A Class of Polynomial Rooting Algorithms for Joint Azimuth/Elevation Estimation Using Multidimensional Arrays [1]

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Abstract

Polynomial rooting techniques for efficient high resolution estimation of direction parameters from linear arrays are well documented in the literature. These techniques are limited, however, to cases of estimating a scalar direction parameter (say, either azimuth or elevation). This paper introduces a methodology for extending the polynomial rooting philosophy to the case of multidimensional arrays, which will be used to estimate jointly both azimuth and elevation parameters of the signal directions. It will be shown via simulation that the resolution capabilities of the **P***olynomial* **R***oot* **I***ntersection for* **M***ultidimensional* **E***stimation (PRIME) class of algorithms is superior to the spectral algorithms they supplant, and that the variance of the direction estimates is equal to that of the corresponding spectral algorithms. It is shown analytically that the mean squared error of the PRIME estimates can be asymptotically equal to that of spectral MUSIC estimates. Finally, some extensions are discussed.*

1 Introduction

Estimating the directions of multiple signals using an array of sensors has been studied extensively in the literature. Classical Maximum Likelihood (ML) estimation of the signal direction parameters can be done with arrays of arbitrary geometry [1], but the resulting computational complexity of this algorithm has limited its use. In addition, when there exist two closely spaced signals of much different power and the array has been imperfectly calibrated, ML techniques tend to exhibit a strong signal capture problem, with both estimates placed close to the strong signal. Other sub-optimal algorithms of note, such as Schmidt's MUSIC algorithm [2], eliminate some of the computational complexity of the ML algorithm, but in their spectral forms lack resolution capability under moderate to low signal to noise (SNR) conditions or situations when the array calibration is less than optimal [3].

These problems were addressed for the case of estimating a scalar direction parameter with linear arrays in the ROOT-MUSIC algorithm, which turned the spectral search of the MUSIC algorithm into a polynomial rooting problem [3]. Straightforward extension of this technique to multidimensional (such as planar) arrays does not work. Due to the multidimensional nature of the sensor distributions, an arbitrary steering vector cannot be approximated as having components which are functions of only one complex variable. If a polynomial is formed using some superresolution paradigm, it will be a function of more than one variable, and thus there exist multiple loci of zero solutions, giving an uncountably infinite number of good 'root' solutions.

This paper proposes a novel technique for calculating a finite number of 'roots' which satisfy the multivariable superresolution polynomials. By constructing *multiple* independent multivariate polynomials, a finite set of simultaneous solutions can be calculated. These simultaneous solutions must include solutions which have source angle information. The resulting class of algorithms, named **P**olynomial **R**oot **I**ntersection for **M**ultidimensional **E**stimation (PRIME), are compared in simulation against some spectral counterparts. It is shown theoretically and via simulation that the variance of PRIME estimates can be equivalent to that of spectral algorithms, and it is shown via simulation that the resolution capabilities of PRIME are much better than its spectral counterparts. An efficient method for calculating the intersection points of the polynomial solutions is presented. Some candidate methods of constructing the multiple polynomials are discussed.

2 Problem Formulation

In this section we will formally introduce the problem to be solved. For simplicity, we limit ourselves to the discussion of two dimensional (planar) arrays, with sensors that lie on a rectangular grid, receiving narrow band signals. Consider a planar array of M identical sensors lying in the x, y plane, where the coordinates of the phase center of the array are $\{0,0\}$. Define the *direction cosine* vector U as:

$$U = [\, \sin\theta\cos\phi,\ \cos\theta\cos\phi,\ \sin\phi\,]^T, \tag{1}$$

where θ is the azimuth angle in the x, y plane and ϕ is the elevation angle from the x, y plane. For an array response vector $V(U)$, the entry corresponding to a sensor at the k, lth position from the phase center will be:

$$x_{k,l}(t) = (e^{j\frac{d_x\omega}{c}u_x})^k\,(e^{j\frac{d_y\omega}{c}u_y})^l\,s(t). \tag{2}$$

$s(t)$ is the (complex) signal amplitude at time t. The terms d_x and d_y correspond to the lengths between the grid lines in meters, the term ω is the frequency of radiation of the signal

[1]This work was sponsored by the Advanced Research Projects Agency and the Department of the Air Force under contract number F19628-90-C-0002

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in radians per second, and c is the speed of propagation of waves in the medium.

Define two complex variables, z and w such that

$$z \stackrel{\text{def}}{=} e^{j\frac{d_x \omega}{c} u_x} \quad (3)$$

$$w \stackrel{\text{def}}{=} e^{j\frac{d_y \omega}{c} u_y}. \quad (4)$$

The term z^k corresponds to the phase delay in the signal due to the displacement of the sensor from the phase center in the grid direction along the x axis, and the term w^l corresponds to the phase delay in the signal due to the displacement of the sensor from the phase center along the y axis. Thus, each element of the vector $V(U)$ can be expressed as a polynomial in z and w. It should also be noted that, for $|z| = |w| = 1$, the vector $V^H(z,w) = V^T(z^{-1}, w^{-1})$. A general class of 'superresolution' direction estimation algorithms can be expressed as finding the arguments which give local minima to a function of the form:

$$||f(R_{xx})V(U)||^2, \quad (5)$$

where R_{xx} is the sampled covariance matrix of sensor outputs. Typical realizations of the function $f(R_{xx})$ include a projection matrix onto the noise subspace (MUSIC), ${R_{xx}}^{-1/2}$ (MLM), and ${e_1}^H {R_{xx}}^{-1}$ (MEM). The magnitude of the polynomial $z^K w^L V^T(z^{-1}, w^{-1}) f(R_{xx})^H f(R_{xx}) V(z,w) \stackrel{\text{def}}{=} g(z,w)$ is equal to $||f(R_{xx})V(U)||^2$ when $|z| = |w| = 1$. The superresolution algorithms can be expressed as finding the values of z and w which give local minima to the polynomial $g(z,w)$ under these conditions.

3 PRIME Superresolution Algorithms

The above methodology, when applied to a linear array, gives a polynomial which is a function of z only. In this case, if the constraint $|z| = 1$ is lifted, some values of z can be found which cause the polynomial to equal zero. The arguments of these values of z are used to estimate the azimuths of the signals. This technique, for obvious reasons, is called rooting. Many variants of rooting algorithms exist [4], but the most prominent by far is ROOT-MUSIC.

When the array is no longer linear, the straightforward application of rooting disappears, as the polynomial is now in two variables. If it is possible to form two independent complex polynomials (i.e., with no non-trivial common factors), say $g_1(z,w)$ and $g_2(z,w)$ such that both g_1 and g_2 are derived from a superresolution functional, then there would exist two complex equations and two complex unknowns. The polynomial equations

$$g_1(z,w) = 0 \quad (6)$$

$$g_2(z,w) = 0 \quad (7)$$

have a finite set of z, w pairs which solve both equations simultaneously. Assuming that the polynomials were formed correctly, the arguments of some of these z, w pairs must correspond to azimuth/elevation parameter pairs of the signal environment. This forms the basis for the PRIME method of direction of arrival estimation.

3.1 Polynomial Construction

There are several methods for generating the two independent polynomials $g_1(z,w)$ and $g_2(z,w)$. One is to consider the collection of antennas as the intersection of two dissimilar subarrays (see figure 1 for an example). 'Dissimilar' implies that one subarray is not simply a shifted version of the other subarray. If the two subarrays are each used to generate a polynomial, then typically these two polynomials will be independent, and the intersections of their zero solutions will include the z, w pairs which have the direction of arrival information in their arguments.

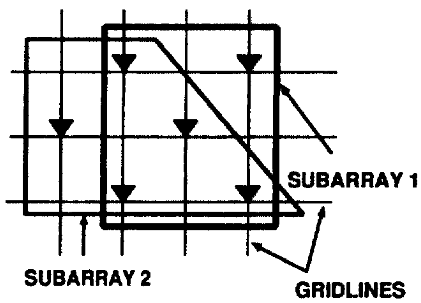

Figure 1: Two subarrays from a rectangular planar array.

Choosing the subarray configuration is an important part of this method. For greatest accuracy of the direction estimates, the aperture of each subarray should be as large as possible. In addition, if the number of antennas in the first and second subarray are denoted as M_1 and M_2, respectively, then the maximum number of signals which can be unambiguously estimated will be

$$\min\{M_1, M_2\} - 1 \quad (8)$$

for unipolar arrays. The maximum number of signal directions which can be estimated using this technique is $M - 2$ for a unipolar array. If the receiver channels can be switched between elements, then the number of receivers needed is equal to the larger of M_1 or M_2. With proper system design, a unipolar array can estimate directions for one less signal than there are receiver channels in the system. This may be an advantage for sparse array systems which have larger costs for receiver hardware than for sensor elements, where directional ambiguity problems may be avoided by using switched arrays.

Another polynomial construction method is most applicable to noise subspace projection algorithms. In this method, two non-identical (but possibly intersecting) subspaces of the noise subspace are chosen. The two polynomials are constructed by using the projections onto the two subspaces of the noise subspace. This method requires that the noise subspace must have a dimension of at least 2 for unipolar arrays.

Thus, with M sensors and receivers, $M-2$ signal directions can be estimated with a unipolar array.

A simple application example of PRIME-MUSIC is the four element square array. If the top left element of this array is designated the phase center, and the elements are ordered in a clockwise fashion from the phase center, then the steering vector for this array can be written as:

$$V(z,w) = [\,1,\ w,\ zw,\ z\,]^T. \tag{9}$$

Assume a two dimensional noise subspace. The two spanning vectors for this space will be denoted as A and B. Thus, the two multivariate polynomial equations which must be satisfied for PRIME-MUSIC are:

$$\begin{aligned} A^H V(z,w) &= 0 \qquad (10)\\ B^H V(z,w) &= 0. \qquad (11)\end{aligned}$$

These equations can be re-written in the form:

$$\begin{aligned} z &= \frac{-(a_1^* + a_2^* w)}{(a_4^* + a_3^* w)} \qquad (12)\\ w &= \frac{-(b_1^* + b_4^* z)}{(b_2^* + b_3^* z)}. \qquad (13)\end{aligned}$$

Making the correct substitutions gives us a quadratic equation for z. These solutions can be plugged into equation (13) to give the corresponding values of w. Hence, *in the four element square array case*, PRIME-MUSIC is a very simple algorithm, requiring only the initial four-by-four SVD and the solution of a quadratic equation.

4 Polynomial Root Intersections

The heart of the PRIME algorithm is the formation of polynomial root intersections. For the cases treated here, root intersection means finding the common zeros of two polynomials in two unknowns. Several procedures are relevant. First, numerical techniques can be employed if good initial guesses are available. A more sophisticated, global approach utilizes root tracking and homotopy (*e.g.,* linear interpolation) of the polynomial coefficients from a canonical, decoupled set of equations to the equations of interest (see [5]). This is most appropriate when the polynomials have high degree. For lower degree polynomials, a simpler procedure provides all intersection points. This procedure relies on a set of classical techniques often called *elimination theory.* An exposition of the ideas involved is provided below.

Consider two polynomials $f(x)$ and $g(x)$ of a single variable with, for example, complex coefficients:

$$\begin{aligned} f(x) &= a_n x^n + \ldots + a_1 x + a_0\\ g(x) &= b_m x^m + \ldots + b_1 x + b_0.\end{aligned}$$

Let $\{\gamma_i\}$ and $\{\delta_i\}$ denote the roots of f and g respectively. The *resultant* ([6]) of the two polynomials is given by

$$R(f,g) \stackrel{\text{def}}{=} a_n^m b_m^n \prod_{ij} (\gamma_i - \delta_j).$$

It can be shown that $R(f,g)$ is a polynomial in the coefficients $\{a_i\}$, $\{b_j\}$ of the polynomials f and g that vanishes if and only if $f(x)$ and $g(x)$ have a common root, provided at least one of a_n and b_m is nonzero. There are several different but equivalent expressions for the resultant. One of the simplest, due to Sylvester, expresses $R(f,g)$ as the determinant of the $(m+n)\times(m+n)$ matrix

$$\begin{pmatrix} a_n & a_{n-1} & \ldots & a_0 & 0 & \ldots & 0\\ 0 & a_n & a_{n-1} & \ldots & a_0 & \ldots & 0\\ \vdots & & & & & & \\ 0 & \ldots & 0 & a_n & \ldots & a_1 & a_0\\ b_m & b_{m-1} & \ldots & b_0 & 0 & \ldots & 0\\ 0 & b_m & b_{m-1} & \ldots & b_0 & \ldots & 0\\ \vdots & & & & & & \\ 0 & \ldots & 0 & b_m & \ldots & b_1 & b_0 \end{pmatrix}.$$

PRIME must find the intersections of two-variable polynomials $g_1(z,w)$ and $g_2(z,w)$. Viewing these as polynomials in w with coefficients that are polynomials in z, the resultant, denoted $R_w(z)$, is a polynomial in z whose roots are the z-coordinates of the common solutions of $g_i(z,w)=0$. Similarly, the resultant $R_z(w)$ is a polynomial in w expressing the w-coordinates. For small degrees, the resultants R_z and R_w can be rooted and the resulting z-w coordinates paired in all possible ways forming a list of candidate intersections that can be plugged back into the original equations. Alternatively, R_z can be rooted with the resulting roots plugged back into the original equations. These equations must share common roots, which can be determined by rooting them.

Not all common roots of $g_i(z,w)=0$ correspond to physical angles. Physically relevant roots can be selected by evaluating a direction-finding statistic such as MUSIC for each common root or by utilizing prior information about signal angles of arrival.

If $R_w(z)$ (or $R_z(w)$) vanishes identically, the polynomials $g_i(z,w)=0$ have a common nontrivial factor (assuming, for example, that the coefficient of the highest w-order term of $g_1(z,w)$ or $g_2(z,w)$ is a nonzero polynomial in z). In this case, the solution sets intersect along the curve defined by the common factor. This is not the typical case encountered in applications. However, it can happen that the $g_i(z,w)$ have a common data-independent factor (typically of the form $z^L w^N$), which can be eliminated from both polynomials.

The number of intersection points of the root loci $g_i(z,w)=0$ depends on the polynomial degrees, the array geometry, and any symmetries inherent in the polynomial coefficients, which, in turn, depend on the polynomial selection procedure. Let d_i denote the degrees of g_i. A bound on the number of intersection points can be based on Bezout's theorem, which states that the number of common solutions of the homogenized polynomials $v^{d_i} g_i(z/v, w/v)$, for $i=1,2$, taking multiplicity into account and including roots at infinity, is $d_1 \cdot d_2$. The polynomials typically used for direction-finding share degenerate roots at infinity (*i.e.,* common solutions involving only the highest order terms of each polynomial). Accounting for the multiplicity of these roots at infinity, a bound on the number of roots (including multiplicity) in the finite (z,w)-plane can be formulated. Alternatively, a result of Kouchnirenko [7] can be used to count the number of common roots of polynomials built using the subspace division method. For example, a rectangular $m \times n$ array provides $2(m-1)(n-1)$ roots, generically, with both z and w non-zero.

5 Direction-Finding Accuracy

Let $V(\alpha,\beta)$ denote the array response of a planar array as a function of the "angles" $\rho = (\alpha,\beta)$. For applications, one can think of $\alpha = 2\pi u_x d_x/\lambda$ and $\beta = 2\pi u_y d_y/\lambda$ where (u_x, u_y, u_z) are the direction cosines for a signal arriving with wavelength λ at a planar x-y array whose elements lie on a rectangular grid with x-spacing d_x and y-spacing d_y.

The MUSIC spectral function can be written as

$$\frac{V^H(\rho)V(\rho)}{V^H(\rho)\hat{P}_N V(\rho)}$$

where $\hat{P}_N$ denotes the noise space projector based on the sample covariance matrix. The ρ-coordinates of local maxima provide angle estimates. Let $R = [E_S E_N]\Lambda[E_S E_N]^H$ express the eigenanalyzed true covariance matrix with ordered eigenvalues in the diagonal matrix Λ and associated normalized eigenvectors in the columns of the signal subspace matrix E_S and the noise subspace matrix E_N. Asymptotically in the number of samples L, the noise subspace projector can be approximated as

$$\hat{P}_N \approx P_N + E_S B E_N^H + E_N B^H E_S^H$$

where B is a $M \times (M-S)$ array with independent, mean zero, complex circular Gaussian entries of covariance

$$\mathrm{E}[|B_{jk}|^2] = L^{-1}\frac{\lambda_j \lambda}{(\lambda_j - \lambda)^2} \tag{14}$$

for $1 \le j \le S$ and $1 \le k \le M-S$ where

$$\lambda_1 \ge \cdots \ge \lambda_S > \lambda_{S+1} = \cdots = \lambda_M \equiv \lambda$$

are the ordered eigenvalues of the spatial covariance and S is the signal subspace dimension.

Let $\rho_\Delta \stackrel{\text{def}}{=} \rho - \rho_0$, where ρ_0 is the true direction of arrival (MUSIC spectral peak), express the estimation error. Define

$$D \stackrel{\text{def}}{=} L^{-1}\mathrm{diag}(\frac{\lambda_1\lambda}{(\lambda_1-\lambda)^2}, \ldots, \frac{\lambda_S\lambda}{(\lambda_S-\lambda)^2}).$$

Let V_α and V_β denote the α and β partial derivatives of $V(\alpha,\beta)$. Also let $\Re(\cdot)$ denote the real part of its argument. Using the asymptotic approximation of the noise space projector given above and the notation just introduced, the asymptotic variance of the estimation error can be written (see also [8])

$$\mathrm{E}[\rho_\Delta\rho_\Delta^T] \approx \frac{1}{2}(V^H E_S D E_S^H V)\left\{\Re\begin{pmatrix} V_\alpha^H P_N V_\alpha & V_\alpha^H P_N V_\beta \\ V_\beta^H P_N V_\alpha & V_\beta^H P_N V_\beta \end{pmatrix}\right\}^{-1}.$$

Let $z = e^{i\alpha}$ and $w = e^{i\beta}$ be the unit circle elements corresponding to the "angles" $\rho = (\alpha,\beta)$. Below, z and w are allowed to take values off the unit circle. In this case, their phases correspond to angle estimates. The true angles of a signal are denoted $\rho_0 = (\alpha_0,\beta_0)$. Let $(z_0,w_0) = (e^{i\alpha_0}, e^{i\beta_0})$.

PRIME utilizes arrays whose elements lie on a planar grid. The elements can be indexed by integer pairs. Let the $(j,k)^{\text{th}}$ element have response $\gamma_{jk}z^j w^k$, where

$$-(N-1)/2 \le j,k \le (N-1)/2.$$

The γ_{jk} are not assumed to have any pattern variations. The array response vector has apparent size N^2 (*e.g.,* when N is even). However, some of the γ_{jk} are allowed to vanish. This is taken into account by dropping these elements from the array response vector so that its actual length is denoted M, as above. The array response can then be written as a vector $P(z,w)$ with monomial entries satisfying $P(e^{i\alpha}, e^{\beta}) \stackrel{\text{def}}{=} e^{iN_z\alpha}e^{iN_w\beta}V(\alpha,\beta)$ for appropriately chosen N_z, N_w. To rephrase 2-D spectral MUSIC accuracy results in terms of the monomial vector $P(z,w)$ and its derivatives, define

$$\begin{aligned} G &\stackrel{\text{def}}{=} \begin{pmatrix} e^{i\alpha_0} & 0 \\ 0 & e^{i\beta_0} \end{pmatrix} \\ \delta &\stackrel{\text{def}}{=} V^H(\alpha_0,\beta_0)E_S D E_S^H V(\alpha_0,\beta_0) \\ &= P^H(z_0,w_0)E_S D E_S^H P(z_0,w_0) \\ C_N &\stackrel{\text{def}}{=} G^H(P_z P_w)^H P_N (P_z P_w) G. \end{aligned}$$

Then the asymptotic accuracy of 2-D MUSIC can be written

$$\mathrm{E}[\rho_\Delta\rho_\Delta^T] \approx \frac{\delta}{2}(\Re C_N)^{-1}.$$

This expression can be simplified slightly when the array is *centro-symmetric.* Centro-symmetry holds when $|\gamma_{jk}| = |\gamma_{-j-k}|$. For example, if the γ_{jk} are all either one or zero then the elements are paired about a common phase center, each element of a pair at opposite ends of a diameter through the phase center. Of course there can be an element at the phase center which can be considered paired with itself. For centro-symmetric arrays and signals that are *not* fully correlated, C_N is real.

One version of PRIME is based on a polynomial built from the denominator of the MUSIC function. Let $F(z,w) \stackrel{\text{def}}{=} z^J w^K P^T(z^{-1},w^{-1})\hat{P}_N P(z,w)$, where J and K are chosen to make $F(z,w)$ a polynomial. With infinite samples (*i.e.,* $\hat{P}_N = P_N$), $F(z,w)$ has double roots in z at $z = z_0$ for fixed $w = w_0$. Similarly, $F(z,w)$ has double roots in w for fixed $z = z_0$. Thus the partials F_z and F_w have a common root at (z_0,w_0). Intersecting these partials is called the *partial derivative method.* Angle estimates (α,β) are obtained from the angular components of (z,w). The estimates have the same asymptotic accuracy as those of 2-D spectral MUSIC as described above.

Another implementation of PRIME is based on choosing *noise subspace probes* (a special case of subspace division PRIME-MUSIC). This method provides polynomials of smaller degree than those obtained from partial derivatives. For two M-vectors, q_i, define two polynomials

$$g_i(z,w) \stackrel{\text{def}}{=} q_i^H \hat{P}_N P(z,w).$$

The common roots provide estimates of (z_0, w_0). Let $\rho_\delta \stackrel{\text{def}}{=} (z-z_0, w-w_0)$ represent the estimation error. Let P_E denote the projector onto the 2-dimensional subspace of the (true) noise subspace spanned by $P_N q_i$. Also let

$$C_E \stackrel{\text{def}}{=} G^H (P_z\, P_w)^H P_E (P_z\, P_w) G.$$

Then, asymptotically, the estimation error is complex circular Gaussian with variance

$$\mathrm{E}[\rho_\delta \rho_\delta^H] \approx \delta G C_E^{-1} G^H.$$

Note that $C_E \leq C_N$ (and hence $C_N^{-1} \leq C_E^{-1}$) since $P_E \leq P_N$ as positive semi-definite hermitian matrices. Equality is achieved if the span of q_i and the span of $P_z(z_0, w_0)$, $P_w(z_0, w_0)$ project onto the same two dimensional subspace of the noise space.

The variance of the angle estimates is expressed by the 2×2 real matrix $\delta/2\, \Re(C_E^{-1})$. Now,

$$\frac{\delta}{2}(\Re C_N)^{-1} \leq \frac{\delta}{2}\Re(C_N^{-1}) \leq \frac{\delta}{2}\Re(C_E^{-1}),$$

where the left-hand and right-hand expressions represent the asymptotic error variances of 2-D spectral MUSIC and PRIME based on noise space probes, respectively. It follows that the asymptotic accuracy of 2-D MUSIC is, in general, better. However, for the proper choice of noise space probes and centro-symmetric planar arrays, the accuracies are identical. In particular, when the array is centro-symmetric and the noise subspace is two-dimensional, the accuracies are identical.

6 Simulation Results

This section will address results of using two PRIME-MUSIC implementations with a five element array. Two four-element subarrays are chosen (see Figure 2) for the subarray method, but the implementation used assumes that there are five receivers available. Thus, the data set is identical to that used with the subspace partition PRIME-MUSIC algorithm. We compare the results against those of a spectral MUSIC algorithm. Three emitters are simulated, separated by a fraction of a Rayleigh beamwidth. One of the emitters is 10 dB lower than the other two. The array is shown in Figure 2. The three sources are arranged such that there is roughly 1/6th of a beamwidth from one source to either of the others. Each trial consisted of taking 100 looks of data. 100 trials were conducted for each algorithm under a given set of conditions. The noise power in the simulation was varied, and the number of independent trials in which each algorithm resolved the weakest signal was calculated. The results of these experiments are plotted in Figure 3. It is obvious from the simulation results that both the PRIME-MUSIC algorithm based on two four-element subarrays and the the PRIME-MUSIC algorithm based on noise subspace partitioning both outperform the spectral MUSIC algorithm. In order to achieve the same probability of resolution, the spectral MUSIC algorithm requires roughly 6 dB greater SNR. It is also interesting to note that the two variants of the PRIME-MUSIC algorithm both have equivalent probability of resolution curves. Thus, for at least this case there seems to be little difference in the resolution capabilities of the two PRIME algorithm variants.

A second set of trials were conducted at sufficiently low noise levels such that all three algorithms resolved the weak signal. The standard deviation of the estimates for the weak signal were calculated based on 100 trials for each algorithm at each noise level. The results are shown in Figure 4. These results indicate that the standard deviations of the three algorithms are equivalent. This indicates that the PRIME algorithms are using the data in a manner as efficient as the spectral MUSIC algorithm. Thus, the additional resolution power of the PRIME-MUSIC algorithms does not incur a cost in estimate accuracy. The equivalence of the standard deviations is analogous to that of ROOT-MUSIC estimates to spectral MUSIC estimates for line arrays [9].

7 Further Work

There are a number of extensions to the preceding work that are worth mentioning very briefly. Polarization diverse arrays are of considerable practical interest and can benefit from PRIME techniques similar to those discussed above. Non-planar arrays also can use the PRIME methodology based, for example, on multi-resultant concepts [6]. The gain pattern of a calibrated array can be modeled (through a truncated Fourier series) in terms of a larger, synthetic array whose elements lie on a lattice. PRIME can be applied to the synthetic array. Thus the limitations on array geometry are less stringent than first appears. The rectangular lattices considered here can be replaced with more general lattices. In some cases it is possible to characterize the array layout with more than one kind of lattice. Finally, polynomial selection is an important component of PRIME implementations and can be studied in a variety of ways. For example, more than two polynomials can used with planar arrays in order to help with root selection and possibly enhance resolution/accuracy.

8 Conclusion

We have presented a new class of direction estimation algorithms for planar arrays which allows a polynomial rooting approach to joint azimuth/elevation estimation. The PRIME algorithms recognize that the traditional barriers to the advantages of polynomial rooting techniques can be eliminated by simultaneously solving two polynomials of two complex variables. A method for reducing this to solving a small number of single variable polynomial equations has been shown to be computationally efficient.

Expressions for the asymptotic variance of PRIME estimates shows that under some benign conditions, the variance of PRIME algorithms will be equal to that of 2 dimensional spectral MUSIC. The enhanced resolution capabilities of the PRIME-MUSIC algorithm over the spectral MUSIC algorithm were demonstrated via simulation of a five element array with two strong interference signals and one weaker signal of interest. In addition, the simulation showed that the standard deviation of the estimates provided by PRIME-MUSIC are equivalent to those given by spectral MUSIC, if spectral MUSIC is capable of resolving the sources, as was predicted. Thus, by moving to PRIME-MUSIC from spectral MUSIC, there is a gain in both resolution and computational efficiency, with no loss of accuracy in the resulting estimates.

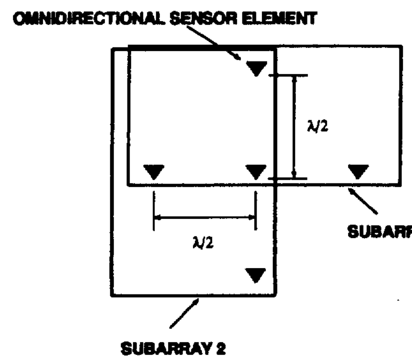

Figure 2: 5 element planar array used in simulation.

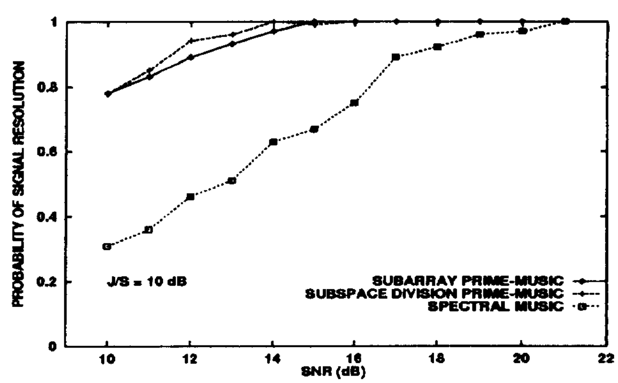

Figure 3: Probability of weak signal resolution in presence of two strong interferers.

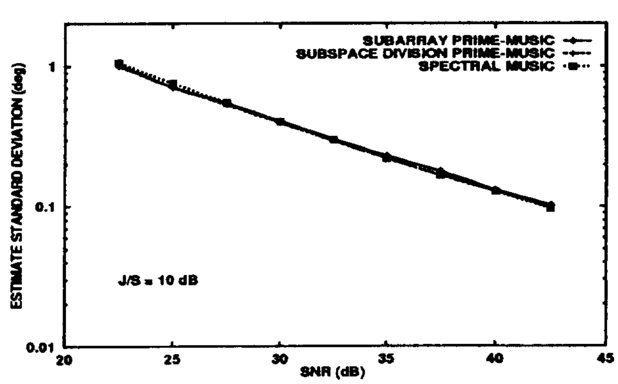

Figure 4: Standard deviation of weak signal direction estimate.

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