Taylor Series Expansion and Modified Extended Prony Analysis for Localization

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Abstract

In the multiple source localization problem, many inverse routines use a rooting of a polynomial to determine the source locations. We present a rooting algorithm for locating an unknown number of three-dimensional, near-field, static sources from measurements at an arbitrarily spaced three-dimensional array. Since the sources are near-field and static, the spatial covariance matrix is always rank one, and spatial smoothing approaches are inappropriate due to the spatial diversity. We approach the solution through spherical harmonics, essentially replacing the point source function with its Taylor series expansion. We then perform a modified extended Prony analysis of the expansion coefficients to determine the number and location of the sources. The full inverse method is typically ill-conditioned, but a portion of the algorithm is suitable for synthesis analysis. We present a simulation for simplifying point charges limited to a spherical region, using an array of voltage potential measurements made outside the region.

1.0 Introduction

In the multiple source localization problem, many inverse routines require a search through a low-dimensional (e.g. MUSIC) or multi-dimensional signal subspace. Rather than searching for minima, alternative approaches use a rooting of a polynomial (e.g. Root-MUSIC) to determine the source locations. These rooting algorithms are typically specialized for independent far-field sources arriving as plane waves at a uniform linear array, whose elements are spaced at half wavelengths. We present here a rooting algorithm for locating an unknown number of three-dimensional, near-field, static sources from measurements at an arbitrarily spaced three-dimensional array. Since the sources are near-field and static, the spatial covariance matrix is always rank one, and spatial smoothing approaches are inappropriate due to the spatial diversity. We approach the solution through spherical harmonics, essentially replacing the point source function with its Taylor series expansion. We then perform a modified extended Prony analysis (MEPA) of the expansion coefficients to determine the number of sources and to form the coefficients of polynomials whose roots yield the three-dimensional locations.

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2.0 One-dimensional model

We use the one-dimensional point charge model to illustrate the new procedure, then in subsequent sections expand to the three-dimensional localization problem. The scalar model is

$$\phi(r) = \frac{q}{r-l} \qquad (1)$$

where q is the charge, l is its location in one dimension, and r is the observation point in one dimension. We acquire the data at several locations, and assume several sources. Thus the general measurement vector to source vector relationship for m measurements and p sources is

$$\begin{bmatrix} \phi(r_1) \\ \cdots \\ \phi(r_m) \end{bmatrix} = \begin{bmatrix} \frac{1}{r_1 - l_1} & \cdots & \frac{1}{r_1 - l_p} \\ \cdots & \cdots & \cdots \\ \frac{1}{r_m - l_1} & \cdots & \frac{1}{r_m - l_p} \end{bmatrix} \begin{bmatrix} q_1 \\ \cdots \\ q_p \end{bmatrix}. \qquad (2)$$

Our "conventional" approach would be to assume a model order, factor out the linear terms q_i, then begin a nonlinear search as a function of the l_i. If we had temporal independence in the source intensities q_i over many time slices of data, we could use MUSIC to locate the sources (see [1] for an application of MUSIC to an analogous modeling problem in magnetoencephalography). Here we assume a single time slice, such that no temporal diversity information is available. We show below a three step procedure the effectively replaces the nonlinear search with a polynomial factorization.

2.1 Taylor polynomial

The n-th order Taylor polynomial for the i-th source in our model as a function of l_i and expanded about $l_i = 0$ is

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$$T_n(l_i) = \frac{q_i}{r} + \frac{q_i}{r^2} l_i + \ldots + \frac{q_i}{r^{n+1}} l_i^n . \qquad (3)$$

For p sources, each expanded as a n-th order Taylor polynomial, the combined polynomial becomes

$$T_n(l_1, \ldots, l_p) = \frac{\sum_{i=1}^{p} q_i}{r} + \frac{\sum_{i=1}^{p} q_i l_i}{r^2} + \frac{\sum_{i=1}^{p} q_i l_i^2}{r^3} + \ldots + \frac{\sum_{i=1}^{p} q_i l_i^n}{r^{n+1}} \qquad (4)$$

We can decompose this representation into two parts, useful below in the MEPA steps. We replace each numerator with the variable x, such that

$$\begin{bmatrix} x^{(0)} \\ \ldots \\ x^{(n)} \end{bmatrix} = \begin{bmatrix} l_1^0 & \ldots & l_p^0 \\ \ldots & \ldots & \ldots \\ l_1^n & \ldots & l_p^n \end{bmatrix} \begin{bmatrix} q_1 \\ \ldots \\ q_p \end{bmatrix} . \qquad (5)$$

Then our polynomial can be represented as

$$T_n(l_1, \ldots, l_p) = \frac{x^{(0)}}{r} + \frac{x^{(1)}}{r^2} + \frac{x^{(2)}}{r^3} + \ldots + \frac{x^{(n)}}{r^{n+1}} . \qquad (6)$$

2.2 MEPA

2.2.1 Model approximation

We now substitute our Taylor polynomial in (4) as an approximation to our model. We select a polynomial order m_l strictly less than the number of sensors m and greater than or equal to two times the number of sources (we have two parameters per source). Our model becomes

$$\begin{bmatrix} \phi(r_1) \\ \ldots \\ \phi(r_m) \end{bmatrix} \approx \begin{bmatrix} \frac{1}{r_1} & \ldots & \frac{1}{r_1^{m_1+1}} \\ \ldots & \ldots & \ldots \\ \frac{1}{r_m} & \ldots & \frac{1}{r_m^{m_1+1}} \end{bmatrix} \begin{bmatrix} \sum_{i=1}^{p} q_i \\ \ldots \\ \sum_{i=1}^{p} q_i l_i^{m_1} \end{bmatrix} \qquad (7)$$

$$\begin{bmatrix} \phi(r_1) \\ \ldots \\ \phi(r_m) \end{bmatrix} \approx \begin{bmatrix} \frac{1}{r_1} & \ldots & \frac{1}{r_1^{m_1+1}} \\ \ldots & \ldots & \ldots \\ \frac{1}{r_m} & \ldots & \frac{1}{r_m^{m_1+1}} \end{bmatrix} \begin{bmatrix} x^{(0)} \\ \ldots \\ x^{(m_1)} \end{bmatrix} \qquad (8)$$

The immediate advantage of (8) over (2) is that the transfer matrix now contains only known parameters, and the unknown vector $\boldsymbol{x}$ is easily found as a linear solution to $\boldsymbol{Ax} = \boldsymbol{b}$. We present next the three steps for MEPA that will solve the above approximation for each q_i and l_i. We defer considerations of numerically forming the matrix, which can become poorly conditioned for poor choices of units. A full discussion of scalar MEPA can be found in [2].

2.2.2 Step one

Form and solve (8) for the unknown linear parameters $\boldsymbol{x}$. The set of equations should be over or exactly constrained, and the coordinate origin of the system should be translated such that condition of the matrix does not become unduly large, nor require a large series expansion order to account for the sources. See [3](Section 2.8) for algorithms to avoid explicitly forming the higher powers of $(1/r)$. If suitable conditioning can not be obtained, we must seek alternative approaches, such as multiple expansion points.

2.2.3 Step two

We now have a system of equations that can be described from (5) as

$$\begin{bmatrix} x^{(0)} \\ \ldots \\ x^{(m_1)} \end{bmatrix} = \begin{bmatrix} l_1^0 & \ldots & l_p^0 \\ \ldots & \ldots & \ldots \\ l_1^{m_1} & \ldots & l_p^{m_1} \end{bmatrix} \begin{bmatrix} q_1 \\ \ldots \\ q_p \end{bmatrix} \qquad (9)$$

where the location l_i and the intensity q_i are unknown.

The matrix has the special Vandermonde structure we will exploit using MEPA [2]. We form the system of equations

$$\begin{bmatrix} x^{(p)} \\ \ldots \\ x^{(m_1)} \end{bmatrix} = \begin{bmatrix} x^{(p-1)} & \ldots & x^{(0)} \\ \ldots & \ldots & \ldots \\ x^{(m_1-1)} & \ldots & x^{(m_1-1-p)} \end{bmatrix} \begin{bmatrix} a_1 \\ \ldots \\ a_p \end{bmatrix} \qquad (10)$$

and solve for the linear parameters a. We note that since we specified $m_l >= 2p$, this set of equations should be exactly or over constrained. We then root the p-th order polynomial

$$p(l) = 1 + (-a_1) l^{-1} + \ldots + (-a_p) l^{-p} . \qquad (11)$$

The roots of this polynomial are the locations l_i.

2.3 Step three

Insert the derived locations l_i into (9) and solve the linear set of equations for the q_i. Again, see [3](Section 2.8) for discussions on efficiently finding the solution.

2.4 Final fit

In the presence of noise, the above steps will "color" the original white noise assumed added to the noiseless signal data. Thus we use the estimates q_i and l_i as initial starting points in a full "conventional" nonlinear least-squares search to achieve a final fit in (2).

3.0 Green's function expansion

We now extend these one dimensional results to the three dimensional localization problem. The solution $1/|\boldsymbol{r}-\boldsymbol{r}'|$ is the Green's function for the source at $\boldsymbol{r}'$ and the observation point at $\boldsymbol{r}$. The voltage potential observed at location $\boldsymbol{r}$ for a charge at $\boldsymbol{r}'$ is

$$\phi(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}, \qquad (12)$$

where $\rho(\boldsymbol{r}')$ is the scalar charge density. From [4], Eq. (3.70),

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi) \qquad (13)$$

where $\boldsymbol{r}$ in spherical coordinates has distance r from the origin, angle ϕ from the x-axis, and angle θ from the z-axis, such that the Cartesian transformations are $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. The notation $r_<$ ($r_>$) denotes the smaller (larger) of r and r'. The *spherical harmonics* Y_{lm} are orthonormal functions over the unit sphere related to the *associated Legendre functions* P_l^m by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\phi}. \qquad (14)$$

We can also express (13) directly in terms of the associated Legendre polynomials,

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r_<}{r_>}\right)^l(-1)^{|m|}P_l^{-m}(\cos\theta')e^{-im\phi'}P_l^m(\cos\theta)e^{im\phi} \qquad (15)$$

Assuming that $r > r'$, we are particularly interested in the $r'^m P_m^m(\cos\theta')e^{im\phi'}$ terms, since they can be expressed as

$$r'^m P_m^m(\cos\theta')e^{-im\phi'} = (-1)^m(2m-1)!!\,(r'\sin\theta' e^{-i\phi'})^m \qquad (16)$$

The unknown parameters in (16) are the complex number $r'\sin\theta' e^{i\phi'}$ raised to the mth power. We note that in Cartesian coordinates, $r'\sin\theta' e^{-i\phi'} = (x'-iy')$. The other terms involve mixed products of these two parameters and the third parameter z, discussion of which we defer.

An alternative expansion could be worked out in rectangular coordinates. As presented and compared in [5](page 1277), the expansions are analogous; however, the rectangular coordinates expansion will yield additional terms that are essentially unobservable, complicating our construction of inverse solutions.

The approach is therefore to form the system of equations from multiple sensor sites (using spherical to Cartesian transformations to keep all equations in a common framework), solve for the multipole moments, extract the terms of interest (16), then perform MEPA to extract the x and y locations. We presently extract the z location by a change of coordinates, such that it effectively maps into x or y.

4.0 Sequence extraction

We discuss in detail the parameters extracted from the expansion coefficient sequence, using the example of two sources and a sixth order expansion, or equivalently 49 terms. The terms of interest to us correspond to $m = l$ and $m = -l$ for $l = 0,...,6$. Indexing the terms in (15) from 1-49, we desire indices

ndx1 = [1, 4, 9, 16, 25, 36, 49],

ndx2 = [1, 2, 5, 10, 17, 26, 37].

These particular expansion coefficients allow us to form two complex series analogous to (4), one series corresponding to the positive m terms, the other to the negative. The x and y locations of the sources form the complex location $x - iy$. We can also extract out terms corresponding to $+/-(m-1)$, which also form a similar sequence, except that the linear source strength is multiplied by the unknown constant z. The sources of these series remain the same $x - iy$:

ndx3 = [3 8 15 24 35 48],

ndx4 = [3 6 11 18 25 38].

For illustration purposes, we then fill out (10) using the index numbers in *ndx1* instead of the true values,

$$\boldsymbol{b}_1 = \begin{bmatrix}9\\16\\25\\36\\49\end{bmatrix} = \begin{bmatrix}4 & 1\\9 & 4\\16 & 9\\25 & 16\\36 & 25\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \boldsymbol{A}_1\boldsymbol{a} \qquad (17)$$

Similarly, we fill out matrices $\boldsymbol{A}_2$, $\boldsymbol{A}_3$, and $\boldsymbol{A}_4$. We can then concatenate these matrices into one system of equations, since in each case we are seeking the same a_1 and a_2. We have thus extracted 26 terms from the original 49 terms in the expansion to form a combined *18 x 2* set of equations.

Solving for our example yields a_1 and a_2. Rooting the polynomial coefficients $(1, -a_1, -a_2)$ yields the x and y locations of the two sources.

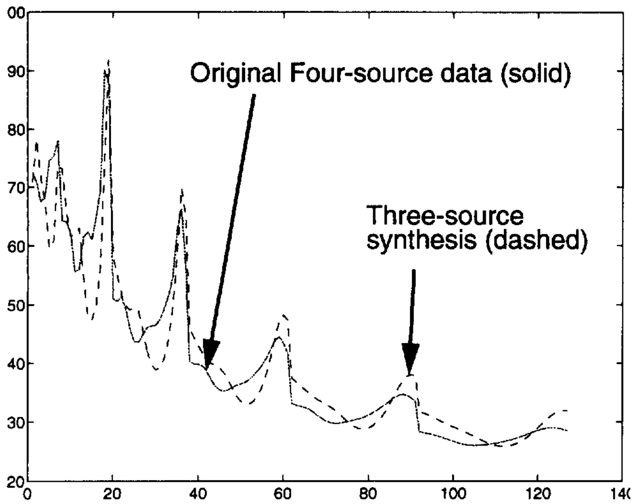

FIGURE 1. Comparison of original four source data with estimated three source equivalent.

5.0 Forward synthesis

In this example, we show how this method may be used to simplify multiple sources to a parsimonious representation. We assume as distribution of 127 sensors about a hemisphere of radius 12 cm. The sensors essentially measure the voltage of the sources in a medium of infinite homogeneity, i.e., no boundary conditions are imposed, such that the simplified model in (12) holds. We placed four sources at

$$l_1 = (-1, -3, 8) \qquad l_2 = (3, 3, 7.5)$$
$$l_3 = (4, -4, 9) \qquad l_4 = (-3, 1, 8.5) \tag{18}$$

Using a sixth order expansion, the resulting 10 x 4 matrix A (analogous to (17)) yields the correct x and y locations. Rotating the coordinates x,y,z into y,z,x yields the correct y and z locations as well. However, the condition numbers are over 1000, indicating a simpler solution is possible. We form (17) assuming there are only three locations. Using this time both forwards and backwards approaches (as in MEPA [2]) and rotating through x-y-z to y-z-x to z-x-y coordinates, we can form an estimate of a three source configuration,

$$l_1 = (4.6, -3.6, 8.7) \qquad l_2 = (1.8, 0.1, 6.9)$$
$$l_3 = (-2.0, 3.7, 7.9) \tag{19}$$

If we assign these three sources intensities of 0.8, 2.5, and 0.7 respectively (found through a least-squares fit), then Fig. 1 compares the original data set with the synthesized set.

6.0 Discussion

As with any Vandermonde matrix, this method is extremely noise sensitive in the inverse procedure, due to interdependencies in the Taylor series expansion, and we are researching procedures for reducing the condition of the transfer matrix, while still retaining the parameters of interest. The method has already proved quite useful in the forward synthesis problems, where we extract parsimonious subsets of discrete sources that adequately represent continuous distributions of sources. In this noiseless forward modeling synthesis, we essentially proceed directly to Step Two, since the multipole moments found in Step One can be synthesized.

We emphasize that the complexity of this rooting approach is driven by the static dependent nature of the source intensities, the near-field spatially-diverse signature across the array, and the arbitrary sensor spacing. These feature preclude the use of spatial smoothing techniques and rooting techniques which rely on uniform linear arrays. The algorithm complexity and noise sensitivity notwithstanding, the MEPA analysis of the extracted multipole sequence can prove useful in examining the parsimonious set of discrete sources that can represent the measured data.

Future efforts of this work will focus on adapting the analysis to the electroencephalography (EEG) and magnetoencephalography problems (MEG). Over-simplified, EEG involves the gradient of the (12), and MEG involves a vector form of (12), the curl of the vector potential. The preliminary results here indicate that a full inverse procedure may be difficult, due to (1) the ill-conditioned nature of a large series expansion vs. (2) the error introduced into the series coefficients for truncated series. Nevertheless, the ability to extract parsimonious sets of sources from complex distributions of sources should prove useful in simulation analysis.

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