The Zak Transform

The Zak transform of a signal $f(x)$ is defined as follows:

$$F(x, \omega) = \lambda^{1/2} \sum_{k=-\infty}^{\infty} f[\lambda(x + k)] \exp[-j2\pi\omega k] \quad (3)$$

where $-\infty < x, \omega < \infty$ and $\lambda > 0$ is a fixed parameter. The properties of the Zak transform required for this paper can be found in [7]

Frames

We now give a very short review of the theory of frames, focusing only on the definitions and results which will be used in the derivations. For more complete reviews the interested reader is referred to [11].

Definition 1 A sequence $\{h_n\}$ in a Hilbert space H constitutes a frame if there exist positive numbers A, B called frame bounds, $0 < A \leq B < \infty$ such that for all $h \in H$ we have

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$$A\|h\|^2 \le \sum_n |<h,h_n>|^2 \le B\|h\|^2$$

Definition 2 Given a frame $\{h_n\}$ in a Hilbert space H the frame operator S is defined as:

$$Sh = \sum_n <h,h_n> h_n \quad (4)$$

The following two corollaries summarize the results required in this paper.

Corollary 1 1. S is a bounded linear operator with $AI \le S \le BI$.

2. S is invertible and $B^{-1}I \le S^{-1} \le A^{-1}I$.

3. $\{S^{-1}h_n\}$ is a frame with bounds B^{-1}, A^{-1}, called the dual frame of $\{h_n\}$.

$AI \le S$ means $<AIh,h> \le <Sh,h>$ for all $h \in H$, where I is the identity operator.

Corollary 2 Every $h \in H$ can be written as

$$h = \sum_n <h,S^{-1}h_n> h_n \quad (5)$$

(5) which expresses h as a linear combination of the frame elements will play an important roll in deriving the Gabor representation.

Frames and the Gabor representation

In [2] Daubechies has shown that if $\{g_{mn}\}$ constitute a frame in $L^2(\mathcal{R})$, the dual frame has the form

$$S^{-1}g_{mn}(x) = \tilde{\gamma}_{mn}(x) = \tilde{\gamma}(x-na)\exp(j2\pi mbx) \quad (6)$$

where S is the frame operator

$$Sh = \sum_{mn} <h,g_{mn}> g_{mn} \quad (7)$$

and $\tilde{\gamma} = S^{-1}g$.

For rational oversampling factor $1/ab = q/p$ Zibulski and Zeevi [7] have shown that the Zak transform of $\tilde{\gamma}(x)$ with $\lambda = 1/b$, $\tilde{\Gamma}(x,\omega)$, satisfies the following equation.

$$\sum_{i=0}^{p-1}\sum_{l=0}^{q-1} G(x-l\frac{p}{q},\omega)G^*(x-l\frac{p}{q},\omega-a\frac{i}{p})\tilde{\Gamma}(x,\omega-\frac{i}{p}) = pG(x,\omega) \quad (8)$$

For $p=1$ (8) is reduced to

$$\tilde{\Gamma}(x,\omega) = \frac{G(x,\omega)}{\sum_{l=0}^{q-1}|G(x-\frac{l}{q},\omega)|^2} \quad (9)$$

A procedure for solving (8) for $p>1$ is outlined in [7]. We will assume $p=1$ restricting ourselves to oversampling by integers, *i.e.*, $1/ab = 1,2,3,\ldots$.

Clearly, $\tilde{\gamma}(x)$ satisfies the reconstruction formula (5). Therefore, for any $y(t)$ in $L^2(\mathcal{R})$

$$y(t) = \sum_{mn} <y,\tilde{\gamma}_{mn}> g_{mn}(t) \quad (10)$$

The coefficients of the Gabor representation of $y(t)$ are obtained by comparing (10) with (1). We get.

$$C_{mn} = <y,\tilde{\gamma}_{mn}> \quad (11)$$

Using the Poisson-sum formula Wexler and Raz have shown that the relation (10) leads to the biorthogonality constraint

$$\frac{1}{ab}\int_{-\infty}^{\infty} g(t)\tilde{\gamma}^*(t-n/b)\exp[-j2\pi m(1/a)t]dt = \delta(n)\delta(m) \quad (12)$$

Thus the dual frame is biorthogonal to the set of window functions. Note that for the ovesampled case the solution of (12) is non-unique [8]. The dual frame method gives us one possible solution for the biorthogonality constraint.

3. The dual frame for the one sided exponential window

In this section we derive the dual frame for the integer oversampled Gabor representation with one sided exponential window. The dual frame is a solution to the biorthogonality constraint (12). For the case of critical sampling our result coincides with the corresponding biorthogonal function obtained in [10]. Then we develop an efficient procedure for computing the Gabor coefficients using the dual frame.

The starting point for the derivation is (9) which expresses the Zak transform of the dual frame $\tilde{\Gamma}(x)$ in terms of the Zak transform of the window function $G(x)$ for the case of integer oversampling ratio. The following results hold therefore for integer oversampling, *i.e.*, $1/ab = 1,2,3,\ldots$.

Let $g(t)$ be the one-sided exponential function

$$g(t) = \sqrt{2\alpha}\exp(-\alpha t)u(t) \quad (13)$$

where $u(t)$ is the unit step function.

Using the result [7] that for integer oversampling factor $q \geq 1$, $\{g_{mn})$ is a frame if only if

$$0 \leq A \leq \sum_{l=0}^{q-1} |G(x - \frac{l}{q}, \omega)|^2 \leq B < \infty \qquad \text{a.e.} \quad (14)$$

where $G(x,\omega)$ is the Zak transform of $g(t)$, we have shown that for integer oversampling, $\{g_{mn}\}$ is a frame in $L^2(\mathcal{R})$. We therefore can apply (9) to $g(t)$. First, we compute the Zak transform of $g(t)$ as required by (9). The Zak transform is evaluated with parameter $\lambda = 1/b$. We get,

$$G^*(x,\omega) = \sqrt{\frac{2\alpha}{b}} \frac{\exp[-\frac{\alpha}{b}(x - \lfloor x \rfloor)] \exp(-\lfloor x \rfloor j2\pi\omega)}{1 - \exp(-\alpha/b + j2\pi\omega)} \qquad (15)$$

where * denotes the complex conjugate operator and $\lfloor x \rfloor$ is the largest integer not greater than x. Also,

$$|G(x - l/q, \omega)|^2 = \frac{2\alpha H_l^2(x)}{b|G_1(\omega)|^2} \qquad (16)$$

where

$$G_1(\omega) = 1 - \exp[-(\alpha/b + j2\pi\omega)] \qquad (17)$$

$$H_l(x) = \exp[-\frac{\alpha}{b}(x - l/q - \lfloor x - l/q \rfloor)] \qquad (18)$$

for $l = 0, \ldots, q-1$

Let

$$G_2(x,\omega) = \exp(\lfloor x \rfloor j2\pi\omega)$$

We can write $G(x,\omega)$ as

$$G(x,\omega) = \frac{\sqrt{2\alpha} H_0(x) G_2(x,\omega)}{\sqrt{b} G_1(\omega)} \qquad (19)$$

Substituting (19) and (16) into (9) we obtain the Zak transform of the dual frame. Now we can apply the inverse Zak transform with $\lambda = 1/b$ to get the desired dual frame. We get

$$\begin{aligned} \tilde{\gamma}(x) &= \sqrt{b} \int_0^1 \tilde{\Gamma}(xb,\omega) d\omega \\ &= \frac{b H_0(xb)}{\sqrt{2\alpha} \sum_{l=0}^{q-1} H_l^2(xb)} \\ &\quad [\delta(\lfloor xb \rfloor) - \exp(-\frac{\alpha}{b}) \delta(\lfloor xb + 1 \rfloor)] \qquad (20) \end{aligned}$$

After some computations we get the following expression for the dual frame.

$$\tilde{\gamma}(x) = \begin{cases} -k \exp\{\alpha[x - \frac{2(l+1)}{bq}]\} & -1 + \frac{l}{q} \leq xb < -1 + \frac{l+1}{q} \\ k \exp\{\alpha[x - \frac{2(l+1)}{bq}]\} & \frac{l}{q} \leq xb < \frac{l+1}{q} \\ 0 & \text{otherwise} \end{cases} \qquad (21)$$

where $l = 0, \ldots, q-1$ and

$$k = \frac{b[\exp(\frac{2\alpha}{bq}) - 1]}{\sqrt{2\alpha}[1 - \exp(-\frac{2\alpha}{b})]} \qquad (22)$$

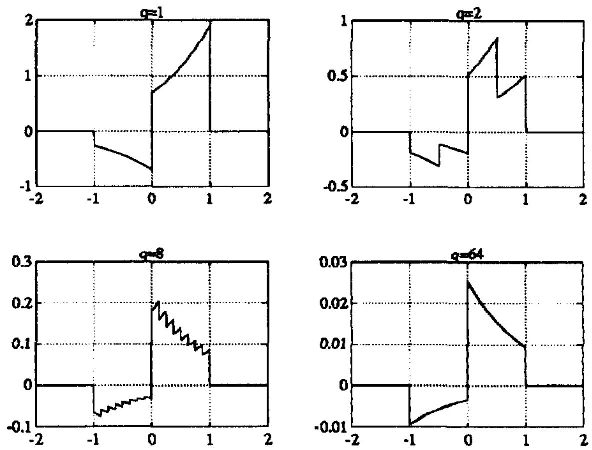

Figure 1 : The dual frame $\tilde{\gamma}(x)$ for the one-sided exponential window. $\alpha = 1$

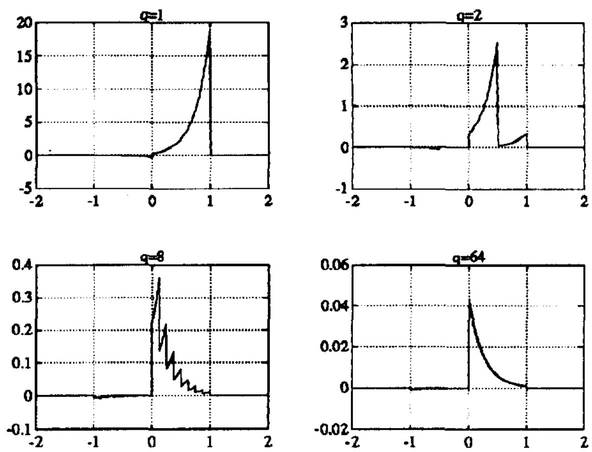

Figure 2 : The dual frame $\tilde{\gamma}(x)$ for the one-sided exponential window. $\alpha = 4$

As mentioned earlier $\tilde{\gamma}(x)$ is zero outside $[-\frac{1}{b}, \frac{1}{b})$. Within this interval it is piecewise continuous, with $2q$ equilength subintervals, in each of which $\tilde{\gamma}(x)$ is continuous.

In the following figures we show the function $\tilde{\gamma}(x)$ for several values of the oversampling q and the parameter α. In all figures, $b = 1$. Note that for a fixed q, changing b corresponds to changing the scaling of the time axis, such that $\tilde{\gamma}(x)$ is supported on $[-\frac{1}{b}, \frac{1}{b})$. In Figure 1 we plot $\tilde{\gamma}(x)$ for $\alpha = 1$, while Figure 2 is for $\alpha = 4$.

Having computed $\tilde{\gamma}(x)$ the Gabor coefficients can be computed by (11). We get

$$C_{mn} = \exp(-j\frac{2\pi mn}{q}) \int_{-1/b}^{1/b} y(t+na)\tilde{\gamma}^*(t)\exp[-j2\pi mbt]dt \quad (23)$$

In practice the coefficients $\{C_{mn}\}$ can be approximated by finite sums. Suppose we sample both the signal and the function $\tilde{\gamma}(t)$ at intervals $1/(Lb)$ starting from $t = -1/b$. Then,

$$C_{mn} \simeq \frac{1}{L}\exp(-j\frac{2\pi mn}{q}) \sum_{n'=-L}^{L-1} y(\frac{n'}{Lb}+na)\tilde{\gamma}^*(\frac{n'}{Lb})\exp(-j\frac{2\pi mn'}{L}) \quad (24)$$

Assume that L/q is an integer and let $y_s(l) = y(l/Lb - 1/b)$ and $\tilde{\gamma}_s(l) = \tilde{\gamma}(l/Lb - 1/b)$. y_s and $\tilde{\gamma}_s$ are sampled and shifted (by $1/b$) versions of y and $\tilde{\gamma}$ respectively. Then we can rewrite (24) as follows

$$C_{mn} \simeq \frac{1}{Lb}\exp(-j\frac{2\pi mn}{q}) \quad (25)$$
$$\sum_{k=0}^{1}\sum_{r=0}^{L-1} y_s(kL+r-nL/q)\tilde{\gamma}_s^*(kL+r)\exp(-j\frac{2\pi mr}{L})$$

The second summation is computed using L point FFT.

4. Numerical Examples

In this section we demonstrate the effect of oversampling on the performance of the above procedure. We assume that the received signal has the following form

$$y(t) = s(t) + v(t) = \sum_{i=1}^{4} A_i \exp[-\alpha(t-t_{0i}) + j2\pi f_i(t-t_{0i})]u(t-t_{0i}) + v(t) \quad (26)$$

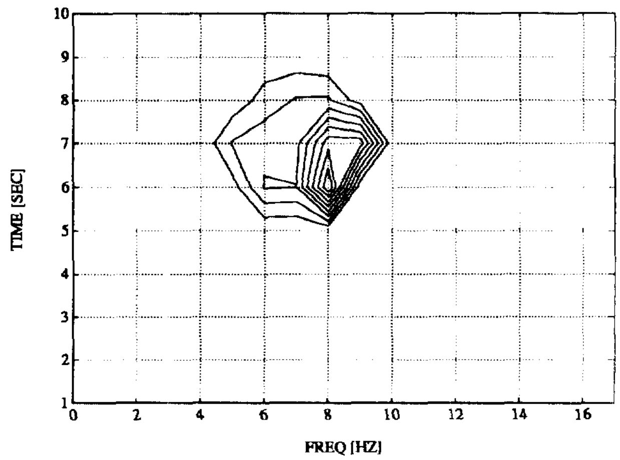

Figure 3 : Contour plot of the Gaborgram with critical sampling $a = b = 1$.

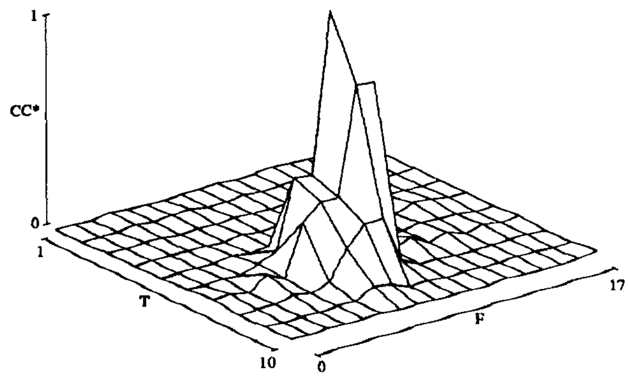

Figure 4 : The Gaborgram with critical sampling $a = b = 1$.

where $s(t)$ is a four component transient signal, $u(t)$ is the unit step function and the decay parameter is $\alpha = 1.25$. $\{A_i, t_{0i}, f_i\}$ are the amplitude, time of arrival and frequency of the i-th transient. $v(t)$ is the background noise which is assumed to be a zero mean white complex Gaussian random process with variance σ^2. The SNR is defined as $10\log(1/\sigma^2)$. In the following examples $\mathbf{A} = (1, 1.2, 1, 1.2)^T$, $\mathbf{t}_0 = (6, 6, 7.5, 7.5)^T$ seconds, $\mathbf{f} = (6.5, 8, 6.5, 8)^T$ Hz and SNR $= -3$dB. Note that three of the components do not lie on the critically sampled Gabor lattice. The received signal is sampled at 128 Hz and observed for a sixteen second interval.

The performance is illustrated by plotting the squared absolute value of the Gabor coefficients as a function of time and frequency. Such a plot is usually called a Gaborgram. Figures 3–4 show the Gabor-

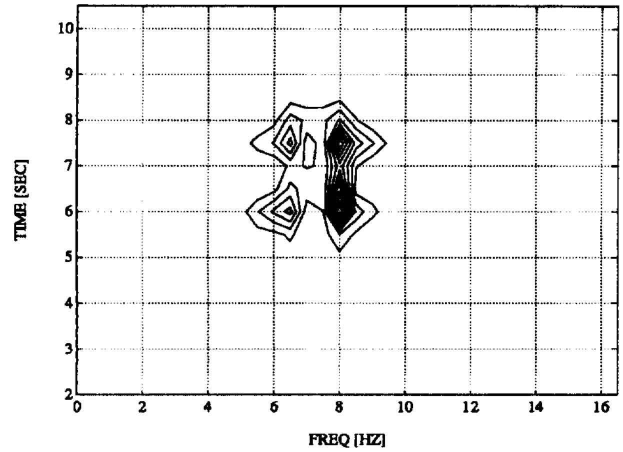

Figure 5 : Contour plot of the Gaborgram with oversampling $a = b = 0.5$.

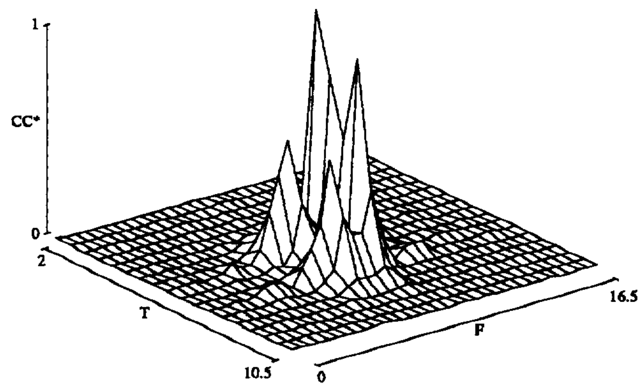

Figure 6 : The Gaborgram with oversampling $a = b = 0.5$.

gram obtained for critical sampling, while figures 5–6 show the Gaborgram for oversampling by a factor of 4 with $a = b = 0.5$. For each case we show a contour plot and a three dimensional plot of the Gaborgram.

As may be expected, the Gaborgram in the critical sampling case was not able to resolve the four transients. Recall that for critical sampling with $a = b = 1$ the Gabor functions (synthesis and analysis) correspond to integer values of time and frequency. Three of the transients comprising the received signal do not lie in the integer time-frequency grid, causing "non-integer" mismatch between the transient signal and the Gabor function. For oversampling with $a = b = 0.5$ the Gabor functions correspond to a grid sampled at integer multiples of 0.5 seconds in time and integer multiples of 0.5 Hz in frequency. In this case there is no mismatch between the transients and the Gabor basis functions and the Gaborgram resolves all four transients.

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