Estimation of the Spatial and Temporal Frequencies of Multiple Sinusoids using a Sparse Linear Array *

Donald W. Tufts Hongya Ge R. Kumaresan

Dept. of Electrical Engineering
University of Rhode Island
Kingston, RI, 02881

Abstract

In this paper, we extend our previous sparse linear prediction (SLP) method [1,2,3] *for two-dimensional (2-D) applications. The temporal and spatial frequencies of multiple sinusoids are simultaneously estimated in pairs, rather than separately, from the outputs of a sparse linear array. Possible ambiguities due to the large interelement spacings can be resolved by properly choosing the interelement spacings and by incorporating different SLP polynomials, the coefficients of which can be calculated in parallel. The temporal and spatial frequencies are simultaneously estimated from the minima of a 2-D objective function, which can be calculated using a 2-D DFT of the properly mapped 2-D SLP coefficient matrix.*

1 Introduction

Two-dimensional (2-D) signal processing is of importance in sonar, radar, geophysics, and tomographic imaging applications. Two of the important problems in 2-D signal processing are (1) to obtain spatial and temporal frequencies in pairs, rather than as separate lists to be paired and (2) to resolve closely spaced 2-D signals with fewer computations. The traditional 2-D periodogram requires many sensor-element channels and its spectral resolution is limited. Due to computational simplicity and potential for high resolution, linear prediction (LP) methods have received much attention, especially in one-dimensional (1-D) signal processing [5,6].

In order to further reduce the amount of computation and provide insensitivity to noise, we proposed a sparse linear prediction (SLP) method [1,2,3], which was motivated by the results of Lank, Reed and Pollon [4], analyzed its ambiguity removal property, and applied it to estimate the frequencies of temporal and spatial sinusoids. In [3], the temporal frequencies are treated as nuisance parameters and are suppressed by the algorithm. In practice, if the temporal frequencies were not known in advance, one could separately estimate them and pair them with the estimated spatial frequencies.

Alternatively, using the recommendations of Jackson and Chien for reducing bias [7], Kumaresan and Tufts proposed a computationally simple method for jointly estimating pairs of spatial and temporal frequencies of multiple sinusoids from dense, Nyquist-rate 2-D space/time data [8]. Efron and Tufts improved the performance of this method, at the expense of more computation, using smaller support regions for linear prediction and singular value decomposition of data matrices [9].

In the work of this paper, we reduce the required number of sensors channels (e. g. antennas, hydrophones, or microphones) required for the two-dimensional linear prediction (2-D LP) methods of Kumaresan, Tufts, and Efron. That is, we use sparse sampling in space and we propose a two-dimensional sparse linear prediction (2-D SLP) method. After solving for the SLP coefficient vector, we then map the SLP coefficient vector into a 2-D SLP coefficient matrix. Then the temporal and spatial frequencies of the source signals can be obtained from the roots of the 2-D Sparse Linear Prediction (SLP) polynomial. Similar to the case of the sparse linear array [3], ambiguity in spatial frequency occurs due to the use of large interelement spacings. This ambiguity can be resolved by combining different 2-D SLP polynomials obtained from different sets of SLP equations. We further ease the computation of rooting a 2-D polynomial by using the 2-D DFT on the properly mapped 2-D

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SLP coefficient matrices.

2 2-D sparse linear prediction and ambiguity

In this section, we specify our 2-D SLP method for *simultaneously* resolving and estimating the temporal and spatial frequencies of multiple 2-D sinusoidal signals. To ease the understanding of the 2-D SLP method, we consider the special case of using the three-sensor sparse linear array proposed in [3]. We first show how to estimate both temporal and spatial frequencies of a single source signal unambiguously using 2-D SLP. We then notice that the same three-sensor arrangement and the 2-D SLP method can be used to estimate the temporal and spatial frequencies of two source signals as long as the temporal frequencies of the two source signals are not equal. The available 2-D data, which consists of N snapshots from a three-sensor sparse linear array, can be described by the following model:

$$y(n,m) = \sum_{i=1}^{2} c_i\, e^{j\,(\omega_i\, n+\nu_i\, m)} + w(n,\, m)\,,$$

$$\text{time sample points: } n = 0,\, 1,\, \ldots,\, N-1\,;$$

$$\text{space sample points: } m = 0,\, K_1,\, K_2\,. \quad (1)$$

where the c_i's are the unknown complex scalars; $w(n,m)$ is independent identically distributed zero-mean complex noise with variance $\sigma_w^2 = 2\sigma^2$; n is a time index associated with the unknown temporal frequencies ω_i's; and m is a space index associated with the unknown spatial frequencies ν_i's; the K_i's are the spacings between the ith sensor and the reference sensor. Note that the available data consists of noisy samples of a linear combination of 2-D sinusoids, which is densely sampled in time and sparsely sampled in space.

Our problem is to estimate the unknown pairs of spatial and temporal frequencies $(\nu_i,\, \omega_i)$'s, of the 2-D sinusoids from the available $N \times 3$ data array. Since these frequencies are the only non-linearly entering parameters, once they are estimated, the linearly entering complex scalars can be easily estimated by a linear fitting [10, 11].

The formation of the 2-D SLP scheme is shown in Figure 1. In this scheme, we first use data from the sensor at $m = K_1$ and the reference sensor at $m = 0$ to establish the forward and backward sparse linear prediction (FBSLP) equations [5, 6]. We can use the data inside the region specified by dashed lines, i. e. two widely spaced columns of the densely sampled data of [7, 8, 9], to set up each FBSLP equation. By moving the dashed line specified region downward, more FBSLP equations can be obtained. Two quadrant prediction-error filters $H_1(z_1, z_2)$ and $H_2(z_1, z_2)$ are used to make the estimates less biased as suggested in [7, 8, 9].

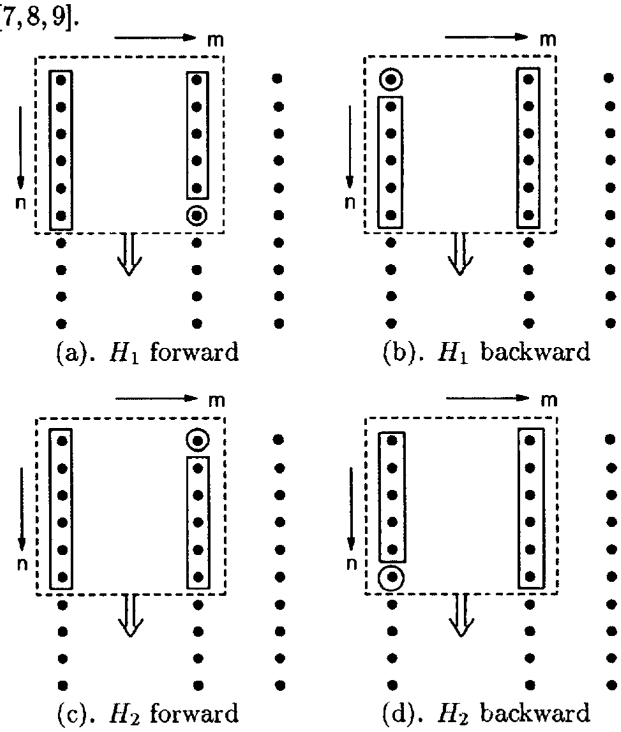

Figure 1. *Data supports of the filters $H_1(z_1, z_2)$ and $H_2(z_1, z_2)$. Equations in formulas (2), (3) are established using the data inside the region formed by dashed lines, which is moving downward. Each FBSLP equation uses the data inside solid-lined rectangles to predict the data value inside the circle.*

Let us assume that the the data inside the region formed by dashed lines has L rows and two columns. We also define the following vectors of time samples from the mth sensor,

$$\mathbf{y}_m(i) = \begin{bmatrix} y(i,\, m) \\ y(i+1,\, m) \\ \vdots \\ y(i+L-1,\, m) \end{bmatrix}, \quad \tilde{\mathbf{y}}_m(i) = J \cdot \mathbf{y}_m(i)$$

$$m = 0,\, K_1,\, K_2;\;\; i = 0,\, 1,\, \ldots,\, N-L.$$

where J is the exchange matrix. When J is applied to a vector, it simply flips the vector upside down.

Then the FBSLP equations can be written as follows,

$$\begin{bmatrix} \tilde{\mathbf{y}}^T_{K_1}(0) & \tilde{\mathbf{y}}^T_0(0) \\ \tilde{\mathbf{y}}^T_{K_1}(1) & \tilde{\mathbf{y}}^T_0(1) \\ \vdots & \vdots \\ \tilde{\mathbf{y}}^T_{K_1}(N-L) & \tilde{\mathbf{y}}^T_0(N-L) \\ \mathbf{y}^H_0(0) & \mathbf{y}^H_{K_1}(0) \\ \mathbf{y}^H_0(1) & \mathbf{y}^H_{K_1}(1) \\ \vdots & \vdots \\ \mathbf{y}^H_0(N-L) & \mathbf{y}^H_{K_1}(N-L) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -a_1 \\ -a_2 \\ \vdots \\ -a_{2L-1} \end{bmatrix} = \mathbf{0} \tag{2}$$

or $\qquad \mathbf{Y}_1 \cdot \mathbf{a} = \mathbf{0}$

$$\begin{bmatrix} \mathbf{y}^T_{K_1}(0) & \mathbf{y}^T_0(0) \\ \mathbf{y}^T_{K_1}(1) & \mathbf{y}^T_0(1) \\ \vdots & \vdots \\ \mathbf{y}^T_{K_1}(N-L) & \mathbf{y}^T_0(N-L) \\ \tilde{\mathbf{y}}^H_0(0) & \tilde{\mathbf{y}}^H_{K_1}(0) \\ \tilde{\mathbf{y}}^H_0(1) & \tilde{\mathbf{y}}^H_{K_1}(1) \\ \vdots & \vdots \\ \tilde{\mathbf{y}}^H_0(N-L) & \tilde{\mathbf{y}}^H_{K_1}(N-L) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -b_1 \\ -b_2 \\ \vdots \\ -b_{2L-1} \end{bmatrix} = \mathbf{0} \tag{3}$$

or $\qquad \mathbf{Y}_2 \cdot \mathbf{b} = \mathbf{0}$

where T denotes transpose, and H denotes complex conjugate transpose.

The sparse linear prediction error (SLPE) filter transfer functions can be formed based on the vectors **a** and **b** as follows,

$$\begin{aligned} H_1(z_1, z_2) &= \sum_{i=0}^{L-1} \sum_{k=0}^{K_1} a_{ik}\, z_1^{-i}\, z_2^{-k}, \\ H_2(z_1, z_2) &= \sum_{i=0}^{L-1} \sum_{k=0}^{K_1} b_{ik}\, z_1^{i}\, z_2^{-k}, \end{aligned} \tag{4}$$

Recalling that we only use data from the sparse linear array in setting up the FBSLP equations, therefore vectors **a** of formula (2) and **b** of formula (3) give us the non-zero elements of the the 2-D SLP coefficient matrices $A = [a_{ik}]$ and $B = [b_{ik}]$ as shown in the following formulas.

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 & -a_L \\ -a_1 & 0 & \cdots & 0 & -a_{L+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ -a_{L-1} & 0 & \cdots & 0 & -a_{2L+1} \end{bmatrix}_{L\times(K_1+1)} \tag{5}$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 & -b_L \\ -b_1 & 0 & \cdots & 0 & -b_{L+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ -b_{L-1} & 0 & \cdots & 0 & -b_{2L+1} \end{bmatrix}_{L\times(K_1+1)} \tag{6}$$

Note that the mapping between matrices A, B and vectors **a**, **b** is consistent with the mapping, which is used to map the 2-D sparse linear array outputs inside the region specified by dashed lines (see Figure 1) into a 1-D vectors used for solving vectors **a**, **b** from the FBSLP equations of formulas (2) and (3).

Combining $H_1(z_1, z_2)$ and $H_2(z_1, z_2)$, the temporal and spatial frequencies of the source signals correspond to the coordinates of zeros of the following function [7, 8, 9],

$$G_1(e^{j\omega}, e^{j\nu}) = \sum_{i=1}^{2} \left|H_i(e^{j\omega}, e^{j\nu})\right|^2 \tag{7}$$

Note that in order to use the same 2-D FFT to evaluate $H_i(e^{j\omega}, e^{j\nu})$'s, we need to flip rows of the 2-D FFT of the B matrix (except the first zero-frequency row) for $G_1(e^{j\omega}, e^{j\nu})$ to have its zeros correctly located.

As in [3] the sparse locations of the sensors will cause ambiguities in estimating the spatial frequencies. Following the ideas used in [3], we can also use the outputs of the a third sensor at $m = K_2$ and the reference one to set up other sets of FBSLP equations similar to equations (2) and (3); forming 2-D SLP polynomials similar to (4) with A and B given by (5) and (6); and finally calculating another function $G_2(e^{j\omega}, e^{j\nu})$ like that of formula (7), *with* K_1 *in formula (2) through formula (6) being replaced by* K_2. Note also that because we do not have the other (K_2-1) columns of output data between the reference sensor and the second sensor. We will have a different set od ambiguities in spatial frequencies.

We can resolve these ambiguities by properly choosing the interelement spacings K_1 and K_2 of the sparse linear array and by combining both $G_1(e^{j\omega}, e^{j\nu})$ and $G_2(e^{j\omega}, e^{j\nu})$ together.

3 Resolving ambiguities due to the use of sparse linear array

As mentioned before, each time when we form the

FBSLP equations by using the outputs from the ith sensor and the reference one of the sparse linear array, we do not have $(K_i - 1)$ columns of data in between, which would be available in a filled array which is Nyquist sampled in space. This corresponds to having $(K_i - 1)$ columns of zeros in the 2-D SLP coefficient matrices A and B. This will cause ambiguity in estimating the spatial frequencies. In Figures 2 and 3 we show the possible ambiguities by plotting 2-D contour plots of the function $G_i(e^{j\omega}, e^{j\nu})$, $(i = 1, 2)$. It is assumed that we have 10 snapshots from the three-element sparse linear array. The interelement spacings of the sparse linear array are chosen to be relatively prime, say $K_1 = 5$, $K_2 = 8$. $L = 7$ is used. Therefore, the coordinates of the common pairs of minima of the function $G_1(e^{j\omega}, e^{j\nu})$ and $G_2(e^{j\omega}, e^{j\nu})$ give us the temporal and spatial frequencies of the 2-D sinusoids. Or equivalently, we can get pairs of temporal and spatial frequencies of the 2-D sinusoids from the maxima of the following combined function,

$$P(e^{j\omega}, e^{j\nu}) = \frac{1}{G_1(e^{j\omega}, e^{j\nu}) + G_2(e^{j\omega}, e^{j\nu})} \tag{8}$$

As a matter of fact, in the case of single source signal without noise, the 2-D SLP coefficient matrix can be calculated analytically from the following vector **a** and **b**,

$$\mathbf{a} = \frac{1}{2L-1}\begin{bmatrix} e^{j\omega_1} \\ e^{j\omega_1 2} \\ \vdots \\ e^{j\omega_1 (L-1)} \\ e^{j\nu_1 K_1} \\ e^{j\omega_1} e^{j\nu_1 K_1} \\ \vdots \\ e^{j\omega_1 (L-1)} e^{j\nu_1 K_1} \end{bmatrix}_{(2L-1)\times 1}$$

$$\mathbf{b} = \frac{1}{2L-1}\begin{bmatrix} e^{-j\omega_1} \\ e^{-j\omega_1 2} \\ \vdots \\ e^{-j\omega_1 (L-1)} \\ e^{j\nu_1 K_1} \\ e^{-j\omega_1} e^{j\nu_1 K_1} \\ \vdots \\ e^{-j\omega_1 (L-1)} e^{j\nu_1 K_1} \end{bmatrix}_{(2L-1)\times 1}$$

It can be seen that information about the temporal and spatial frequencies is contained in vectors **a** and **b**. Figure 2 below shows the occurrence of ambiguities in each $G_i(e^{j\omega}, e^{j\nu})$'s. Note that after combining both $G_1(e^{j\omega}, e^{j\nu})$, and $G_2(e^{j\omega}, e^{j\nu})$, no more ambiguity appears in $P(e^{j\omega}, e^{j\nu})$.

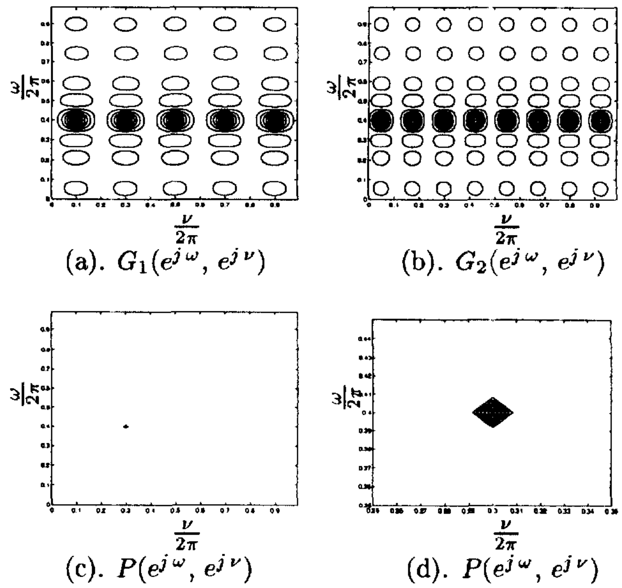

Figure 2. *Contour plots of* $G_i(e^{j\omega}, e^{j\nu})$, $(i = 1, 2)$ *and* $P(e^{j\omega}, e^{j\nu})$. *The temporal and spatial frequencies of the single source signal is at* $(\frac{\nu_1}{2\pi}, \frac{\omega_1}{2\pi}) = (0.3, 0.4)$. *Different scales are used in (c) and (d).*

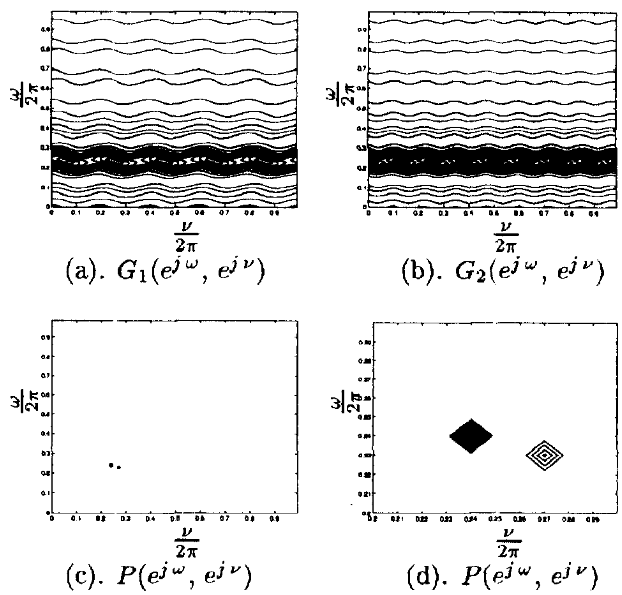

Figure 3. *Contour plots of* $G_i(e^{j\omega}, e^{j\nu})$, $(i = 1, 2)$ *and* $P(e^{j\omega}, e^{j\nu})$. *The frequencies of the two source signals are at* $(\frac{\nu_1}{2\pi}, \frac{\omega_1}{2\pi}) = (0.24, 0.24)$ *and* $(\frac{\nu_2}{2\pi}, \frac{\omega_2}{2\pi}) = (0.23, 0.27)$. *Note that ambiguities appear in both* $G_1(e^{j\omega}, e^{j\nu})$, *and* $G_2(e^{j\omega}, e^{j\nu})$, *but do not appear in* $P(e^{j\omega}, e^{j\nu})$ *Different scales are used in (c) and (d).*

Figure 3 shows the case of two closely spaced 2-D sinusoids. After resolving the spatial ambiguities by combining two 2-D SLP polynomials, the high resolution property of LP method is preserved in Figures 3(c), (d). Further simulations also show that the proposed method also works for the case of two widely spaced sinusoids, as long as the temporal frequencies are not equal.

4 Discussions and Conclusions

In this paper, we proposed a 2-D sparse LP method of resolving and estimating frequencies of the 2-D sinusoidal signals from the outputs of a sparse linear array. By properly mapping, we only need to solve sets of sparse LP equations to get the fewer non-zero elements of the 2-D SLP coefficient matrices with a few computation. We also ease the computation of rooting the 2-D polynomials by only searching for the maxima of the 2-D function $P(e^{j\omega}, e^{j\nu})$ with the help of the computationally efficient zero-padded 2-D FFTs of the properly mapped 2-D SLP coefficient matrices.

Note that if we choose $L = N$, the above 2-D SLP method is simply a sparse version of 2-D K-P method [8], which provides the highest resolution with minimum number of equations. To combat the effect of noise, a smaller value of L is chosen, for example $L = \frac{2}{3}N$, then the SLP coefficients can be obtained by solving more equations. We can further improve the estimation performance by using the singular value decomposition of the FBSLP data matrices of equations (2) and (3); by collecting more snapshots from the sparse linear array.

The same algorithm can also be used for the estimation of the angles of arrival of the source signals. Since once we have estimated pairs of temporal and spatial frequencies (ω_i, ν_i)'s, the angles of arrival θ_i's can be obtained simply by a nonlinear transformation,

$$\theta_i = \arccos(\frac{c}{d_0}\frac{\nu_i}{\omega_i})$$

where c is the medium propagation speed, and d_0 the the unit spacing corresponding to half-wavelength of the highest temporal frequency within the bandwidth of applications.

If we need to estimate pairs of temporal and spatial frequencies of more than two source signals, we need to add extra sensor channels. Details still need to be worked out.

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