A New Method for Estimating the Weight of Very Heavy Objects

Steven D. Gray
John A. Parmentola

The MITRE Corporation, 202 Burlington Rd, Bedford, MA 01730

Abstract

We propose and study a new technique for estimating the weight of very heavy objects. This new technique is based on measurements made by a device, a gravity gradiometer, that is capable of measuring subtle changes in the gravitational field produced by an object with respect to distance. The physical observables, known as gravity gradients, depend upon the mass density distribution of an object and the relative distance between the object and the measurement device. We show that for very heavy objects gravity gradient measurements can be used to form least squares estimates for the object's mass. This technology is potentially important in the implementation of arms control agreements or for application in the transportation industry.

1 Introduction

The purpose of this paper is to propose the use of gravity gradiometers in portable weighing systems. Weighing very heavy vehicles to within 3% of their weight is potentially an important capability for the verification of weight limits in future arms control agreements, where weight is used as a discriminator. Currently, portable weighing systems exist which consist of two separate scales having an accuracy of 0.5% when used in a static weighing mode. In a weight-in-motion mode, tests have shown that these scales can provide weight estimates which are at approximately 5% of the vehicles weight for speeds of 3 miles per hour. With these scales, the weight of a very heavy truck (on the order of 80,000 pounds) can be estimated through axle by axle measurement.

For potential future applications, there is interest in portable weighing systems that can estimate the weight of a moving truck to within 3%. One possible approach is based upon the use of a piezoelectric sheet which is in contact with the truck while it is moving [1]. Another method employs fiber optic cable [1]. There are many variables affecting the accuracy of these measurement methods, including the nature of the tire tread and the vehicle dynamics. These parameters vary from truck to truck, making the problem of weight estimation on the move through contact difficult.

A method that avoids contact might provide a suitable solution to this problem. In this paper, we will present a methodology for weight estimation based on gravity gradient measurements when the gravity gradiometer and the truck are stationary. However, because the gravity gradiometer does not make contact with the truck, it is possible to generalize our techniques to dynamic weight measurement by measuring the acceleration between the gravity gradiometer and the truck. Furthermore, for a vehicle traveling sufficiently slowly, the weight estimation method we will describe satisfies the 3% requirement with high confidence for sufficiently heavy objects.

2 Problem Formulation

We consider a scenario where the gravity gradiometer and the object to be weighed are both stationary. From a verification and arms control perspective, it is important that the time required for estimating the weight of an object be kept to a minimum in order to avoid interference with normal military operations. Therefore minimizing the number of measurements needed to obtain an acceptable level of precision is an important objective. Precision is important and is greatly affected by the amount of noise that contributes to the gravity gradiometer measurement process. Both of these issues are important in the development of an efficient algorithm that minimizes the measurement time and also the effect of noise on estimating the weight of an object.

2.1 Multipole Expansion

The multipole expansion was selected as a technique for determining the mass, due to the linear relationship between the multipoles and the scalar gravitational potential function $\Phi(\mathbf{x})$. The expansion is defined for the scalar gravitational potential of an arbitrary mass distribution with density $\rho(\mathbf{x})$ such that

$$\Phi(\mathbf{x}) = G \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}} \qquad (1)$$

where the $Y_{lm}(\theta,\phi)$ are the well-known spherical harmonic functions and the q_{lm}'s are the lth order multipole moments defined by [2]

$$q_{lm} = \int Y^*_{lm}(\theta',\phi')r'^l \rho(\mathbf{x}')d\mathbf{x}'. \quad (2)$$

Because of a well known property of the spherical harmonics

$$q_{l,-m} = (-1)^m q^*_{lm}, \quad (3)$$

not all the q_{lm}'s are independent. It is important to note that q_{00} is linearly related to the mass of the object. The coordinate vector $\mathbf{x}$ is measured from the origin of the chosen coordinate system to the point of observation. To find the gravity gradient tensor, $\Phi(\mathbf{x})$ must be differentiated with the gradient operator twice. For the purpose of this discussion we limit the gravity gradient tensor to the diagonal yy component. The differential operator in the y-direction can be expressed in spherical coordinates such that

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\sin\theta\sin\phi + \cos\theta\sin\phi\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}. \quad (4)$$

To find the y-component of the gradient of the y-component of the gravity field we take the second derivative of the potential $\Phi(\mathbf{x})$ with respect to y, hence

$$g_{yy} = \frac{\partial^2\Phi(\mathbf{x})}{\partial y^2}. \quad (5)$$

Although tedious to calculate, (5) can be written as

$$g_{yy} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} q_{lm}F_{lm}(\theta,\phi,r) \quad (6)$$

where

$$F_{lm}(\theta,\phi,r) = \frac{4\pi G}{2l+1}\frac{\partial^2}{\partial y^2}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}.$$

Because the zeroth order multipole, q_{00}, is linearly related to the mass, gravity gradient measurements can be used to determine the mass. For example, let

$$\mathbf{g}_{yy} = \begin{bmatrix} g_{yy}(1) \\ \vdots \\ g_{yy}(n) \end{bmatrix} \quad (7)$$

be a vector of gravity gradient measurements along the y-axis. Assume now, that the multipole expansion is truncated to n terms. The relationship between $\mathbf{g}_{yy}$ and the q_{lm}'s in the expansion is as follows,

$$\mathbf{g}_{yy} = \mathbf{M}\mathbf{q}. \quad (8)$$

$\mathbf{M}$ is a $n \times L$ matrix composed of $F_{lm}(\theta,\phi,r)$ values for the n spatial measurement points and L multipole terms, and $\mathbf{q}$ is a vector of the q_{lm}'s. The solution is straightforward provided the matrix $\mathbf{M}$ is positive definite. However, for the particular mass distribution we have considered and the measurement points we have chosen, $\mathbf{M}$ in (8) is ill-conditioned. Any perturbation of the gravity gradients from their true values will cause gross errors in determining $\mathbf{q}$ by simple matrix inversion when $\mathbf{M}$ is ill-conditioned. Finding a reasonable method of solving for the vector $\mathbf{q}$ under noisy conditions is the primary technical goal of this paper. It is important to note that even if $\mathbf{M}$ is well-conditioned, the noise contribution to the mass estimate can be amplified by making measurements at larger distances. Hence it is important to keep the signal-to-noise ratio large to reduce mass estimation error due to measurement noise.

2.2 Pseudoinverse

Careful examination of the singular values for $\mathbf{M}$ illustrates how small errors in the gradient measurements can cause gross inaccuracies in the solution of (8). Suppose the largest singular value in $\mathbf{M}$ is greater than one and several of the remaining singular values are less than 10^{-10}, implying an approximate linear dependence between the n equations relating the q_{lm}'s and g_{yy}'s. Solution by direct inversion of $\mathbf{M}$ will amplify any inaccuracies in the g_{yy} measurements by an amount inversely proportional to the smallest singular value. The resulting solution in many cases will be meaningless.

One way to combat the problems associated with the ill-conditioning is to use condition number limiting. One method for imposing limits on the singular values of a matrix is to perform a singular value decomposition of $\mathbf{M}$ and limit the maximum singular value spread. Then, the pseudoinverse is used to solve for $\mathbf{q}$. The pseudoinverse for $\mathbf{M}$ is defined as follows

$$\mathbf{M}^{-1} = \mathbf{V}\mathbf{M}_d^{-1}\mathbf{U}^{\mathrm{H}} \quad (9)$$

where $\mathbf{M}_d$ is a diagonal matrix containing the singular values of $\mathbf{M}$, and $\mathbf{U}$ and $\mathbf{V}$ are row orthonormal matrices. The condition limited pseudoinverse is defined as

$$\tilde{\mathbf{M}}^{-1} = \mathbf{V}\mathbf{M}_c^{-1}\mathbf{U}^{\mathrm{H}} \quad (10)$$

where $\mathbf{M}_c$ is a diagonal matrix with small singular values replaced by a limiting value. Computing the pseudoinverse from a subset of the largest singular values will add stability to the solution of the q_{lm}'s even when noise is introduced in the gravity gradient measurements.

Consider now, an over determined solution of the vector $\mathbf{q}$. In this case, the matrix $\mathbf{M}$ will not be square. An over determined solution can aid in finding the vector of q_{lm} values when $\mathbf{M}$ is approximately singular. Thus, the general approach to finding the mass of an object is to: first, truncate the multipole expansion to L terms; second, take n gravity gradient measurements where $n > L$; third, determine $\mathbf{M}$ by calculating the L terms in the multipole expansion for the n different gravity gradient measurements (This is done by evaluating the multipole expansion for different values of r corresponding to the distance between the origin of the coordinate system and the n measurements.); fourth, solve for the pseudoinverse and limit the smallest singular values; fifth, solve for $\mathbf{q}$ by the product of $\tilde{\mathbf{M}}^{-1}$ and $\mathbf{g}_{yy}$. The mass is found by extracting the first element of the vector $\mathbf{q}$.

2.3 Linear Least Squares Estimate for q

The gravity gradiometer is a device that measures the local gravity gradients produced by a mass distribution. Similar to any device designed to measure a physical quantity, a non-zero amount of measurement error or noise is added to each true gravity gradient value. This error is a function of gravity gradiometer design parameters, the integration time of measurement, and background noise. For the sake of simplicity, the noise is assumed to be centered around the true mass. Rather than make an explicit assumption about the noise distribution, $\mathbf{q}$ will be determined using a Linear Least Squares Estimator (LLSE). Assuming N measurements are made at each of n positions of the gravity gradiometer, the LLSE for the multipole vector can be expressed as

$$\widehat{\mathbf{q}} = \frac{1}{N}\sum_{j=1}^{N}\mathbf{V}\mathbf{M}_c^{-1}\mathbf{U}^{\mathrm{H}}\mathbf{g}_{yy}(j) = \frac{\mathbf{V}\mathbf{M}_c^{-1}\mathbf{U}^{\mathrm{H}}}{N}\sum_{j=1}^{N}\mathbf{g}_{yy}(j). \tag{11}$$

It is interesting to note that $(1/N)\sum_{j=1}^{N}\mathbf{g}_{yy}(j)$ is the sample average for $\mathbf{E}\{\mathbf{g}_{yy}\}$. Expression (11) is the LLSE for $\mathbf{q}$ independent of the distribution of the noise. If the noise is Gaussian distributed, the LLSE is the minimum variance estimate for $\mathbf{q}$ [4].

The LLSE for the mass is determined from $\widehat{q}_{00}$, the first component of $\widehat{\mathbf{q}}$, and is given by

$$\widehat{M} = K_v\widehat{q}_{00} \tag{12}$$

where $K_v = (1 \times 10^{-9})\sqrt{4\pi}/G^1$, which ensures that $\widehat{M}$ is in kilograms.

2.3.1 Noise Amplification by the Pseudoinverse

In most cases of interest, the noise present in the measurement of the gravity gradient is small compared to the signal. However, this is not true when the noise is multiplied by the pseudoinverse of the matrix $\mathbf{M}$. An alternate way of representing this decomposition shown in [3] is in terms of the column vectors of $\mathbf{V}$, $\mathbf{U}$ and the eigenvalues μ_i of $\mathbf{M}_d$ such that

$$\mathbf{M} = \sum_{i=1}^{n}\mu_i\mathbf{u}_i\mathbf{v}_i^{\mathrm{H}}. \tag{13}$$

Defining a gradient measurement, $\mathbf{g}_{yy} = \mathbf{g}'_{yy} + \mathbf{n}$, as the noise free gradient vector $\mathbf{g}'_{yy}$ plus noise vector $\mathbf{n}$,

$$\mathbf{q} = \sum_{i=1}^{n}\frac{\mathbf{u}_i^{\mathrm{H}}\mathbf{g}'_{yy}}{\mu_i}\mathbf{v}_i + \sum_{i=1}^{n}\frac{\mathbf{u}_i^{\mathrm{H}}\mathbf{n}}{\mu_i}\mathbf{v}_i \tag{14}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ are column vectors of $\mathbf{U}$ and $\mathbf{V}$, and μ_i are singular values of $\mathbf{M}$ (μ_i are also the eigenvalues of $\mathbf{M}_d$). It is obvious from (14) that small singular values amplify the noise greatly.

Defining

$$\mathbf{n}_m = \sum_{i=1}^{n}\frac{\mathbf{u}_i^{\mathrm{H}}\mathbf{n}}{\mu_i}\mathbf{v}_i \tag{15}$$

as the vector of noise present in the solution for $\mathbf{q}$. The variance of the noise in the mass estimate is found by assuming the components of $\mathbf{n}_m$ are zero mean, independent and identically distributed (iid). Then, the variance of the noise in the first component of $\mathbf{q}$ is given by

$$\sigma_m^2 = E[(n_m(1))^2] = \sigma^2\left(\sum_{i=1}^{n}\frac{v_i(1)^2}{\mu_i^2}\mathbf{u}_i^{\mathrm{H}}\mathbf{u}_i\right) \tag{16}$$

where the measurement noise $\sigma^2 = E[n(i)^2]\ \forall i$. Finally, the estimation error variance for the LLSE of the mass, $\widehat{M}$ is

$$\sigma_{\widehat{M}}^2 = K_v^2\sigma_m^2/N.$$

2.3.2 Confidence of Mass Estimate

As shown in (12), $\widehat{M}$ is found by taking a set of measurements, solving (11) and extracting the first element of $\widehat{\mathbf{q}}$. The resulting estimate will have a noise component, as discussed above, with a variance proportional to the variance of the input noise and the sum of one over the singular values squared. For ease in analysis, we assume that the measurement noise is iid Gaussian with a variance σ^2. The estimation error is also Gaussian with a variance σ_m^2. The goal is now one of defining the confidence in the mass estimate for the nN gravity gradient measurements needed to form the mass estimate.

By defining α to be the confidence that $\hat{M}$ is within δ of the true mass, M, it is possible to determine how many measurements are needed to ensure any predefined precision in the mass estimate. Mathematically, this means that

$$\mathcal{P}[|\widehat{M} - M| \leq \delta] = 2\ \mathrm{erf}\left(\frac{\delta}{\sigma_y}\right) = \alpha. \tag{17}$$

[1] $G = 6.67 \times 10^{-11}$ Newton meter^2/kilogram^2

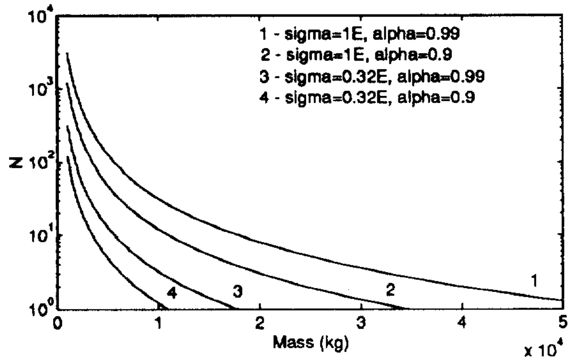

Figure 1: N vs. Mass for $\delta = 0.03m$ and a Starting Measurement of 1.25 meters

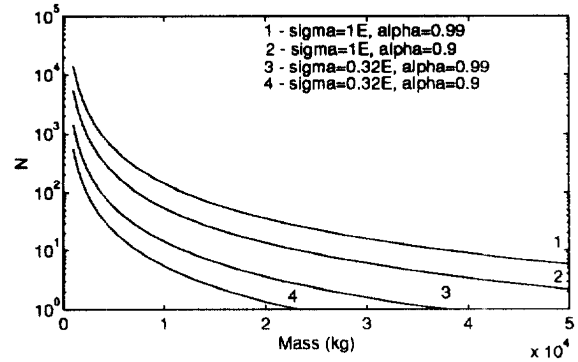

Figure 2: N vs. Mass for $\delta = 0.03m$ and a Starting Measurement of 2.25 meters

Solving for the number of measurements in terms of the error variance yields

$$N = \left[\frac{K_v \sigma_m \beta_\alpha}{\delta}\right]^2 \tag{18}$$

where $\beta_\alpha = \text{erf}^{-1}(\alpha/2)$.

As noted earlier, the precision for the mass estimate must be within 3% of the true mass. Substituting $\delta = 0.03M$ into (18),

$$N = \left[\frac{K_v \sigma_m \beta_\alpha}{0.03M}\right]^2 \tag{19}$$

is the functional relationship between the mass and the number of gravity gradient measurements needed at each spatial coordinate of the multipole expansion to achieve a confidence of α. It is important to note, that geometrically dependent "noise" or bias is introduced in the mass estimate due to a finite number of terms, L, in the multipole expansion. This geometrical or expansion bias is caused by a difference between the true gravity gradient and the gravity gradient calculated with the multipole expansion. However, the expansion bias is a deterministic quantity that can be calculated numerically for a particular object.

To gain an understanding of how large N must be to ensure a prescribed confidence, consider the curves shown in figures 1 and 2. A sphere with constant volume is used as the test object. For a gravity gradient measurement accuracy of 1 or 0.1 Eötvös (E), and a condition limit of 10^5 for **M**, the value of σ_m^2 was determined by simulation. Figures 1 and 2 are plots of N for a confidence of 99% and 90% as a function of mass where the starting point is specified relative to the geometric center of the object. These figures illustrate the effect of using a gradiometer with a noise variance $\sigma^2 \in \{1E, 0.1E\}$ and also the effect of moving the initial starting point of the gravity gradient measurements further away from the object. Because the gravity gradients for the sphere decrease at a rate of $1/r^3$, moving the initial measurement point further from the object decreases the signal-to-noise ratio. We also have chosen a grid spacing for gravity gradient measurements of 0.25 meters.

Examining the two figures, the amount of noise enhancement varies from figure 1 to figure 2. Numerical estimates for σ_m^2 show that $\sigma_m^2 \approx 150\sigma^2$ for figure 1 and $\sigma_m^2 \approx 675\sigma^2$ for figure 2. Examination of the singular value spread for **M**), when the starting measurement was 2.25 meters from the center of the sphere, showed a significant increase in the condition number when compared to a starting measurement of 1.25 meters. A condition limit of 10^5, was large enough to not effect **M** when the starting distance was 1.25 meters, but the condition number for **M** was approximately 10^7 when the starting distance was increased to 2.25 meters. The amount of noise enhancement is an important factor effecting the number of measurements need to achieve confidence. The goal is control the measurement environment such that the signal-to-noise ratio is great enough to meet confidence when $N = 1$, implying that only n gravity gradient measurements at different points in space are required. However, the signal-to-noise ratio is effected by the objects shape (number of multipole terms needed in the expansion), the distance from the object (singular value spread in **M**, and the object's mass. We obviously cannot control all three.

3 Simulation Results

In the results that follow, we chose the number of multipole terms by trial-and-error. The expansion size was increased until the "noise free" estimate of the mass stabilized. This technique for determining the expansion size works well in simulation experiments where noise free gravity gradient signals are generated numerically. However, noisy observations of the gravity gradients may require a more sophisticated technique for determining the expansion size. The precision of the mass estimates in the presence of measurement noise is reported as the absolute difference between the estimate and the true mass divided by the true mass for LLSE.

We use a series of gravity gradiometer measurements which are equally spaced and radially outward from an object where the initial measurement is assumed to be made fairly close to the object and subsequent measurements are made further away. This is in some sense a worst case in that the signal-to-noise ratio is monotonically decreasing as the measurements are made further and further away from the object.

The object considered in this simulation study is a cylinder. The number of terms in the multipole expansion was set by trial and error to be 21. However, because of conjugate symmetry the number of unknowns in $\mathbf{q}$ was reduced to 12 where

$$\mathbf{q} = \begin{bmatrix} q_{00} \\ q_{11} - q_{11}^{*} \\ q_{22} + q_{22}^{*} \\ q_{20} \\ q_{33} - q_{33}^{*} \\ q_{31} - q_{31}^{*} \\ q_{44} + q_{44}^{*} \\ q_{42} + q_{42}^{*} \\ q_{40} \\ q_{55} - q_{55}^{*} \\ q_{53} - q_{53}^{*} \\ q_{51} - q_{51}^{*} \end{bmatrix} \quad (20)$$

The estimation accuracy results for a cylinder with a radius and length of 1 meter as a function of density are presented in table 1. The variance of the measurement noise was set at $\sigma^2 = 1\text{E}^2$. The initial measurement point was taken at 1.25 meters from the centerline of the cylinder and the number of measurements, N, used in the simulation was set at 10.

As shown in table 1, estimation precision decreases as the density for the cylinder also decreased. This relationship is true because the signal-to-noise ratio is inversely proportional to the mass of the cylinder. Therefore, this method is best applied in cases where the mass density is large and the overall mass of the object is also large.

Density (kg/m^3)	% *Difference*
6000	1.6
3000	2.2
1500	4.2
750	8.9
500	12.1

Table 1: LLSE Accuracy for Nonuniform Cylinder, $\sigma^2 = 1\text{E}^2$, Radius and Length of 1 Meter.

4 Conclusions

In this paper we have proposed a viable alternative to conventional weighing techniques. Provided the object is very heavy, a gravity gradiometer can be used as a sensor for an accurate portable weighing system. Unlike a spring-balance scale, the gravity gradiometer does not have to be in direct contact with the object during the weighing process. For example, a gravity gradiometer could be positioned next to a production line of treaty-limited items without disturbing routine operations.

We have shown that for objects with approximately the same length to width scale, the multipole expansion for the gravitational potential and hence the gravity gradients are an efficient method for estimating the mass of a treaty-limited item. Moreover, for an object of very high density the multipole expansion can be used to determine its mass to within 3% with high confidence. However, the further the object diverges from spherical symmetry the greater the number of multipole terms required. Eventually, the number of terms will become prohibitively high requiring an alternate technique.

References

[1] Private Conversation with LCDR Randall Johnston, USN, Defense Nuclear Agency, 1992.

[2] Jackson, J.D.,*Classical Electrodynamics*, John Wiley & Sons, Inc., New York, 1962.

[3] Golub, G.H. and C.F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore MD, 1983.

[4] Bickel, P.J. and K.J. Doksum, *Mathematical Statistics*, Holden-Day Inc., Oakland, CA, 1977.