A Reduced Sufficient Statistics Tracking Algorithm for Infrared Images

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Abstract

The problem of tracking a target using a sequence of infrared (IR) images is addressed. A Bayes-closed estimation algorithm developed by Kulhavý is shown to be well-suited to the IR tracking problem. Due to the form of the model for the radiation intensity pattern on the IR focal plane array, closed-form expressions are found for the reduced sufficient statistics (RSS) which are used to approximate the true posterior density in the Kulhavý algorithm. An estimate of the target state is then derived via a reconstruction formula from the RSS. For comparison, both a previously developed IR tracking algorithm based on an extended Kalman filter (EKF) and the new RSS-based algorithm are used to track a target through a sequence of IR images. It is shown that the RSS algorithm can maintain track in high velocity scenarios where the EKF diverges.

1 Introduction

Several techniques have been used to solve the problem of tracking an object, such as an air-to-air missile, using a sequence of infrared (IR) images as measurements. Early methods employed a simple correlation algorithm to estimate the two-dimensional position offsets from frame to frame [1]. The offsets could then be used to keep the system centered on the target. Later, an extended Kalman filter (EKF) based tracker was developed to exploit knowledge of the size, shape, and motion characteristics of the target to improve performance [1]. The shape of the intensity pattern on the IR image plane was assumed to be well modeled by a bivariate Gaussian function. Unfortunately, robustness studies showed a degradation in performance when the filter design parameters varied from the actual parameters [2]. Thus, adaptive estimation of the height and variances of the Gaussian intensity pattern were implemented. Further research has investigated ways of dynamically estimating the target shape function for use in combination with an EKF tracker [3]. Finally, multiple model adaptive estimation was used to enhance the ability of the EKF to track maneuvering targets [4, 5]. In this method, multiple filters, each tuned to represent different target dynamics, are run in parallel, and the filter states are probabilistically weighted to form an estimate.

In this paper, we apply a reduced sufficient static (RSS) estimation algorithm developed by Kulhavý [6, 7] to the IR image tracking problem. In the RSS method, the true posterior density of the target's state is fitted to a parameterized model density. Various model densities such as point-mass, piecewise-constant, and Gaussian sum can be used as discussed in [6]. Numerical integration or an approximation method is typically needed to determine the RSS vector used to represent the model density. However, choosing a Gaussian sum as our model density yields a closed-form expression for the RSS vector, when a Gaussian intensity pattern is assumed. To estimate the target's state, the weighting coefficients of the Gaussian sum model density must be determined from the RSS vector. Unfortunately, an exact solution for the coefficients requires a numerical search algorithm. To avoid an exhaustive search, an approximate reconstruction algorithm as developed in [8] will be used.

A review of the IR tracking problem, and the EKF tracker formulation will be given in Section 2. The RSS algorithm is briefly reviewed in Section 3, and applied to the IR tracking problem in Section 4. Section 5 compares the performance of the EKF and RSS trackers. Conclusions are presented in Section 6.

*This work was supported in part by the SDIO/IST program managed by the Office of Naval Research under Contract No. N00014-91-J-4122

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2 IR Tracking and EKF Formulation

The purpose of the IR tracking system is to assist in keeping a target centered in the IR array field of view. To do so, a tracking algorithm must determine the two-dimensional pointing errors from the array center. The intensity pattern on the IR focal plane array for a point target or an air-to-air missile can be well modeled by a bivariate Gaussian function as shown in Figure 1. The target intensity model is thus

$$I_{target}(x', y') = I_{max} e^{-\frac{1}{2}\left[\left(\frac{x'}{\sigma_v}\right)^2 + \left(\frac{y'}{\sigma_{pv}}\right)^2\right]} \quad (1)$$

where x', and y' are measured along the principle axes of the ellipse [1,2]. To generate simulated IR images, the three-dimensional position and velocity of a target, $(x(k), y(k), z(k))$ and $(\dot{x}(k), \dot{y}(k), \dot{z}(k))$, can be used to project the target onto the IR array. The angle the 3-D velocity vector makes with the image plane determines the aspect ratio of the target, σ_v/σ_{pv}, and the velocity of the target is assumed to be aligned with the major axis of the ellipse.

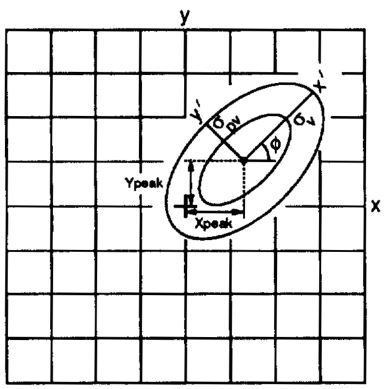


Figure 1: Target intensity pattern on IR image plane

To determine the intensity value $z_{l,m}(k)$ at a given pixel in the lth row and mth column of the IR array, the target intensity function is integrated over the pixel area, and independent IR and background noises are added. The measurement equation is thus

$$z_{l,m}(k) = \frac{1}{A_p} \iint_{\text{l,mth pixel area}} I_{target}(x'(x,y), y'(x,y))\,dx\,dy + n_{l,m}(k) + b_{l,m}(k) \quad (2)$$

where A_p is the pixel area, and $n(k)$ and $b(k)$ represent the IR and background noise respectively. In the above integral, each point (x, y) in the IR array (Figure 1) is mapped to the ellipse coordinates using

$$x'(x, y) = (x - x_{peak})\cos\phi + (y - y_{peak})\sin\phi \quad (3)$$
$$y'(x, y) = (y - y_{peak})\cos\phi - (x - x_{peak})\sin\phi$$

where (x_{peak}, y_{peak}) locates the center of the Gaussian function, and ϕ is the rotation angle measured from the x array axis to the x' axis of the ellipse .

To reduce complexity, an 8×8 subset of the IR array is used as the measurement vector. Arraying the 64 pixel intensities in a vector yields the following nonlinear measurement model

$$\mathbf{z}(k) = \mathbf{h}(x_{peak}(k), y_{peak}(k)) + \mathbf{n}(k) + \mathbf{b}(k). \quad (4)$$

Given the nonlinear measurement equation, an EKF filter has previously been developed to track the target [2]. At a minimum, an estimate of the target's velocity and position on the IR array are needed to predict its position one sample time ahead, and orient the Gaussian function. Therefore, a minimal state vector for the EKF is $\mathbf{x}(k) = [x_{peak}(k), y_{peak}(k), \dot{x}(k), \dot{y}(k)]^T$ when the atmospheric jitter which was added to the state vector in [2] is ignored.

The Gaussian intensity model is a function of not only the predicted position of the target, but also I_{max}, σ_v, and σ_{pv}. These three parameters could be added to the state vector or estimated separately. Separate estimation techniques have been found to perform adequately [2]. I_{max} was estimated by time averaging the maximum pixel in the 8×8 array which has been adjusted to remove a bias factor. The parameters σ_v and σ_{pv} were estimated by an approximation to their maximum-likelihood estimates. The same estimation techniques will be used for these parameters in the RSS algorithm.

Using separate estimates for I_{max}, σ_v, and σ_{pv}, the state equation for the EKF assumes the familiar form

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(k) + \mathbf{v}(k), \quad (5)$$

where the sampling time, T, for the IR array is typically $\frac{1}{30Hz}$, and $\mathbf{v}(k)$ models slight changes in the assumed constant velocities. The information formulation of the Kalman filter is used to eliminate the need for large matrix inversions. Also, the EKF measurement equation was simplified in two ways. The integral in (2) was approximated by the integrand evaluated at the center of each pixel, and the combined effects of $\mathbf{n}(k)$ and $\mathbf{b}(k)$ were assumed to be well represented by a single vector of white Gaussian noise with covariance $\mathbf{R} = E\left[\mathbf{w}(j)\mathbf{w}(k)^T\right] = \begin{cases} \sigma_n^2 \mathbf{I} & j = k \\ 0 & j \neq k. \end{cases}$ The RSS algorithm will also use the two IR measurement equation simplifications.

3 Estimation using RSS

The goal of RSS estimation is to project the posterior density orthogonally onto a model density. Using concepts of statistical differential geometry developed by Amari [9], Kulhavý has shown that orthogonal projection minimizes the Kullback-Liebler distance between the two densities when a mixture model density is used [7]. In the following brief review of the RSS method, the posterior density is denoted by $p(\boldsymbol{\theta}|Z^k)$ where $\boldsymbol{\theta} \in \mathcal{R}^N$ is a parameter vector observed at discrete time instants through observations $\mathbf{z}(k) \in \mathcal{R}^M$ which form part of a cumulative measurement set $Z^k = \{\mathbf{z}(k), \mathbf{z}(k-1), \ldots, \mathbf{z}(0)\}$. The mixture model density, a weighted sum of parameterized basis functions, is represented by

$$p(\boldsymbol{\theta};\alpha) = \sum_{i=1}^{n} \alpha_i \tilde{g}(\boldsymbol{\theta}-\boldsymbol{\theta}_i) + g(\boldsymbol{\theta}-\boldsymbol{\theta}_{n+1}), \quad (6)$$

where $\tilde{g}(\boldsymbol{\theta}-\boldsymbol{\theta}_i) = g(\boldsymbol{\theta}-\boldsymbol{\theta}_i) - g(\boldsymbol{\theta}-\boldsymbol{\theta}_{n+1})$. The form of (6) guarantees a properly normalized model density.

The distance minimizing relationship between the posterior density and model density is given by [6]

$$\int_{-\infty}^{\infty} \tilde{g}(\boldsymbol{\theta}-\boldsymbol{\theta}_i) \log \sum_{i=1}^{n+1} \alpha_i g(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\, d\boldsymbol{\theta} =$$
$$\int_{-\infty}^{\infty} \tilde{g}(\boldsymbol{\theta}-\boldsymbol{\theta}_i) \log p(\boldsymbol{\theta}|Z^k)\, d\boldsymbol{\theta} \text{ for } i = 1,2,\ldots,n. \quad (7)$$

The right hand side of the above equation is thus a reduced sufficient statistic, $\boldsymbol{\chi}(k)$, for determining the α_i weights. Replacing $p(\boldsymbol{\theta}|Z^k)$ with its Bayesian update, $p(\boldsymbol{\theta}|Z^k) = \frac{1}{c} p(\mathbf{z}(k)|\boldsymbol{\theta}) p(\boldsymbol{\theta}|Z^{k-1})$, gives a recursive formula for the RSS vector elements [6].

$$\boldsymbol{\chi}_i(k) = m(\mathbf{z}(k)) + \boldsymbol{\chi}_i(k-1), \quad (8)$$

where

$$m(\mathbf{z}(k)) = \int_{-\infty}^{\infty} \tilde{g}(\boldsymbol{\theta}-\boldsymbol{\theta}_i) \log p(\mathbf{z}(k)|\boldsymbol{\theta})\, d\boldsymbol{\theta}. \quad (9)$$

To obtain the parameter estimate from the RSS vector, the model density weights $p(\boldsymbol{\theta};\alpha)$ are found using (7). The approximate minimum-variance parameter estimate is then $\hat{\boldsymbol{\theta}} = \sum_{i=1}^{n+1} \alpha_i \boldsymbol{\theta}_i$. A closed-form expression for the model weights is obtainable only if disjoint basis functions are used [7]. An exhaustive search over the α_i is required when overlapping basis functions such as Gaussians are employed.

The RSS vector recursion (8) was derived in [8] for the following nonlinear measurement model

$$\mathbf{z}(k) = \mathbf{h}_k(\boldsymbol{\theta}) + \mathbf{n}(k), \quad (10)$$

where $\mathbf{n}(k) \in \mathcal{R}^M$ is a white Gaussian vector sequence, such that $E\{\mathbf{n}(k)\mathbf{n}(j)^T\} = \mathbf{R}\delta_{k,j}$, with $\delta_{k,j}$ representing the Kronecker delta function. To derive the recursion, the function (9) must be determined for the specific form of the model's likelihood. Using the log-likelihood for the nonlinear measurement model, $m(\mathbf{z}(k))$ reduces to

$$m(\mathbf{z}(k)) = \mathbf{z}(k)^T \tilde{\mathbf{h}}_i - \frac{1}{2}\tilde{H}_i \quad (11)$$

where the terms $\tilde{\mathbf{h}}_i \in \mathcal{R}^M$ and $\tilde{H}_i$ are given by

$$\tilde{\mathbf{h}}_i = \int_{-\infty}^{\infty} \tilde{g}(\boldsymbol{\theta}-\boldsymbol{\theta}_i) \mathbf{R}^{-1} \mathbf{h}_k(\boldsymbol{\theta})\, d\boldsymbol{\theta}, \quad (12)$$
$$\tilde{H}_i = \int_{-\infty}^{\infty} \tilde{g}(\boldsymbol{\theta}-\boldsymbol{\theta}_i) \mathbf{h}_k(\boldsymbol{\theta})^T \mathbf{R}^{-1} \mathbf{h}_k(\boldsymbol{\theta})\, d\boldsymbol{\theta}.$$

Substituting $m(\mathbf{z}(k))$ as given by (11) into (8) gives the following recursive RSS vector update

$$\chi_i(k) = \mathbf{z}(k)^T \tilde{\mathbf{h}}_i - \frac{1}{2}\tilde{H}_i + \chi_i(k-1) \text{ for } i = 1,2,\ldots,n. \quad (13)$$

4 IR Image Tracking using RSS

The parameter vector to be estimated using the RSS method is $\boldsymbol{\theta} = [x_0, y_0, \dot{x}, \dot{y}]^T$, where $(x_0 = x_{peak}(0), y_0 = y_{peak}(0))$ represents initial target position, and $(\dot{x}, \dot{y})$ represents target velocity. To write the measurement equation (2) explicitly in terms of $\boldsymbol{\theta}$, we rotate and translate the Gaussian intensity function's coordinates (x', y') to be aligned with the image array coordinates (x, y) (see Figure 1), and get the equivalent measurement equation

$$z_{l,m}(k) = \frac{1}{A_p} \iint_{\text{l,mth pixel area}} h_k(\boldsymbol{\theta}, x, y)\,dx\,dy + n_{l,m}(k) + b_{l,m}(k), \quad (14)$$

where

$$h_k(\boldsymbol{\theta}, x, y) = I_{max} \times e^{-\frac{1}{2}\begin{bmatrix} x-(x_0+kT\dot{x}) \\ y-(y_0+kT\dot{y}) \end{bmatrix}^T \mathbf{C}^{-1} \begin{bmatrix} x-(x_0+kT\dot{x}) \\ y-(y_0+kT\dot{y}) \end{bmatrix}}, \quad (15)$$

$((x_0 + kT\dot{x}), (y_0 + kT\dot{y}))$ is used as an estimate of $(x_{peak}(k), y_{peak}(k))$, and

$$\mathbf{C} = \begin{bmatrix} \sigma_v^2 \cos^2\phi + \sigma_{pv}^2 \sin^2\phi & \sin\phi\cos\phi(\sigma_v^2 - \sigma_{pv}^2) \\ \sin\phi\cos\phi(\sigma_v^2 - \sigma_{pv}^2) & \sigma_v^2\cos^2\phi + \sigma_{pv}^2 \sin^2\phi \end{bmatrix}. \quad (16)$$

Estimating the integral in (14) by the integrand evaluated at the center of each pixel, the measurement equation can be written as

$$z_{l,m}(k) = h_k(\boldsymbol{\theta}, x_l, y_m) + n_{l,m}(k) + b_{l,m}(k) \quad (17)$$

where (x_l, y_m) is the center of the l, mth pixel in the 8×8 IR sub-array. Arraying the 64 pixel intensities in a vector yields

$$\mathbf{z}(k) = \mathbf{h}_k(\boldsymbol{\theta}) + \mathbf{n}(k) + \mathbf{b}(k), \quad (18)$$

where the combined effects of $\mathbf{n}(k)+\mathbf{b}(k)$ are modeled by a single vector with covariance $\mathbf{R} = \sigma_n^2\mathbf{I}$.

With the measurement equation defined in terms of $\boldsymbol{\theta}$, the RSS algorithm can be applied to the IR image tracking problem by determining the quantities $\tilde{\mathbf{h}}_i$ and $\tilde{H}_i$ found in the update equation (13), and finding a method for solving the reconstruction equation. Generally, an exact solution for $\tilde{\mathbf{h}}_i$ and $\tilde{H}_i$ will require numerical integration of the equations in (12). However, the fact that both $h_k(\boldsymbol{\theta})$, and our basis functions are Gaussian permits an exact solution for $\tilde{\mathbf{h}}_i$ and $\tilde{H}_i$ in the IR tracking problem.

To solve for $\tilde{\mathbf{h}}_i$ and $\tilde{H}_i$, we use equation (12) noting that $g(\boldsymbol{\theta} - \boldsymbol{\theta}_i)$ is a Gaussian function with mean $\boldsymbol{\theta}_i = [x_{0_i}, y_{0_i}, \dot{x}_i, \dot{y}_i]$ and covariance $\mathbf{R}_g$ set equal to $\mathbf{R}_g = diag\left[\sigma_{x_0}^2, \sigma_{y_0}^2, \sigma_{\dot{x}}^2, \sigma_{\dot{y}}^2\right]$ when a Gaussian sum is used for the model density. Using the fact that $\mathbf{R}_g$ is diagonal, the multidimensional integrals in (12) can be recognized as the correlation of Gaussian functions and thus be simplified. The element of $\tilde{\mathbf{h}}_i$ dependent on the l, mth pixel can be shown to be

$$\tilde{h}_i(x_l, y_m) = h'(\boldsymbol{\theta}_i, x_l, y_m) - h'(\boldsymbol{\theta}_{n+1}, x_l, y_m), \quad (19)$$

where

$$h'(\boldsymbol{\theta}_i, x_l, y_m) = \frac{I_{max}|\mathbf{C}|^{\frac{1}{2}}|\mathbf{C}''|^{\frac{1}{2}}}{\sigma_n^2|\mathbf{C}'|^{\frac{1}{2}}|\mathbf{C}'''|^{\frac{1}{2}}} \times$$
$$e^{-\frac{1}{2k^2T^2}\begin{bmatrix} x_l - (x_{0_i} + kT\dot{x}_i) \\ y_m - (y_{0_i} + kT\dot{y}_i) \end{bmatrix}^T \mathbf{C}'''^{-1} \begin{bmatrix} x_l - (x_{0_i} + kT\dot{x}_i) \\ y_m - (y_{0_i} + kT\dot{y}_i) \end{bmatrix}}, \quad (20)$$

$\mathbf{C}' = \mathbf{C} + \begin{bmatrix} \sigma_{x_0}^2 & 0 \\ 0 & \sigma_{y_0}^2 \end{bmatrix}$, $\mathbf{C}'' = \frac{\mathbf{C}'}{k^2T^2}$, and $\mathbf{C}''' = \mathbf{C}'' + \begin{bmatrix} \sigma_{\dot{x}}^2 & 0 \\ 0 & \sigma_{\dot{y}}^2 \end{bmatrix}$. Similarly the solution for $\tilde{H}_i$ is

$$\tilde{H}_i = \frac{1}{\sigma_n^2}\sum_{l,m}\left[H'(\boldsymbol{\theta}_i, x_l, y_m) - H'(\boldsymbol{\theta}_{n+1}, x_l, y_m)\right], \quad (21)$$

where

$$H'(\boldsymbol{\theta}_i, x_l, y_m) = \frac{I_{max}^2|\mathbf{P}|^{\frac{1}{2}}|\mathbf{P}''|^{\frac{1}{2}}}{|\mathbf{P}'|^{\frac{1}{2}}|\mathbf{P}'''|^{\frac{1}{2}}} \times$$
$$e^{-\frac{1}{2k^2T^2}\begin{bmatrix} x_l - (x_{0_i} + kT\dot{x}_i) \\ y_m - (y_{0_i} + kT\dot{y}_i) \end{bmatrix}^T \mathbf{P}'''^{-1} \begin{bmatrix} x_l - (x_{0_i} + kT\dot{x}_i) \\ y_m - (y_{0_i} + kT\dot{y}_i) \end{bmatrix}}, \quad (22)$$

$\mathbf{P} = \frac{1}{2}\mathbf{C}$, $\mathbf{P}' = \mathbf{P} + \begin{bmatrix} \sigma_{x_0}^2 & 0 \\ 0 & \sigma_{y_0}^2 \end{bmatrix}$, $\mathbf{P}'' = \frac{\mathbf{P}'}{k^2T^2}$, and $\mathbf{P}''' = \mathbf{P}'' + \begin{bmatrix} \sigma_{\dot{x}}^2 & 0 \\ 0 & \sigma_{\dot{y}}^2 \end{bmatrix}$.

Finally, to avoid a numerical search, a more practical reconstruction technique must be developed. In [8], an approximate reconstruction algorithm was achieved by using a first order approximation for the log terms. Using this approximation, the reconstruction equation can be reduced to the following set of linear equations

$$\sum_{k=1}^{n}\alpha_k\left[g(\boldsymbol{\theta}_i - \boldsymbol{\theta}_k) - e^{\chi_i}(g(\boldsymbol{\theta}_{n+1} - \boldsymbol{\theta}_k) - g(0))\right.$$
$$\left. - g(\boldsymbol{\theta}_i - \boldsymbol{\theta}_{n+1})\right] = e^{\chi_i}g(0) - g(\boldsymbol{\theta}_i - \boldsymbol{\theta}_{n+1}) \quad (23)$$

for $i = 1, 2, \ldots, n$. As the variances of the Gaussian basis functions decrease, the approximation equations converge to

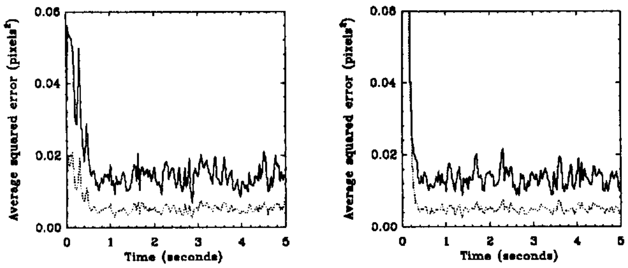
$$\alpha_i = \frac{e^{\chi_i}}{1 + \sum_{k=1}^{n} e^{\chi_k}} \text{ for } i = 1, 2, \ldots, n. \quad (24)$$

5 Performance Analysis

For comparative purposes, both the RSS and EKF tracking algorithms were applied to the same sequence of simulated IR images. Initially, a rather benign trajectory, a target moving with constant velocity in 3-D space, was chosen. However, projecting a constant velocity target onto the IR image plane produces an intensity pattern with some acceleration. The RSS and EKF target motion models are for constant velocity intensity patterns. Therefore, both algorithms require adjustments to track an accelerating target. The EKF algorithm can be easily adjusted by setting the process noise covariance to an artificially large value. To adjust the RSS algorithm, the RSS vector was reset every 5 sample times. Each time the RSS vector is reset, the basis functions are positioned around the current target state estimate. Reseting the RSS algorithm in this manner produces a piece-wise linear approximation to the target trajectory.

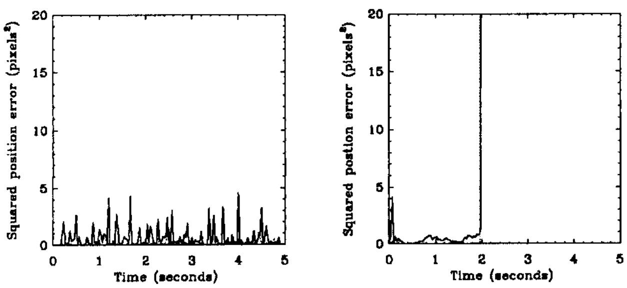
The results of running the adjusted tracking algorithms on the rather benign trajectory are shown in Figure 2. The results were averaged over 50 Monte Carlo runs. As can be seen from Figure 2, both algorithms were capable of very accurate tracking.

Next, a moderate maneuver was added to the benign trajectory after 2 seconds of flight. Even after increasing the process noise covariance, the EKF algorithm consistently lost track shortly after the onset of the maneuver. In contrast, the RSS algorithm was able to maintain track when the spacing of the position and velocity values used in the basis functions was increased. The tracking results are shown in Figure 3. Note that the accuracy of the RSS tracking is



(a) RSS algorithm (b) EKF algorithm

Figure 2: Tracking errors for benign trajectory



(a) RSS algorithm (b) EKF algorithm

Figure 3: Tracking errors for moderate maneuver

decreased somewhat due to the increased spacing of the basis functions.

The large number of basis functions employed in the RSS algorithm makes it more computationally expensive than the EKF algorithm. However, the updating of the RSS vector elements could be performed in parallel to reduce the execution time. In addition, methods to reduce the number of basis functions could be investigated. In the simulations, the simplified reconstruction formula (24) was used due to the large number of basis functions.

6 Conclusions

Using reduced sufficient statistics, a new tracking algorithm for IR images has been developed. By representing the image intensity pattern and RSS basis functions by Gaussians, a closed-form recursion for the RSS vector was found. To approximate the true posterior density, a simplified formula was used to reconstruct the model density thus avoiding an exhaustive search. In order for the RSS algorithm to track a maneuvering object through a sequence of IR images, the RSS vector was reset every 5 scans. For comparison, an EKF-based algorithm with process noise set to an artificially high value was applied to the same image sequence. Both methods could track an object with a benign trajectory. However, when the target's trajectory included a maneuver, the EKF algorithm diverged while the RSS algorithm was able to maintain track. RSS estimation provides a robust tracking algorithm for IR images, though further investigation may be necessary to reduce its complexity.

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