Direction-Finding Performance of the Multistage CM Array

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Abstract

The constant modulus (CM) array is a blind adaptive beamformer that steers nulls in the directions of cochannel interferers without requiring a training (pilot) signal. A cascade implementation of the system, known as the multistage CM array, is designed to recover ("copy") several narrowband cochannel signals and provide estimates of their angles of arrival. Each stage is composed of a CM array that "captures" one of the sources and an adaptive signal canceler that removes the source from the array input. This signal cancelation influences the capture and direction-finding performance of the remaining stages. In this paper, we quantify this behavior using a stochastic model of convergence and present computer simulation results for some example source scenarios.

1 Introduction

The first stage of the multistage CM array [1], [2], [3], is shown in Figure 1. The N antenna elements are equally spaced along a line with interelement spacing d. For narrowband sources, the input signal vector $\mathbf{x}(k) = [x_1(k), \ldots, x_N(k)]^T$ can be expressed as

$$\mathbf{x}(k) = \mathbf{A}\mathbf{s}(k) + \mathbf{n}(k), \tag{1}$$

which is a baseband model with source signal vector $\mathbf{s}(k) = [s_1(k), \ldots, s_L(k)]^T$ and noise vector $\mathbf{n}(k) = [n_1(k), \ldots, n_N(k)]^T$. The array response matrix $\mathbf{A}$ has columns (direction vectors) $\{\mathbf{a}_i\}$, $i = 1, \ldots, L$, and components $a_{m,i} = \exp(-j(m-1)\phi_i)$, $m = 1, \ldots, N$, where $j = \sqrt{-1}$. The phase angles are given by $\phi_i = 2\pi(d/\lambda)\sin(\theta_i)$ where $\{\theta_i\}$ are the source angles of arrival (AOAs) and λ is their common wavelength.

We assume that the source signals and noise terms have zero mean and are mutually uncorrelated. Their covariance matrices are $E[\mathbf{s}(k)\mathbf{s}^H(k)] = \mathbf{\Sigma}_s$ and $E[\mathbf{n}(k)\mathbf{n}^H(k)] = \sigma_n^2\mathbf{I}$ where $\mathbf{\Sigma}_s$ is a diagonal matrix with components (source powers) $\sigma_{s_i}^2 = E[|s_i(k)|^2]$, and σ_n^2 is the power of each noise term. Thus, from (1) we may write the input covariance matrix $\mathbf{R}_x = E[\mathbf{x}(k)\mathbf{x}^H(k)]$ as

$$\mathbf{R}_x = \mathbf{A}\mathbf{\Sigma}_s\mathbf{A}^H + \sigma_n^2\mathbf{I}. \tag{2}$$

In most direction-finding algorithms, an estimate of $\mathbf{R}_x$ is needed in order to estimate the columns of $\mathbf{A}$. The CM array, on the other hand, operates directly on the input vector $\mathbf{x}(k)$ in a real-time manner without requiring a direct estimate of $\mathbf{R}_x$. It provides estimates of the sources $\{s_i(k)\}$ as well as their direction vectors $\{\mathbf{a}_i\}$ [3], [4].

The output of the beamformer is computed as the inner product $y(k) = \mathbf{w}^H(k)\mathbf{x}(k)$ where $\mathbf{w}(k) = [w_1(k), \ldots, w_N(k)]^T$ are the CM array weights. They are updated by the constant modulus algorithm (CMA) [5] as follows:

$$\mathbf{w}(k+1) = \mathbf{w}(k) + 2\mu_{\text{cma}}\mathbf{x}(k)y^*(k)\left[1/|y(k)| - 1\right] \tag{3}$$

where $\mu_{\text{cma}} > 0$ is the step size and the superscript * denotes complex conjugate. Observe in the figure that the output is weighted by $\mathbf{u}(k) = [u_1(k), \ldots, u_N(k)]^T$, which corresponds to the adaptive signal canceler. The least-mean-square (LMS) algorithm [6] is employed to adjust these weights as follows:

$$\mathbf{u}(k+1) = \mathbf{u}(k) + 2\mu_{\text{lms}}y^*(k)\mathbf{e}(k) \tag{4}$$

where $\mathbf{e}(k) = \mathbf{x}(k) - \mathbf{u}(k)y(k)$ is an error vector that serves as the "effective" input for the second stage.

Without loss of generality, we assume that the first stage captures the first source signal $s_1(k)$. Note that the resolution of the CM array is identical to that of a conventional weight-and-sum beamformer. The issue addressed here involves the steady-state convergence behavior of the first stage and how its signal canceler influences the direction-finding performance of the second stage. It will be evident from this analysis that

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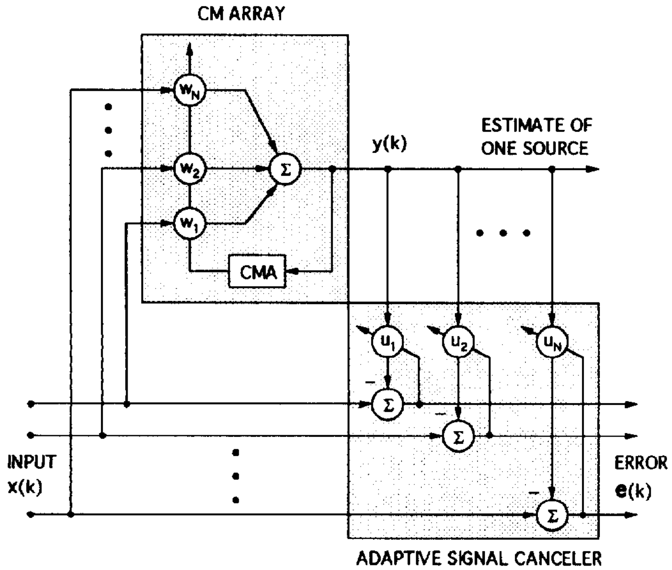

Figure 1: First-stage CM array and adaptive signal canceler with input $\mathbf{x}(k)$ and error $\mathbf{e}(k)$.

similar results may be concluded for all stages in the cascade system. We begin with a summary of the convergence results presented in [3], [4].

2 Effective Array Matrix

From a steady-state convergence analysis, the Wiener weights may be computed from the following orthogonality condition: $E[\mathbf{x}(k)(s_1(k) - y(k))^*] = \mathbf{0}$. This approach yields [3]

$$\mathbf{w} = \sigma_{s_1}^2 \mathbf{R}_x^{-1} \mathbf{a}_1 \tag{5}$$

where $\mathbf{a}_1$ is the first column of $\mathbf{A}$ and $\sigma_{s_1}^2$ is the power of $s_1(k)$. Similarly, the canceler weights at convergence are [3]

$$\mathbf{u} = \mathbf{R}_x \mathbf{w}/\sigma_y^2 = (\sigma_{s_1}^2/\sigma_y^2)\mathbf{a}_1 \tag{6}$$

where $\sigma_y^2 = \mathbf{w}^H \mathbf{R}_x \mathbf{w}$ is the array output variance. Note that we have substituted (5) to obtain the second equality; this result illustrates that the canceler weights may be used to estimate the AOA of $s_1(k)$.

At convergence we may write the error vector as follows:

$$\begin{aligned} \mathbf{e}(k) &= (\mathbf{I} - \mathbf{u}\mathbf{w}^H)\mathbf{x}(k) = \mathbf{T}\mathbf{x}(k) \\ &= \mathbf{A}_e \mathbf{s}(k) + \mathbf{T}\mathbf{n}(k) \end{aligned} \tag{7}$$

where we have defined the data transfer matrix $\mathbf{T} \triangleq \mathbf{I} - \mathbf{u}\mathbf{w}^H$ and the effective array matrix $\mathbf{A}_e \triangleq \mathbf{T}\mathbf{A}$. Substituting $\mathbf{u}$ and $\mathbf{w}$ yields [3]

$$\begin{aligned} \mathbf{A}_e &= \mathbf{A} - \mathbf{a}_1 [\beta_{1,1}, \beta_{1,2}, \ldots, \beta_{1,L}] \\ &= [\mathbf{0}, \mathbf{a}_2 - \beta_{1,2}\mathbf{a}_1, \ldots, \mathbf{a}_L - \beta_{1,L}\mathbf{a}_1] \end{aligned} \tag{8}$$

where $\beta_{1,i} \triangleq (\mathbf{a}_1^H \mathbf{R}_x^{-1} \mathbf{a}_i)/(\mathbf{a}_1^H \mathbf{R}_x^{-1} \mathbf{a}_1)$. The first subscript refers to the captured source $s_1(k)$, while the second subscript denotes any of the L sources. Note that $\beta_{1,1} = 1$ so that the first column of $\mathbf{A}_e$ is zero, indicating that exact cancelation of $s_1(k)$ has taken place. However, we also see that the other columns of $\mathbf{A}_e$ are shifted versions of the original columns of $\mathbf{A}$. Thus, the direction-finding performance of the remaining stages is linked to the magnitude of the shift factors $\{\beta_{1,i}\}$, $i = 2, \ldots, L$.[1]

It was demonstrated in [4] using the matrix inversion lemma [7] that the shift factors may be expressed as

$$\beta_{1,i} = \frac{g_{1,i} - \mathbf{g}_1^H(\mathbf{\Sigma}_{\mathrm{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_i}{N - \mathbf{g}_1^H(\mathbf{\Sigma}_{\mathrm{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_1} \tag{9}$$

where $\mathbf{G} \triangleq \mathbf{A}^H\mathbf{A}$, $\mathbf{g}_i$ is the i^{th} column of $\mathbf{G}$, and $g_{1,i}$ is the first element of $\mathbf{g}_i$. The source signal-to-noise ratios (SNRs) are collected in the diagonal matrix $\mathbf{\Sigma}_{\mathrm{SNR}} = \mathbf{\Sigma}_s/\sigma_n^2$. From this result, we see that the shift factors are influenced by the source AOAs (i.e., the inner products yielding the components of $\mathbf{G}$) and the noise power. In the following sections, we further investigate and quantify the behavior of $\{\beta_{1,i}\}$.

3 Closed-Form Expressions for $\{\beta_{1,i}\}$

In Appendix A, we present a derivation that leads to the following alternative closed-form expression for $\beta_{1,i}$:

$$\beta_{1,i} = \frac{\Delta_{11}g_{1,i} + \Delta_{21}g_{2,i} + \cdots + \Delta_{L1}g_{L,i}}{\Delta_{11}g_{1,1} + \Delta_{21}g_{2,1} + \cdots + \Delta_{L1}g_{L,1}}, \tag{10}$$

where Δ_{ij} is the cofactor of the $(i,j)^{th}$ element of $\mathbf{I} + \mathbf{G}\mathbf{\Sigma}_{\mathrm{SNR}}$. Note that matrix inverses are not needed in this result, unlike the definition of $\beta_{1,i}$ and the expression given in (9). Equation (10) holds for any $N \geq L$.

Consider $\mathbf{I} + \mathbf{G}\mathbf{\Sigma}_{\mathrm{SNR}}$, which we may write as

$$\mathbf{I} + \mathbf{G}\mathbf{\Sigma}_{\mathrm{SNR}} =$$

[1] Similar results can be derived for the shift factors associated with the other stages; thus it is clear that a loss in performance can accumulate from stage to stage if the various shift factors are not close to zero.

$$N\begin{pmatrix} \frac{1}{N}+\frac{\sigma_{s_1}^2}{\sigma_n^2} & \cdots & f(\phi_1,\phi_L)\frac{\sigma_{s_L}^2}{\sigma_n^2} \\ f(\phi_2,\phi_1)\frac{\sigma_{s_1}^2}{\sigma_n^2} & \cdots & f(\phi_2,\phi_L)\frac{\sigma_{s_L}^2}{\sigma_n^2} \\ \vdots & \vdots & \vdots \\ f(\phi_L,\phi_1)\frac{\sigma_{s_1}^2}{\sigma_n^2} & \cdots & \frac{1}{N}+\frac{\sigma_{s_L}^2}{\sigma_n^2} \end{pmatrix} \quad (11)$$

where

$$f(\phi_1,\phi_2) \triangleq \frac{1}{N}\frac{\sin[N(\phi_1-\phi_2)/2]}{\sin[(\phi_1-\phi_2)/2]}e^{j(N-1)(\phi_1-\phi_2)/2}. \quad (12)$$

Note that the cofactors $\{\Delta_{m1}\}$ and thus $\beta_{1,i}$ are all independent of the captured source power $\sigma_{s_1}^2$. We conclude this section with two cases ($L=2$ and $L=3$) which are easily evaluated from (10) and (11). In both cases, the number of antenna elements N is arbitrary (with the restriction that $N \geq L$).

3.1 Case of $L = 2$

For $L=2$, $\mathbf{I}+\mathbf{G}\mathbf{\Sigma}_{\mathrm{SNR}}$ is a 2×2 matrix; thus (10) reduces to

$$\beta_{1,2} = \frac{f(\phi_1,\phi_2)}{1+N(1-|f(\phi_1,\phi_2)|^2)(\frac{\sigma_{s_2}^2}{\sigma_n^2})}. \quad (13)$$

Because $|f(\phi_1,\phi_2)|^2 \leq 1$ it is clear that $|\beta_{1,2}|^2 \leq 1$.

3.2 Case of $L = 3$

For $L=3$, the cofactors are simple 2×2 determinants, and hence it is straightforward to show that

$$\beta_{1,2} = \frac{\left(\frac{\sigma_n^2}{\sigma_{s_2}^2}\right)\left(f(\phi_1,\phi_2)\left(N+\frac{\sigma_n^2}{\sigma_{s_3}^2}\right)-Nf(\phi_1,\phi_3)f(\phi_3,\phi_2)\right)}{\left\{\left(N+\frac{\sigma_n^2}{\sigma_{s_2}^2}\right)\left(N+\frac{\sigma_n^2}{\sigma_{s_3}^2}\right)-N^2|f(\phi_2,\phi_3)|^2 +2N^2 f(\phi_1,\phi_2)f(\phi_2,\phi_3)f(\phi_3,\phi_1) -N|f(\phi_1,\phi_2)|^2\left(N+\frac{\sigma_n^2}{\sigma_{s_3}^2}\right) -N|f(\phi_1,\phi_3)|^2\left(N+\frac{\sigma_n^2}{\sigma_{s_2}^2}\right)\right\}}. \quad (14)$$

A similar formula for $\beta_{1,3}$ is also easily derived. Depending on the source AOAs, $|\beta_{1,i}|^2$ is not restricted to be less than unity[2] (as it was for $L=2$ above).

4 Analysis of $\{\beta_{1,i}\}$

We prove in this section that for a low noise variance, $\beta_{1,i} \approx 0$ provided the source AOAs are well separated [8]. Recall that the Wiener weight vector for the first-stage array is $\mathbf{w} = \sigma_{s_1}^2\mathbf{R}_x^{-1}\mathbf{a}_1$. Let $\{\mathbf{d}_1,\ldots,\mathbf{d}_N\}$ be the eigenvectors of $\mathbf{R}_x$ such that[3]

$$\mathbf{A}^H\mathbf{d}_j = \mathbf{0}, \quad L+1 \leq j \leq N. \quad (15)$$

The $\{\mathbf{d}_j\}$ can always be chosen to form an orthonormal basis in $\mathcal{R}^N$ (because $\mathbf{R}_x$ is Hermitian). Define the square matrix

$$\mathbf{D} \triangleq [\mathbf{d}_1,\ldots,\mathbf{d}_N]. \quad (16)$$

Thus, we may write

$$\mathbf{R}_x = \mathbf{D}(\mathbf{\Sigma}+\sigma_n^2\mathbf{I})\mathbf{D}^H \quad (17)$$

where

$$\mathbf{\Sigma} = \mathrm{diag}(\nu_1,\ldots,\nu_L,0,\ldots,0) \quad (18)$$

is a diagonal matrix containing the eigenvalues of $\mathbf{A}\mathbf{\Sigma}_s\mathbf{A}^H$. From this decomposition, we have

$$\mathbf{R}_x^{-1} = \mathbf{D}\mathrm{diag}\left(\frac{1}{\nu_1+\sigma_n^2},\ldots,\frac{1}{\nu_L+\sigma_n^2},\frac{1}{\sigma_n^2},\ldots,\frac{1}{\sigma_n^2}\right)\mathbf{D}^H, \quad (19)$$

and thus

$$\mathbf{w} = \sigma_{s_1}^2\mathbf{D}\mathrm{diag}\left(\frac{1}{\nu_1+\sigma_n^2},\ldots,\frac{1}{\nu_L+\sigma_n^2},\frac{1}{\sigma_n^2},\ldots,\frac{1}{\sigma_n^2}\right)\mathbf{D}^H\mathbf{a}_1. \quad (20)$$

From the orthogonality condition in (15), it is clear that

$$\mathbf{D}^H\mathbf{a}_1 = \left[\mathbf{a}_1^H\mathbf{d}_1,\ldots,\mathbf{a}_1^H\mathbf{d}_L,0,\ldots,0\right]^H, \quad (21)$$

which yields

$$\begin{aligned}\mathbf{w} &= \sigma_{s_1}^2[\mathbf{d}_1,\ldots,\mathbf{d}_N] \times \left[\frac{\mathbf{d}_1^H\mathbf{a}_1}{\nu_1+\sigma_n^2},\ldots,\frac{\mathbf{d}_L^H\mathbf{a}_1}{\nu_L+\sigma_n^2},0,\ldots,0\right]^T \\ &= \sigma_{s_1}^2\sum_{k=1}^{L}\frac{\mathbf{d}_k^H\mathbf{a}_1}{\nu_k+\sigma_n^2}\mathbf{d}_k. \end{aligned} \quad (22)$$

Recall that $\mathbf{w}$ also satisfies the Wiener-Hopf equation:

$$(\mathbf{A}\mathbf{\Sigma}_s\mathbf{A}^H+\sigma_n^2\mathbf{I})\,\mathbf{w} = \sigma_{s_1}^2\mathbf{a}_1 \quad (23)$$

where we have substituted (2) for $\mathbf{R}_x$.

[2] In fact, $|\beta_{1,i}|^2$ is not necessarily bounded by one for $L \geq 3$.

[3] Equation (15) simply states that the noise subspace is orthogonal to the signal subspace.

Before proceeding with the derivation, we first demonstrate that if a vector $\mathbf{v}$ is a solution of $\mathbf{A}\mathbf{\Sigma}_s\mathbf{A}^H\mathbf{v} = \sigma_{s_1}^2\mathbf{a}_1$, then $\mathbf{a}_i^H\mathbf{v} = 0$ for $i \neq 1$ (this equation corresponds to (23) with $\sigma_n^2 = 0$). Observe that

$$\begin{aligned} \sigma_{s_1}^2\mathbf{a}_1 &= \mathbf{A}\mathbf{\Sigma}_s\mathbf{A}^H\mathbf{v} \\ &= \left[\sigma_{s_1}^2\mathbf{a}_1, \ldots, \sigma_{s_L}^2\mathbf{a}_L\right]\mathbf{A}^H\mathbf{v}. \end{aligned} \quad (24)$$

Since $\mathbf{A}$ is full rank ($= L$), we must have that

$$\mathbf{A}^H\mathbf{v} = [1, 0, \ldots, 0]^T. \quad (25)$$

Thus, all columns of $\mathbf{A}$ except the first column are orthogonal to the solution vector $\mathbf{v}$.

Applying this result to $\mathbf{w}$ in (22) with $\sigma_n^2 = 0$ yields

$$\sigma_{s_1}^2\mathbf{a}_i^H\sum_{k=1}^{L}\frac{\mathbf{d}_k^H\mathbf{a}_1}{\nu_k}\mathbf{d}_k = \delta_{1,i}, \quad (26)$$

where $\delta_{1,i}$ is the Kronecker delta function. Consequently, we find that

$$\begin{aligned} \mathbf{w}^H\mathbf{a}_i &= \sigma_{s_1}^2\sum_{k=1}^{L}\frac{\mathbf{a}_1^H\mathbf{d}_k}{\nu_k+\sigma_n^2}\mathbf{d}_k^H\mathbf{a}_i \\ &= \sigma_{s_1}^2\sum_{k=1}^{L}\frac{\mathbf{a}_1^H\mathbf{d}_k\mathbf{d}_k^H\mathbf{a}_i}{\nu_k+\sigma_n^2} - \\ &\quad \sigma_{s_1}^2\sum_{k=1}^{L}\frac{\mathbf{a}_1^H\mathbf{d}_k\mathbf{d}_k^H\mathbf{a}_i}{\nu_k} + \delta_{1,i} \\ &= \delta_{1,i} - \sigma_n^2\sigma_{s_1}^2\sum_{k=1}^{L}\frac{\mathbf{a}_1^H\mathbf{d}_k\mathbf{d}_k^H\mathbf{a}_i}{\nu_k(\nu_k+\sigma_n^2)}. \end{aligned} \quad (27)$$

The second equality follows because of (26) and the fact that the $\{\nu_k\}$ are real-valued (since they are the eigenvalues of a Hermitian matrix). The result in (27) corresponds to the numerator of $\beta_{1,i}$, i.e., recall that $\beta_{1,i} = (\mathbf{a}_1^H\mathbf{R}_x^{-1}\mathbf{a}_i)/(\mathbf{a}_1^H\mathbf{R}_x^{-1}\mathbf{a}_1) = (\mathbf{w}^H\mathbf{a}_i)/(\mathbf{w}^H\mathbf{a}_1)$. The numerator of $\beta_{1,i}$ is proportional to σ_n^2 (for $i \neq 1$), while the denominator is close to 1 for small σ_n^2. Therefore, $\beta_{1,i} \approx 0$ ($i \neq 1$) for moderate-to-high SNRs and when the sources are not closely spaced. The denominator of (27) is small only when two or more sources are closely spaced, such that one or more of the eigenvalues $\{\nu_k\}$ approaches zero. These results are verified by the examples presented in the next section.

5 Relationship Between the Shift Factors and the Beampatterns

The beampattern of the array weights $\mathbf{w}$ as a function of the angle of arrival θ is defined by

$$B(\theta) \triangleq |\mathbf{w}^H\mathbf{a}(\theta)|^2 \quad (28)$$

where $\mathbf{a}(\theta) = [1, e^{-j\phi}, \ldots, e^{-j(N-1)\phi}]^T$ and $\phi = 2\pi(d/\lambda)\sin(\theta)$ is the corresponding phase. This expression is a measure of the array gain over the range of possible source directions. Figure 2 gives three example beampatterns for $N = 3$ and $L = 2$. They are normalized by the array gain of the captured source at $\theta_1 = -10°$. It is evident from (28) and the discussion following (27) that $\beta_{1,i}$ is related to the beampattern as follows:

$$|\beta_{1,i}|^2 = \frac{B(\theta_i)}{B(\theta_1)}, \quad (29)$$

i.e., the shift factor associated with the i^{th} column of $\mathbf{A}$ (and thus the i^{th} source) is equal to the ratio of the array gain at θ_i to that at θ_1 (corresponding to the captured source). This interpretation clearly illustrates why $\beta_{1,i} \approx 0$: if the sources are not closely spaced, there will be a null at θ_i while the gain at θ_1 will be significantly greater. Figure 3 shows an example of $|\beta_{1,2}|^2$ as a function of θ_2. Note that specific values of θ_2 in Figure 3 correspond to the beampatterns in Figure 2. Thus, we may view the shift factors as a "summary" of the beampatterns over the range of angles of the second source.

For example, observe from Figure 2(a) that when $s_1(k)$ and $s_2(k)$ are not close in angle, the ratio of the corresponding array gains is close to zero; this result is confirmed by the plot of $|\beta_{1,2}|^2$ in Figure 3. However, as θ_2 approaches θ_1 (Figures 2(b) and 2(c)), this ratio increases (as indicated by the parallel lines in the beampattern plots) so that $|\beta_{1,2}|^2$ approaches unity.[4] Thus, the $\{\beta_{1,i}\}$ are also a measure of the resolution of the array for a given source scenario.

6 Concluding Remarks

We have presented an analysis of the direction-finding performance of the multistage CM array. Each stage consists of a CM array that captures one of the source signals, and an adaptive signal canceler that removes it from the input before subsequent processing. The canceler weight vector also provides an estimate of the source direction (angle of arrival). Although

[4] Note that by increasing N we can improve the resolution of the array.

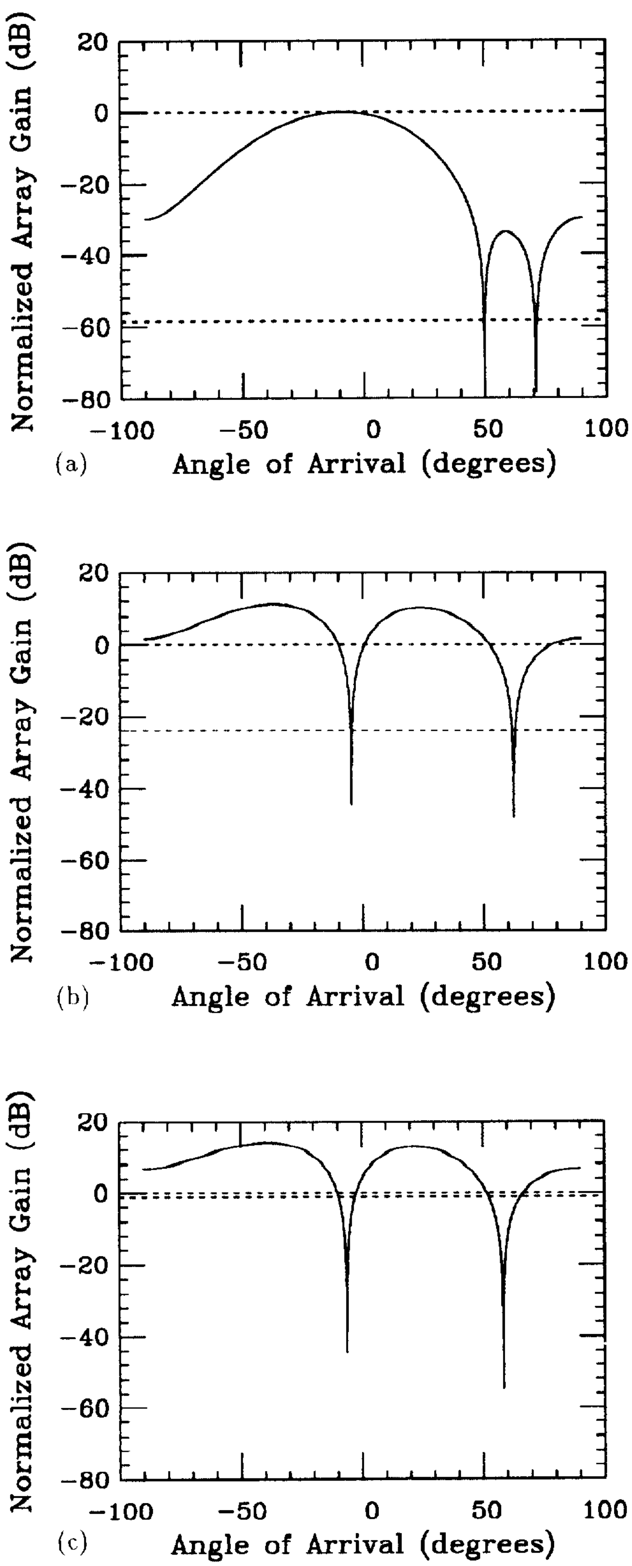

Figure 2: Example beampatterns. $N = 3$, $L = 2$, $\theta_1 = -10°$, $\sigma_n^2 = 0.01$. (a) $\theta_2 = 50°$. (b) $\theta_2 = -5°$. (c) $\theta_2 = -9.5°$.

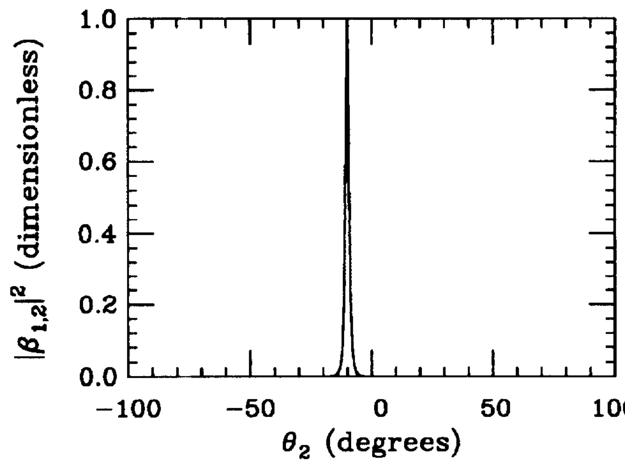

Figure 3: $|\beta_{1,2}|^2$ versus θ_2. $N = 3$, $L = 2$, $\theta_1 = -10°$, $\sigma_n^2 = 0.01$.

we focused on the first stage, similar results may be derived for any stage along the cascade system [8].

Based on a steady-state convergence model, it can be shown that the effective array matrix $\mathbf{A}_e$ for the next stage has a zero column (corresponding to the captured source), while all other columns are shifted versions of those of the original array matrix $\mathbf{A}$. Thus, if the shift factors are significant, the direction-finding performance of the following stages (via the signal cancelers) will be affected; i.e., the estimated angles of arrival will not be accurate.

We demonstrated that the shift factors $\{\beta_{1,i}\}$ are proportional to the noise variance σ_n^2 provided the sources AOAs are not too close. Thus, for moderate-to-high SNRs, we can expect that the AOA estimates obtained by the other stages will be reasonably accurate. We also showed that $|\beta_{1,i}|^2$ provides information about the array gains (beampatterns) at θ_1 and θ_i. The shift factors are a convenient summary of the antenna gain pattern over the range of source angles of arrival; in effect, they represent a measure of the resolution of the CM array.

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A Appendix: Evaluation of $\{\beta_{1,i}\}$

Because $\mathbf{\Sigma}_{\text{SNR}}$ and $\mathbf{G}$ are both Hermitian, we may write the identity

$$(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^H(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1} = \mathbf{I}. \tag{A.1}$$

Expanding this expression yields

$$\mathbf{\Sigma}_{\text{SNR}}^{-1}(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1} + \begin{pmatrix} \mathbf{g}_1^H \\ \mathbf{g}_2^H \\ \vdots \\ \mathbf{g}_L^H \end{pmatrix} (\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1} = \mathbf{I}, \tag{A.2}$$

and postmultiplying by $\mathbf{g}_i$, we have

$$\mathbf{\Sigma}_{\text{SNR}}^{-1}(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_i + \begin{pmatrix} \mathbf{g}_1^H \\ \mathbf{g}_2^H \\ \vdots \\ \mathbf{g}_L^H \end{pmatrix} (\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_i = \mathbf{g}_i. \tag{A.3}$$

Choosing the first element of both sides of (A.3) and rearranging it yields

$$\mathbf{g}_1^H(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_i = \left\{\left[\mathbf{I} - \mathbf{\Sigma}_{\text{SNR}}^{-1}(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1}\right]\mathbf{g}_i\right\}_1 \tag{A.4}$$

where the subscript $_1$ denotes the first element of the quantity in the braces. This implies that the numerator of (9) can be written as

$$g_{1,i} - \mathbf{g}_1^H(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_i = \left\{\mathbf{\Sigma}_{\text{SNR}}^{-1}(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_i\right\}_1. \tag{A.5}$$

Similarly, the denominator of (9) is

$$N - \mathbf{g}_1^H(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_1 = \left\{\mathbf{\Sigma}_{\text{SNR}}^{-1}(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_1\right\}_1. \tag{A.6}$$

Combining these two expressions yields

$$\beta_{1,i} = \frac{\left\{\mathbf{\Sigma}_{\text{SNR}}^{-1}(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_i\right\}_1}{\left\{\mathbf{\Sigma}_{\text{SNR}}^{-1}(\mathbf{\Sigma}_{\text{SNR}}^{-1} + \mathbf{G})^{-1}\mathbf{g}_1\right\}_1}. \tag{A.7}$$

Using the fact that the inverse of a matrix is its adjugate matrix divided by its determinant, (A.7) becomes

$$\beta_{1,i} = \frac{\{\text{adj}(\mathbf{I} + \mathbf{G}\mathbf{\Sigma}_{\text{SNR}})\mathbf{g}_i\}_1}{\{\text{adj}(\mathbf{I} + \mathbf{G}\mathbf{\Sigma}_{\text{SNR}})\mathbf{g}_1\}_1}. \tag{A.8}$$

Denote the cofactor of the $(i,j)^{th}$ element of $\mathbf{I} + \mathbf{G}\mathbf{\Sigma}_{\text{SNR}}$ by Δ_{ij}. Then

$$\text{adj}(\mathbf{I} + \mathbf{G}\mathbf{\Sigma}_{\text{SNR}}) = \begin{pmatrix} \Delta_{11} & \Delta_{21} & \cdots & \Delta_{L1} \\ \Delta_{12} & \Delta_{22} & \cdots & \Delta_{L2} \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{1L} & \Delta_{2L} & \cdots & \Delta_{LL} \end{pmatrix}, \tag{A.9}$$

which, when combined with (A.8), leads to the final result in (10).

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