A Homomorphic Approach to Companding

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Abstract

The procedure of first COMpresssing and then exPANDing a signal is known as "companding." In PCM systems, commonly used in telephone switching networks, samples of an analog speech waveform are encoded as binary words and transmitted. Large dynamic range speech is most efficiently encoded and transmitted if the amplitude of the waveform is compressed before transmission and then expanded at the receiver. This paper describes a homomorphic approach to companding. It reviews and illustrates how these signals are processed and used in speech processing and communication networks.

1. Introduction

A companding system is shown in Fig. 1. The signal is s and the quantization noise and transmission channel noise is n. C is the compressor, E is the expandor, and H is the transmission channel and processing system. For a typical encoding scheme, the digitized code word is a truncated binary representation of the analog sample. Truncation error is very distinct for small signals. This is undesirable for voice since most of the information in speech signals require a wide dynamic range. Companding controls the dynamic range of any signal and improves signal-to-noise ratio (SNR) characteristics.

Given a compressor input x, the compressor output y and expandor output x equal

$$y = C(x), \qquad x = E(y) \tag{1}$$

where C is the compression law and E is the expansion law. For the original signal x to be perfectly recovered

$$x = E(y) = E(C(x)) = C^{-1}(C(x)) = x \tag{2}$$

so C and E must be inverse to each other. $E = C^{-1}$ must also be made single-valued.

2. Compander Laws

A p-law compander has the form $y = x^p$ so $x = y^{1/p}$. x is multiple-valued and is made single-valued using

$$y = sgn(x)\ |x|^p, \qquad x = sgn(y)\ |y|^{1/p} \tag{3}$$

$0 < p < 1$ is generally used for compression and $p > 1$ is used for expansion. Single-valuedness and sign retention is important for recovering the signal. pth roots generate multiple values. Therefore using magnitudes to avoid multiple values destroys the sign. Since expansion and compression is done on the magnitude of the signal, taking roots guarantees that the output will be single-valued. The *sgn* function will restore the sign and thus restore the original value of the signal.

The A-law compression characteristic is defined as

$$\begin{aligned} y &= sgn(x)\ \frac{A|x|}{1+log(A)}, && 0 < |x| < \frac{1}{A} \\ &= sgn(x)\ \frac{1+log(A|x|)}{1+log(A)}, && \frac{1}{A} < |x| < 1 \end{aligned} \tag{4}$$

It can be seen that the A-law characteristic is linear below $1/A$. The inverse characteristic equals

$$\begin{aligned} x &= sgn(y)\ \frac{|y|(1+log(A))}{A}, \qquad 0 < |y| < \frac{1}{1+log(A)} \\ &= sgn(y)\ \frac{exp(|y|(1+log(A))-1)}{A}, \\ &\qquad\qquad \frac{1}{1+log(A)} < |y| < 1 \end{aligned} \tag{5}$$

A-law characteristics usually provides a slightly larger dynamic range. y is the compressed output value and the input signal x is normalized between -1 and 1. The compression parameter A is usually 87.6 as reconmended by the CCITT.

The μ-law compression characteristic is defined as

$$y = sgn(x)\,\frac{log(1+\mu|x|)}{log(1+\mu)} \tag{6}$$

μ is the compression parameter which is 255 in North America. The reason that $\mu = 255$ companding is usually selected over the p and A companders, is simply because it improves the voice quality.

3. Estimation and Homomorphic Filters

Estimation filters separate signal and noise which have been combined using addition. The gain G of a Class 1 uncorrelated Wiener estimation filter equals

$$G = \frac{|F(s)|^2}{|F(s)|^2 + |F(n)|^2} \tag{7}$$

where $F(\;)$ is the Fourier transform operator. Homomorphic filters separate signal and noise which have been combined using some other operation other than addition. A homomorphic transformation $H(\;)$ converts a nonlinear operation $\otimes$ involving signal and noise into a linear operation as

$$\begin{aligned} H(k_1 s(t) \otimes k_2 n(t)) &= k_1 H(s(t)) + k_2 H(n(t)) \\ &= k_1 s_H(t) + k_2 n_H(t) \end{aligned} \tag{8}$$

The transformed signal $H(s(t))$ is denoted as $s_H(t)$ while the transformed noise $H(n(t))$ is denoted as $n_H(t)$. The inverse homomorphic transformation H^{-1} is used to recover the signal. Homomorphic filters can be used in companding systems to recover the signal as shown in Fig. 2. Using an uncorrelated estimation filter, the gain is

$$G = \frac{|F(s_H)|^2}{|F(s_H)|^2 + |F(n_H)|^2} \tag{9}$$

When s_H and/or n_H are complex-valued, better estimation is obtained using two-channel recovery. Expressing s_H and n_H in their real and imaginary parts, then each is processed separately using

$$\begin{aligned} G_r &= \frac{|F(Re(s_H))|^2}{|F(Re(s_H))|^2 + |F(Re(n_H))|^2} \\ G_i &= \frac{|F(Im(s_H))|^2}{|F(Im(s_H))|^2 + |F(Im(n_H))|^2} \end{aligned} \tag{10}$$

Alternatively the magnitude and phase can be processed separately. The rectangular or polar selection is based upon which representation produces the most dissimilarity between s_H and n_H.

4. Companding System

Now consider the companding system in Fig. 1. The signal s is companded using C to yield s_C. Then s_C is combined with additive noise to form the channel input r. The channel output is d and is formed using a linear filter whose impulse response is g or whose transfer function is G. It is often more computationally efficient to form the system G in the frequency transform domain using some suitable transform F (e.g. Fourier). The channel output d is expanded using E to form the system output c. The companding system equations are therefore

$$\begin{aligned} s_C(t) &= C(s(t)) \\ r(t) &= s_C(t) + n(t) \\ d(t) &= g(t) * r(t) = F^{-1}(G(j\omega)F(r(t))) \\ c(t) &= E(d(t)) \end{aligned} \tag{11}$$

where the * denotes convolution. From the input r equation in (11b), the Wiener filter of (7) using companded signal s_C and noise n becomes

$$G = \frac{|F(s_C)|^2}{|F(s_C)|^2 + |F(n)|^2}, \qquad g = F^{-1}(G) \tag{12}$$

For the two-channel system, Eq. 11c is replaced by

$$\begin{aligned} d(t) &= g_r(t) * Re(r(t)) + j g_i(t) * Im(r(t)) \\ &= F^{-1}(G_r(j\omega)F(Re(r(t))) + j\; G_i(j\omega)F(Im(r(t)))) \end{aligned} \tag{13}$$

Therefore substituting in the various companding laws given by (3), (4), and (6), we find the following:

$$\begin{aligned} p-\text{law}: \quad & G = \frac{|F(sgn(s)|s|^p)|^2}{|F(sgn(s)|s|^p)|^2 + |F(n)|^2} \\ A-\text{law}: \quad & G = \frac{|F(sgn(s)\frac{1+log(|As|)}{1+log(A)})|^2}{|F(sgn(s)\frac{1+log(|As|)}{1+log(A)})|^2 + |F(n)|^2} \\ \mu-\text{law}: \quad & G = \frac{|F(sgn(s)\frac{log(1+\mu|s|)}{log(1+\mu)})|^2}{|F(sgn(s)\frac{log(1+\mu|s|)}{log(1+\mu)})|^2 + |F(n)|^2} \end{aligned} \tag{14}$$

5. Homomorphic Companding Systems

An alternative approach is to use the homomorphic companding system in Fig. 2. This system is described by augmenting equations of (11) as

$$\begin{aligned} s_C(t) &= C(s(t)) \\ s_H(t) &= H(s_C(t)) \\ r(t) &= s_H(t) + n(t) \\ d_H(t) &= g(t) * r(t) = F^{-1}(G(j\omega)F(r(t))) \\ d(t) &= H^{-1}(d_H(t)) \\ c(t) &= E(d(t)) \end{aligned} \tag{15}$$

Using this approach for the p-law case, $s_H = log(s_C)$ and $d = exp(d_H)$. In this case, the Wiener estimation filter gain equals

$$G = \frac{|F(log(sgn(s)\ |s|^p))|^2}{|F(log(sgn(s)\ |s|^p))|^2 + |F(n)|^2} \tag{16}$$

from the $r(t)$ equation in (15c). G is best implemented using the two-channel form of (10).

Another approach is to use the amplitude domain companding system shown in Fig. 3. In this system the probability density functions of the signal and noise are processed. In MATLAB, this is accomplished using the "hist" command to empirically determine probability density functions or pdf. In this case, manipulating and augmenting the system equations of (11) gives

$$\begin{aligned} &f_s = pdf(C(s(t))), \qquad f_n = pdf(n(t)) \\ &F_s = F(f_s), \qquad F_n = F(f_n) \\ &r = log(F_s) + log(F_n), \quad F_r = F(f_r) \\ &D_H = G(j\omega)F_r, \qquad d_H = F^{-1}(F_r) \\ &r_d = F^{-1}(exp(d_H)) \\ &d(t) = pdf^{-1}(r_d) \\ &c(t) = E(d(t)) \\ &c_s(t) = sort(c(t)) \end{aligned} \tag{17}$$

where $F(\ )$ denotes the Fourier transform. The uncorrelated Wiener filter G used in this case is

$$\begin{aligned} G &= \frac{|log(F(pdf(s_C)))|^2}{|log(F(pdf(s_C)))|^2 + |log(F(pdf(n)))|^2} \\ g &= F^{-1}(G) \end{aligned} \tag{18}$$

from the r equation in (17). Like (16), G is best implemented using the two-channel form of (10). In the amplitude domain, the pdf's of additive signals are convolved so their Fourier transforms are multiplied and the logarithms of these transforms are added. The inverse pdf is implemented using a random number generator so $d(t)$ appears to be random. However this is not the case. It is simply an unsorted estimate of the compressed signal c. A proper sort algorithm removes the apparent randomness and produces the estimated expanded signal c_s.

6. Simulation Results

The companding systems of Figs. 1, 2, and 3 were simulated using MATLAB. A variety of signals were considered including triangular, sawtooth, sine, sin(x)/x, double pulse, double sawtooth, exponential, ramp, triangular, sine pulse, one-cycle sine, and four-cycle sine pulses. Essentially, Fig. 1 was encoded using (11)-(12), Fig. 2 using (15)-(16), and Fig. 3 using (17)-(18). The p-law was selected for its relative ease. The value of p was set to 0.5. The SNR was set to 0 dB. Three of the results are shown in Figs. 4 (triangular), 5 (sin(x)/x), and 6 (four-cycle sine pulses). Each figure shows the signal, companded signal, total input (companded signal plus white noise), and output (estimated signal). Good recovery of signal is made by the Figs. 1 and 2 systems but only fair recovery by the Fig. 3 system.

To make quantitative comparisons, the rms error of each output relative to the signal was computed using $rms = \sqrt{norm(signal - output)/N}$ and are listed in Table 1. The pdf method has about twice the rms error of the other two methods. of these two, there appears to be no systematic or consistent manner to predict which method is best. The output is noisier using the Fig. 1 method while the output is smoother using the Fig. 2 method.

7. Conclusion

A homomorphic approach to companding has been presented. Three different systems were described with the appropriate equations. p, A, and μ law companders were considered. These three systems were used to compand a variety of signals as listed in Table 1. Simulations showed good signal recovery using the Figs. 1 and 2 approaches. They also showed only fair pdf recovery using the Fig. 3 approach. The first two methods have about the same error performance while the third method about doubles the error. These results are reassuring since any of the three systems can be used for form a companding system. The first two appear preferable. We are encouraged by these results and are performing further companding research using Class 2 and 3 filtering techniques.

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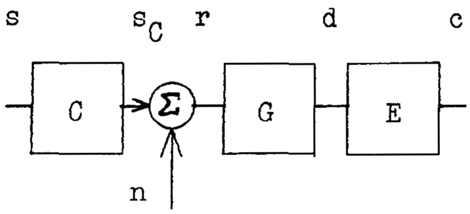

Fig. 1. Companding system including compressor C and expandor E.

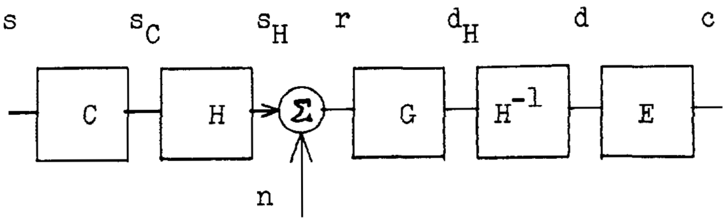

Fig. 2. Homomorphic companding system including homomorphic filters H and H^{-1}.

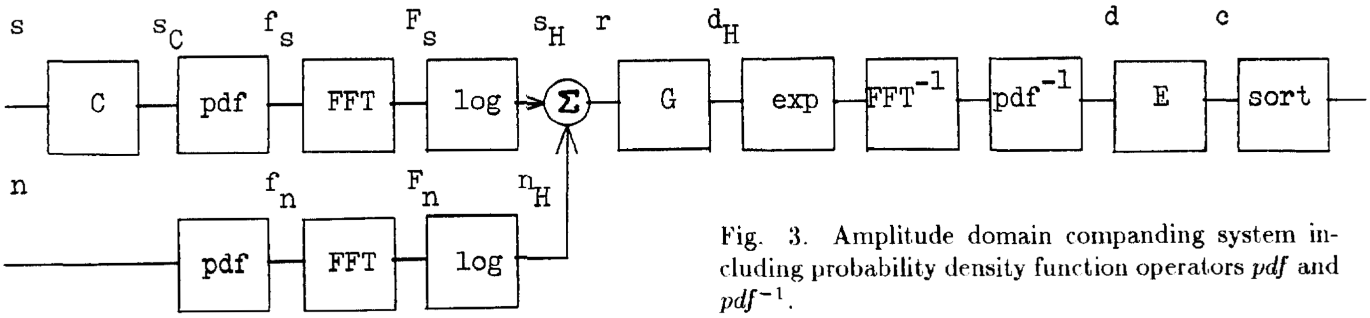

Fig. 3. Amplitude domain companding system including probability density function operators pdf and pdf^{-1}.

Signal type ($1v_{pk}$)	Fig. 1 (s_C)	Fig. 2 ($log(s_C)$)	Fig. 3 (pdf)
square	.178	.103	.231
triangular	.061	.078	.201
sawtooth	.108	.097	.207
sine	.108	.149	.227
six(x)/x	.092	.108	–
double square	.109	.105	.191
double saw	.141	.099	–
exponential	.109	.092	.289
ramp	.139	.100	.287
triangular	.072	.112	–
half sine	.111	.102	.258
1 sine pulse	.120	.207	.290
4 sine pulses	.120	.207	.270

Table 1. RMS error performance of Figs. 1, 2, and 3 for different signal types.

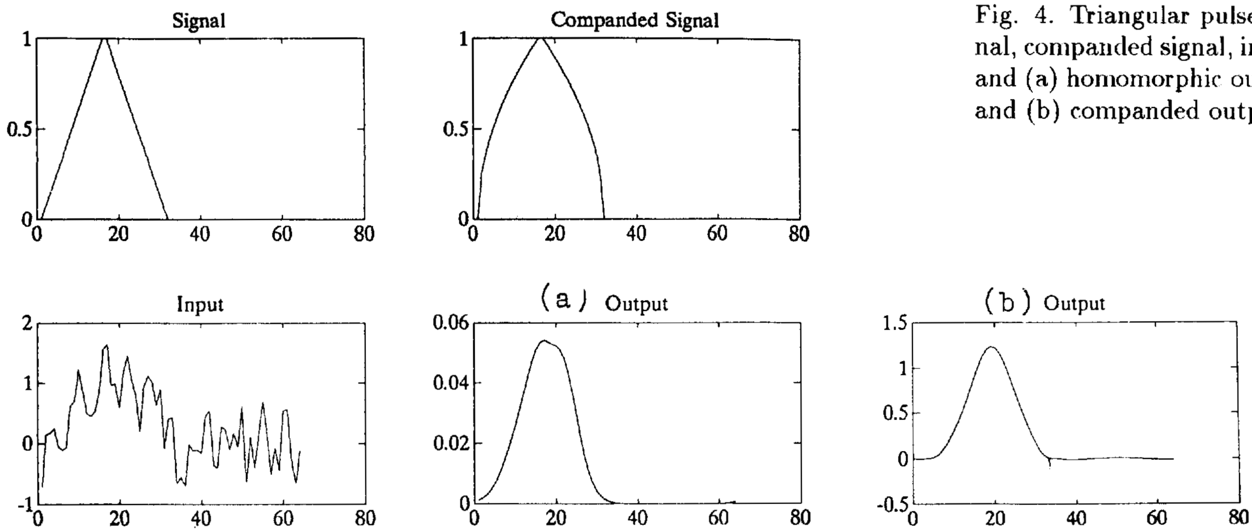

Fig. 4. Triangular pulse signal, companded signal, input, and (a) homomorphic output and (b) companded output.

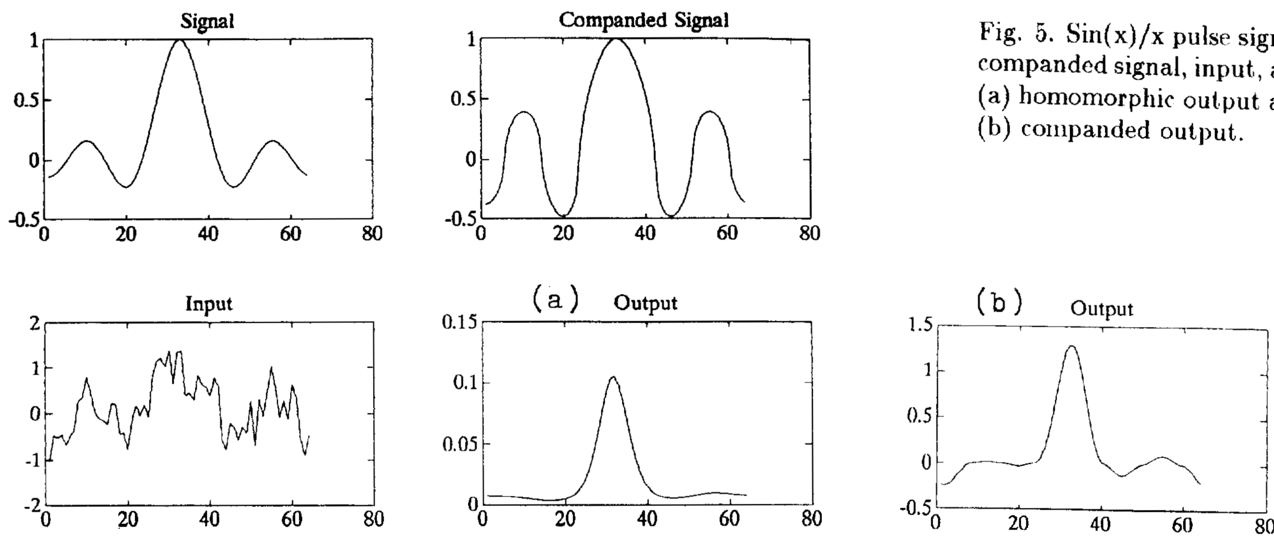

Fig. 5. Sin(x)/x pulse signal, companded signal, input, and (a) homomorphic output and (b) companded output.

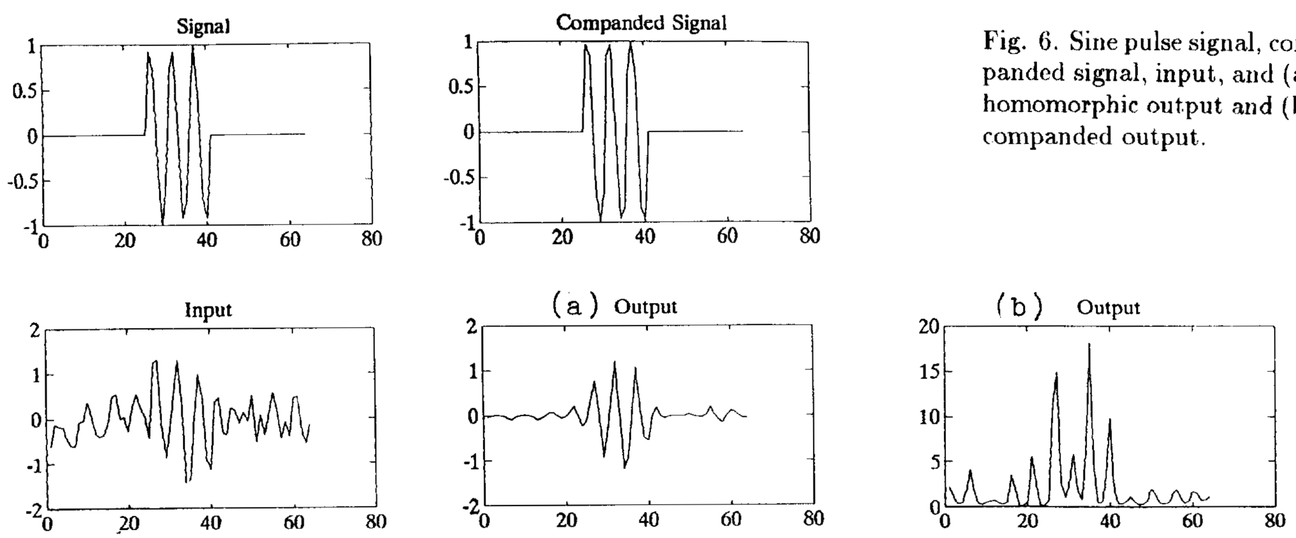

Fig. 6. Sine pulse signal, companded signal, input, and (a) homomorphic output and (b) companded output.