# A State Space Algorithm for Wide-Band Source Localization

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*High-resolution subspace algorithms have proved to be very useful in locating point sources radiating narrow-band waves. Here, we introduce a subspace method for estimating the angles-of-arrival of multiple wide-band signals, such as in acoustic emission. This state space approach is applicable under the assumptions that the sensor array consists of doublets and that the wide-band signals can be modeled as the output of a linear time-invariant system driven by white noise.*

## 1 Introduction

Estimating the directions-of-arrival (DOA) of wide-band emitters by means of a sensor array has many applications, *e.g.*, in audio and sonar. A first group of wide-band algorithms reduces the wide-band estimation problem to a series of related narrow-band problems [1]. A second group of algorithms models the sensor signals as the output of a MIMO linear system driven by white noise. In this class the algorithm of Ottersten and Kailath [2] is especially appealing. It generalizes the computationally efficient ESPRIT algorithm [3] to wide-band signals. The algorithm determines a rational model of the sensor outputs. A restriction is that the sensor array should consist of two identical subarrays displaced by a known constant vector.

Our method is founded on the same assumptions. However, instead of determining a rational model, a state space model of the sensor outputs is constructed. Recently this approach has proved its merits in the field of system identification [4]. A few advantages of the use of state space models w.r.t. rational or polynomial models are a better numerical conditioning and the fact that there is no need for a priori knowledge of the system order.

The algorithm presented here differs from an earlier state space wide-band direction finding algorithm [6], in that it treats the outputs of the two subarrays symmetrically. Also we show how the memory requirements and the computation can be cut down by exploitation of the block Hankel structure of the data matrix.

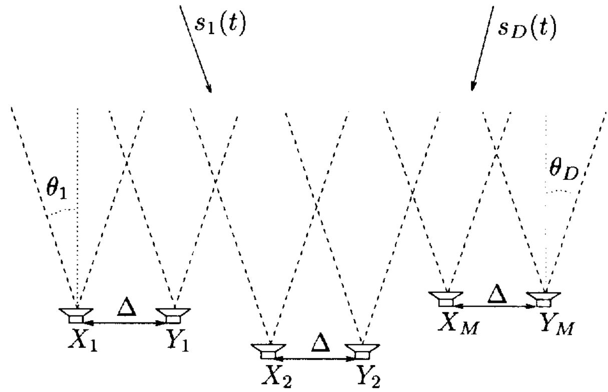


Figure 1: ESPRIT sensor array illuminated by $D$ incident wide-band signals.

## 2 Data model

Figure 1 shows schematically the wide-band direction finding scene. The sensor array consists of two identical subarrays translated over a known vector of length $\Delta$. The sensors are pairwise identical (*i.e.* doublets). The time-difference-of-arrival (TDOA) $\tau_l$ between the X- and the Y-sensor of a doublet for a signal impinging from the far field under an angle $\theta_l$, is related to the direction-of-arrival (DOA) $\tau_l$ by

$$\tau_l = \frac{\Delta \cdot \sin(\theta_l)}{c},$$

where $c$ is the propagation speed. The TDOA can be any real number in the interval $-qT_s \leq -\frac{\Delta}{c} \leq \tau_l \leq \frac{\Delta}{c} \leq qT_s$, where $q \in \mathbb{N}$ and $T_s$ is the sampling period.

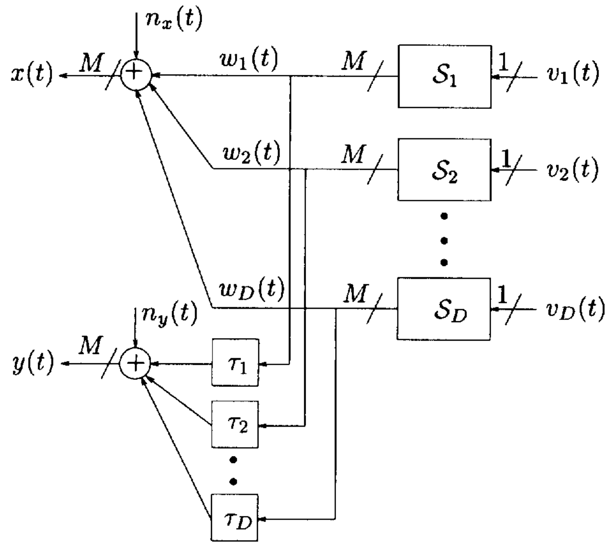


Figure 2: Signal diagram: the $S_l$ boxes represent linear systems, the $\tau_l$ boxes represent delay operators.

The observations due to the $l$th emitter, are modeled as the output of a continuous-time linear system $S_l$ of finite order $n_l$ driven by an independent scalar 'almost' white[1] stochastic process $v_l(t)$ (Figure 2).

$$\begin{bmatrix} \dot{z}_l(t) \\ w_l(t) \end{bmatrix} = \begin{bmatrix} A_l & B_l \\ C_l & D_l \end{bmatrix} \cdot \begin{bmatrix} z_l(t) \\ v_l(t) \end{bmatrix}, \qquad (1)$$

where $A_l \in \mathbb{R}^{n_l \times n_l}, B_l \in \mathbb{R}^{n_l \times 1}, C_l \in \mathbb{R}^{M \times n_l}, D_l \in \mathbb{R}^{M \times 1}$ are the system matrices of the $l$th emitter and $z_l(t) \in \mathbb{R}^{n_l}$ holds the state of $S_l$. The linear system $S_l$ models the dynamics of the $l$th emitter and the array response, which may depend on the angle $\theta_l$. Below, we assume that $A_l$ is given in diagonal form (which is generically possible).

The X-array outputs are collected in the vector $x(t) \in \mathbb{R}^M$,

$$x(t) = \sum_{l=1}^{D} w_l(t) + n_x(t),$$

where $D$ is the total number of emitters and the vector $n_x(t)$ adds independent white noise. The full state space model for $x(t)$ is given by

$$\begin{bmatrix} \dot{z}(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} z(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0_{n \times 1} \\ n_x(t) \end{bmatrix},$$

where the diagonal matrix $A \in \mathbb{R}^{n \times n}$, the matrices $B \in \mathbb{R}^{n \times D}, C \in \mathbb{R}^{M \times n}, D \in \mathbb{R}^{M \times D}$ and the vectors $z(t) \in \mathbb{R}^n, v(t) \in \mathbb{R}^D$ have block components given in eq. (1) and $n = \sum_{l=1}^{D} n_l$ is the total system order.

Similarly the Y-array observations are given by

$$y(t) = \sum_{l=1}^{D} w_l(t - \tau_l) + n_y(t).$$

In terms of the system matrices $y(t)$ can be written as

$$y(t) = C \cdot e^{A(t-t_0)} \cdot \Phi \cdot z(t_0) + f^y(t_0, t) + n_y(t), \quad (2)$$

where $t_0 \leq t - \tau_l$ for $1 \leq l \leq D$, $\Phi \in \mathbb{R}^{n \times n}$ is a diagonal matrix with block entries $\Phi_{ll} = e^{-A_l \tau_l}$, $f^y(t_0, t)$ is a term summing the input contributions

$$f^y(t_0, t) = \sum_{l=1}^{D} \int_{t_0}^{t-\tau_l} h_l(t - \tau_l - \sigma) \cdot v_l(\sigma) \cdot d\sigma,$$

and $h_l(t)$ is the impulse response of the $l$th system

$$\begin{aligned} h_l(t) &= C_l \cdot e^{A_l t} \cdot B_l, & t > 0 \\ &= D_l, & t = 0. \end{aligned}$$

Also $x(t)$ can be expressed in the same way.

$$x(t) = C \cdot e^{A(t-t_0)} \cdot z(t_0) + f^x(t_0, t) + n_x(t), \quad (3)$$

where

$$f^x(t_0, t) = \sum_{l=1}^{D} \int_{t_0}^{t} h_l(t - \sigma) \cdot v_l(\sigma) \cdot d\sigma.$$

Let $t_k = \begin{bmatrix} x(kT_s)^T & y(kT_s)^T \end{bmatrix}^T \in \mathbb{R}^{2M}$ denote the combined output at sampling time $kT_s$, $k \in \mathbb{N}$. Making use of eqs. (2,3), $t_k$ is given by

$$t_k = \begin{bmatrix} C \\ C \cdot \Phi \end{bmatrix} \cdot \mathcal{A}^q \cdot z_{k-q} + \begin{bmatrix} f^x_{(k-q,k)} \\ f^y_{(k-q,k)} \end{bmatrix} + \begin{bmatrix} n^x_k \\ n^y_k \end{bmatrix}, \quad (4)$$

where $\mathcal{A} = e^{AT_s}$ and $t_0 = t - q.T_s$.

The TDOAs appear in the exponent of the block components of $\Phi$, reflecting the exponential decay of the state $z_{k-q}$. Therefore, if we can isolate the term corresponding to the state, the TDOAs can be determined.

## 3 State space direction finding

The state space method outlined below, is related to recent subspace algorithms for system identification

[1] 'It is understood that an 'almost' white signal denotes a continuous-time signal, the correlation time of which is smaller than the sampling period.

[4]. The starting point is the construction of block Hankel matrices.

$$\underbrace{T_{k|k+i-1}}_{Mi\times j} = \begin{bmatrix} t_k & t_{k+1} & \cdots & t_{k+j-1} \\ t_{k+1} & t_{k+2} & \cdots & t_{k+j} \\ \vdots & & & \vdots \\ t_{t+i-1} & t_{t+i} & \cdots & t_{k+i+j-2} \end{bmatrix}.$$

By repeated substitution of eq. (4) this block Hankel matrix can be written as a linear combination of a state matrix, an input matrix and a noise term [4].

$$T_{k|k+i-1} = \Gamma_i \cdot \mathcal{A}^q \cdot Z_{k-q} + F_{(k,k+i-1)} + N_{k|k+i-1}, \quad (5)$$

where $\Gamma_i \in \mathbb{R}^{2Mi\times n}$ is a generalized observability matrix

$$\begin{aligned} \Gamma_i &= \begin{bmatrix} C'^T & (C'\mathcal{A})^T & \cdots & (C'\mathcal{A}^{i-1})^T \end{bmatrix}^T \\ C' &= \begin{bmatrix} C^T & (C\Phi)^T \end{bmatrix}^T, \end{aligned}$$

the matrix $Z_{k-q} \in \mathbb{R}^{n\times j}$ contains a state sequence

$$Z_{k-q} = \begin{bmatrix} z_{k-q} & z_{k-q+1} & \cdots & z_{k-q+j-1} \end{bmatrix},$$

the $(m,n)$th block entry of $F_{(k,k+i-1)}$ is

$$F_{(k,k+i-1)}(m,n) = \begin{bmatrix} f^x_{(k-q+n-1,k+m-1)} \\ f^y_{(k-q+n-1,k+m-1)} \end{bmatrix}$$

and $N_{k|k+i-1}$ is a block Hankel matrix containing the measurement noise.

The observability matrix $\Gamma_i$ is highly structured. Certain submatrices are shift-invariant in $\Phi$ and $\mathcal{A}$. However, first the matrices containing the contributions from the inputs and the measurement noise have to be eliminated. This can be done by an instrumental variable approach. We look for a second matrix such that its row space is orthogonal to the row space of the input and noise terms in $T_{k|k+i-1}$, but has a non-zero projection onto the row space of the state term. A natural choice is a block Hankel matrix with past outputs. We perform a row projection of the matrix $T_{i+2q|2i+2q-1}$ onto the matrix $T_{0|i-1}$. Under the assumption that the inputs and the measurement noise are independent zero-mean (almost) white stochastic processes, and that current states are uncorrelated with current and later inputs, asymptotically ($j \to \infty$) only the first term in the projection are retained [5].

$$T^p_{2q+i|2q+2i-1} = T_{2q+i|2q+2i-1}/T_{0|i-1} = \Gamma_i \cdot G, \quad (6)$$

where $G = \mathcal{A}^q \cdot Z_{q+i}/T_{0|i-1}$ and the notation $A/B$ denotes row projection $A/B = A \cdot B^T \cdot (B \cdot B^T)^{-1}B$.

On condition that $2Mi > n$ the projected matrix $T^p_{2q+i|2q+2i-1}$ has ideally a rank $n$. Therefore, the total system order is revealed in the rank of this matrix. Of course, the rank is $n$ only if the orthogonality of the row spaces holds perfectly. Nevertheless, for finite noisy data records the system order $n$ can be estimated by means of a singular value decomposition (SVD) of $T^p_{2q+i|2q+2i-1}$. An estimate of the signal subspace (span of $\Gamma_i$) is then obtained after truncating this SVD.

Just as the array gain matrix in the narrow-band ESPRIT data model, the matrix $\Gamma_i$ exhibits certain invariance properties. Here, a double invariance is present. A first invariance is due to the doublet structure of the sensor array. Consider the matrices $X_1, Y_1 \in \mathbb{R}^{M(i-1)\times j}$ obtained by partitioning the rows of $T^p_{2q+i|2q+2i-1}$ according to the $X$- and $Y$-array and omitting the trailing block row. Now form the matrix pencil

$$Y_1 - \mu X_1 = \Gamma^x_{i-1} \cdot (\Phi - \mu I_n) \cdot G,$$

where

$$\Gamma^x_{i-1} = \begin{bmatrix} C^T & (C\mathcal{A})^T & \cdots & (C\mathcal{A}^{i-2})^T \end{bmatrix}^T.$$

Its rank-reducing numbers are exactly the eigenvalues of $\Phi$,

$$\mu = e^{-p_l \tau_l},$$

where $p_l$ is an eigenvalue of the continuous-time system matrix $A$. In order to determine the TDOAs, the $p_l$s have to be estimated as well. For this purpose, a second matrix pencil is constructed with outputs at two subsequent sampling times. Let $X_2$ be the matrix containing the rows of $T^p_{2q+i|2q+2i-1}$ corresponding to the $X$-array outputs but now after omission of the leading block row. The second matrix pencil is

$$X_2 - \lambda X_1 = \Gamma^x_{i-1} \cdot (\mathcal{A} - \lambda I_n) \cdot G.$$

Its rank-reducing numbers are the eigenvalues of $\mathcal{A}$,

$$\lambda = e^{p_l T_s}.$$

Once we know the pair $(\lambda, \mu)$ corresponding to the same $p_l$, the TDOA is finally computed as

$$\tau_l = -T_s \cdot \frac{\ln \mu}{\ln \lambda}.$$

In the selection of the matrix pencils, care has been taken such that the pairing of the $\lambda$s and $\mu$s is obtained without additional computation. By selecting matrix pencils with the same left and right factors $\Gamma^x_i$

and $G$, their eigentransformations are identical. Ideally, it suffices to compute the rank-reducing numbers one pencil, and apply the same transformations to the other to obtain its rank-reducing numbers. For noisy data, refined methods to compute the eigenvalue pairs simultaneously are given in [7].

## 4 A fast algorithm

Below the wide-band direction state space direction finding algorithm is described. It consists entirely of matrix decompositions, for which reliable numerical algorithms are available. The projection step can be performed by a QR decomposition (or its transpose the LQ decomposition) of the data matrix $T_{0|2i+2q-1}$. The estimation of the signal subspace is then done by truncating the SVD of the submatrix $L_{31}$. The matrices used in the two matrix pencils are then formed by copying the appropriate rows of $U_s$ (indicated by the selection matrices $J_{x1}, J_{x2}, J_{y1} \in \mathbb{R}^{M(i-1)\times 2Mi}$). Note that $X_1, X_2$ and $Y_1$ are defined slightly differently than in the previous section. They only contain $n$ columns, since the common matrix $G$ may be omitted. The pencil matrices still need to be further reduced to square form. An optimal row compression matrix in total least squares (TLS) sense is given by the dominant column subspace of the 3 matrices. This computation involves a second SVD. Finally the rank reducing numbers of the two matrix pencils are computed as the generalized eigenvalues of the matrix pair $(X_{2p}, X_{1p})$. The same eigentransformations are then applied to the second matrix pencil and the TDOAs are obtained as in eq. (3).

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1. Projection step: LQ decomposition of $T_{0|2i+2q-1}$.

$$\begin{bmatrix} T_{0|i-1} \\ T_{i|i+2q-1} \\ T_{i+2q|2i+2q-1} \end{bmatrix} = \begin{bmatrix} L_{11} & & \\ L_{21} & L_{22} & \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \cdot Q$$

2. Signal subspace determination: SVD of $L_{31}$.

$$L_{31} = \left[\; U_s \mid U_n \;\right] \cdot \left[\begin{array}{c|c} \Sigma_s & \\ \hline & \Sigma_n \end{array}\right] \cdot \left[\begin{array}{c} V_s^T \\ \hline V_n^T \end{array}\right]$$

3. Matrix pencils and column compression

$$\left[\begin{array}{c} X_1 \\ \hline X_2 \\ \hline Y_1 \end{array}\right] = \left[\begin{array}{c} J_{x1} \\ \hline J_{x2} \\ \hline J_{y1} \end{array}\right] \cdot U_s$$

4. Row compression (SVD)

$$\begin{aligned} \left[\; X_1 \mid X_2 \mid Y_1 \;\right] &= \left[\; W_s \mid W_n \;\right] \cdot S \cdot Z \\ \left[\; X_{1p} \mid X_{2p} \mid Y_{1p} \;\right] &= W_s^T \cdot \left[\; X_1 \mid X_2 \mid Y_1 \;\right] \end{aligned}$$

5. Generalized eigenvalue problem

$$\begin{aligned} X_{2p} \cdot X_{1p}^{-1} &= G \cdot \Lambda \cdot G^{-1} \\ M &= G^{-1} \cdot Y_{1p} \cdot X_{1p}^{-1} \cdot G \end{aligned}$$

6. Estimation of the TDOAs

$$\tau = -T_s \ln \, \mathrm{diag}(M \cdot \Lambda^{-1})$$

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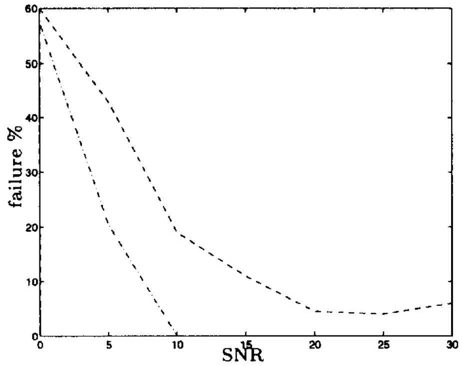
Because of its large dimension, the LQ decomposition of $T_{0|2i+2q-1}$ is the dominant part of the computation. However, the block Hankel structure of this data matrix can be exploited in the computation of the $L$ factor. This matrix is the Cholesky factor of $T_{0|2i+2q-1} \cdot T_{0|2i+2q-1}^T$, which is almost a Toeplitz matrix. Therefore, it has a low displacement rank $(2M + 2)$ for the block shift operator $Z_M \in \mathbb{R}^{4M(i+q)\times 4M(i+q)}$ [8],

$$Z_M = \begin{bmatrix} O & & & O \\ I_M & O & & \\ & \ddots & \ddots & \\ O & & I_M & O \end{bmatrix}.$$

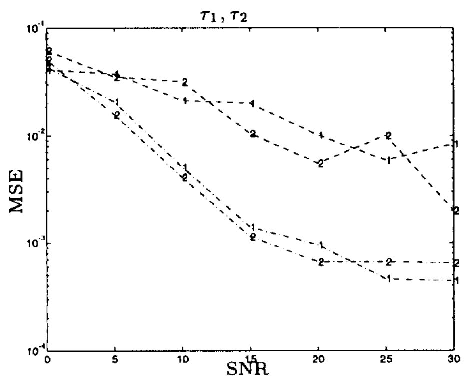
The generators are given by

$$\begin{aligned} W_1 &= T_{0|2i+2q-1} \cdot T_{0|0}^T \cdot R_0^{-1/2} \\ W_2 &= \left[\; O_{1\times M} \mid t_j^T \;\; \cdots \;\; t_{2i+2q+j-2}^T \;\right]^T \\ W_3 &= Z_M \cdot W_1 \\ W_4 &= \left[\; O_{1\times M} \mid t_0^T \;\; \cdots \;\; t_{2i+2q-2}^T \;\right]^T \end{aligned}$$

where $T_{0|0}$ is the first block row of $T_{0|2i+2q-1}$, $R_0 = T_{0|0} \cdot T_{0|0}^T$ and $R_0^{1/2}$ is its Cholesky factor. The generators $W_1, W_2$ have a positive signature, whereas the generators $W_3, W_4$ have a negative signature. The vectors $W_2$ and $W_4$ contain entries of the last, respectively first, column of $T_{0|2i+2q-1}$. These generators for the block Hankel case are a straightforward generalization of scalar Hankel case, considered in [8]. The $L$-factor can be computed by a triangularization algorithm, leading to important savings in memory and computation.




(a) Failure rate versus SNR.
A failure is reported is $\tau \notin [0, 0.8]$.




(b) MSE on $\tau_1, \tau_2$ versus SNR

Figure 3: Number of sensors: dashed line $M = 2$, dash-dot line: $M = 4$. Number of snapshots is 400. $i = 8, q = 1$. Average result over 50 independent runs.

## 5 Simulations

In the simulation we study the performance of the algorithm on a relatively simple scenario. There are two wide-band emitters with transfer functions and TDOAs

$$H_1(s) = 0.894/(s^2 + 0.2s + 2) \quad , \quad \tau_1 = 0.2T_s$$
$$H_2(s) = 1.095/(s^2 + 0.2s + 3) \quad , \quad \tau_2 = 0.6T_s$$

These transfer functions have a considerable spectral overlap. The data are generated by steering white noise through the continuous-time systems at 8 times the sampling rate. The sensor array is a ULA with varying number of sensors. The subarrays are maximally overlapping. Because of the measurement noise on the outputs, the eigentransformations of the two pencils tend to deviate. Therefore, we implemented the Schur refinement method of [7]. Although in theory one doublet suffices, the accuracy of the estimates clearly improves with an increasing number of sensor doublets.

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## References

[1] G. Su, M. Morf, "The signal subspace approach for multiple wide-band emitter location", *IEEE Trans. on ASSP*, Vol. 31, No. 6, December 1983, pp. 1502-1521.

[2] B. Ottersten, T. Kailath, "Direction-of-arrival estimation for wide-band signals using the ESPRIT algorithm", *IEEE Trans. on ASSP*, Vol. 38, February 1990, pp. 317-327.

[3] R. Roy, T. Kailath, "ESPRIT - Estimation of signal parameters via rotational invariance techniques", *IEEE Trans. on ASSP*, Vol. 37, No. 7, July 1989, pp. 984-995.

[4] P. Van Overschee, B. De Moor, "N4SID: Subspace algorithms for the identification of combined deterministic-stochastic systems", *Automatica*, Vol. 30, No. 1, January 1994, pp. 75-93.

[5] F. Vanpoucke, M. Moonen, E. Deprettere, "Direction finding of multiple wide-band emitters using state space modeling", To appear in Signal Processing, p. 25.

[6] F. Vanpoucke, M. Moonen, "A State Space Method for Direction Finding of Wide-Band Emitters", *Signal Processing VII, Proc. Eusipco 1994*, M. Holt *et al.* (eds), Vol. 2, pp. 780-783.

[7] A.J. van der Veen, P. Ober and E. Deprettere, "Azimuth and elevation computation in high resolution DOA estimation", *IEEE Trans. on SP*, vol. 40, no. 7, 1992, pp. 1828-1832.

[8] J. Chun, T. Kailath, H. Lev-Ari, "Fast parallel algorithms for QR and triangular factorization", *SIAM J. Sci. Stat. Comput.*, Vol. 8, No. 6, November 1987, pp. 899-913.