Direction Finding Estimation for Linear Chirp Signal

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Abstract

In this paper, we propose a Fast Maximum Likelihood Estimation (FMLE) to estimate the Direction of Arrival (DOA) θ and the Frequency Sweeping Rate (FSR) α of a linear chirp signal impinging upon an M-elements linear sensor array from a far-field source. The estimators θ and α of a chirp signal can be estimated by globally maximizing the likelihood function on the θ~α plane. The global maximum is located on a ridge of the 2-D θ~α plane. To reduce the heavy 2-D ML computation, we propose an algorithm of 1-D FMLE to search for the global maximum only along the ridge.

1. Introduction

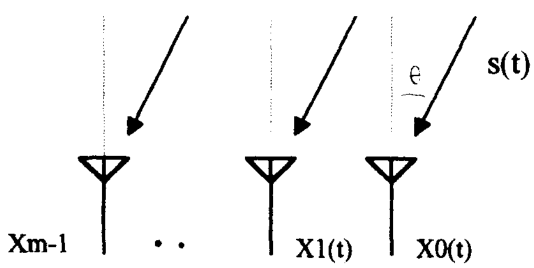

Figure 1: Uniform Linear Array

In the estimation of the direction of arrival (DOA), the signal is often assumed to be sinusoidal and stationary. In this paper, we are interested in estimating both the Frequency Sweeping Rate (FSR) α and DOA θ of a linear chirp signal. The observation from the m^{th} element of the uniformly spaced linear array (ULA) with additive Gaussian noise is given by

$$x_m(t) = s(t + m\tau) + w_m(t), \tag{1}$$

where $m = 0, 1, .., M-1$, and τ is the time delay of the signal traveling from one sensor to the others [1]. The $w(t)$ is band-limited complex white noise and $s(t) = \exp[j(\omega t + \alpha t^2)]$ is the chirp signal with instantaneous frequency $f_i = (\omega + 2\alpha t)/2\pi$. The N observations of $\underline{x}(t)$ can be sampled at nt_s for $0 \le n \le N-1$, and the discrete model from (1) is shown by

$$\underline{x}(n) = \underline{a}(\alpha, \tau, n)s(n) + \underline{w}(n), \tag{2}$$

where $\underline{a}(\alpha, \tau, n) = \begin{bmatrix} 1 \\ e^{j[\omega\tau + \alpha\tau^2 + 2\alpha\tau n]} \\ \cdot \\ e^{j[(M-1)\omega\tau + (M-1)^2\alpha\tau^2 + 2(M-1)\alpha\tau n]} \end{bmatrix}$.

2. The Maximum Likelihood Estimation

The maximization of the logarithm of the likelihood function is a nonlinear, multidimensional maximization problem [2]. The likelihood function of the equation (2) with N snapshots is given by

$$l(\alpha, \theta) = \prod_{n=0}^{N-1} \frac{1}{\pi \det[\sigma^2 I]} \exp\{-\frac{1}{\sigma^2} |\underline{x}(n) - \underline{a}(\alpha, \theta, n)s(n)|^2\}, \tag{3}$$

where $\theta = \sin^{-1}(\frac{C}{d}\tau)$.

The logarithm of the likelihood function, ignoring constant terms is given by

$$L(\alpha, \theta) = -NM \ln\sigma^2 - \frac{1}{\sigma^2} \sum_{n=0}^{N-1} |\underline{x}(n) - \underline{a}(\alpha, \theta, n)s(n)|^2 . \tag{4}$$

The estimation above can be done by searching for the global maximum on the α~θ plane as follows:

$$F(\alpha, \theta) = \sum_{n=0}^{N-1} \underline{x}^H(n)\underline{a}(\alpha, \theta, n)[\underline{a}^H(\alpha, \theta, n)\underline{a}(\alpha, \theta, n)]^{-1} \cdot \underline{a}^H(\alpha, \theta, n)\underline{x}(n). \tag{5}$$

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$$\left[\hat{\alpha},\hat{\theta}\right] = \arg\left\{\max_{\alpha,\theta} [F(\alpha,\theta)]\right\}. \quad (6)$$

3. The Fast DOA & FSR Estimation

In this paper, our interests are estimating the direction of arrival and the frequency sweeping rate of the impinging chirp signal. The ML estimators can be estimated by (6). However, the computation will be expansive. The estimation of the DOA and the FSR can be performed in 2-D search and by proposed fast algorithm by searching only along the locus of the ridge as in the Figure 2(c). In the Figure 2(a), it shows a noiseless chirp signal and in (b), it shows the contour of the 2-D MLE. From the (c), it shows the locus of the proposed ridge. In the (d), it shows the magnitude of the cost function along the ridge, where the maximum is located when $\alpha_0 = 10$. Then, the corresponding θ_0 can be found from (c).

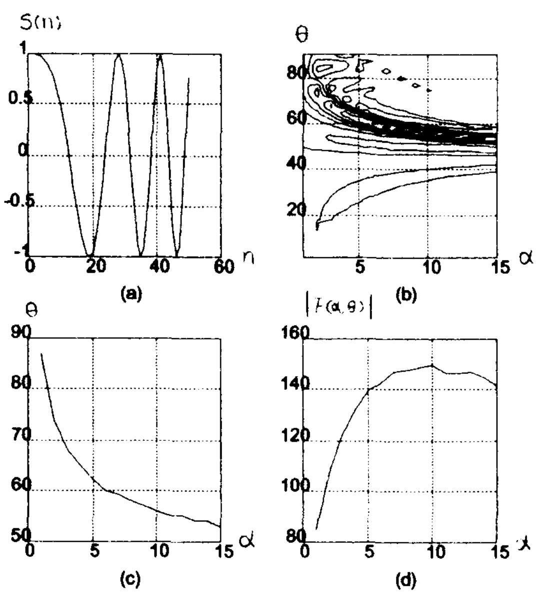

Figure 2: (a) A noiseless chirp signal, (b) the contour of 2-D MLE, (c) the locus of the ridge, (d) the magnitude of the cost function $|F(\alpha,\theta)|$along the ridge.

In this paper, we propose a Fast Maximum Likelihood Estimation (FMLE) that can reduce the computation by searching for the global maximum only along 1-D space [3]. This 1-D space can be described by the relationship of α and τ, seeing the Appendix A, that is

$$\text{Locus: } \tau = \frac{-[\omega+2\alpha n] \pm \sqrt{[\omega+2\alpha n]^2+4\alpha[\omega\tau_0+\alpha_0\tau_0^2+2\alpha_0\tau_0 n]}}{2\alpha}, \quad (7)$$

where α_0 and τ_0 are true values. Since the true values are unknown, we can select another pair of reference point to be plugged into the equation (7) to find the locus of the proposed ridge. Thus, we can choose an arbitrary α_r inside the searching range and then find the τ_r by the 1-D MLE as follows:

$$\tau_r = \arg\left\{\max_{\tau} [F(\alpha_r,\tau)]\right\}. \quad (8)$$

The estimation of the DOA and the FSR can be performed by 2-D searching of (6) or by searching along the 1-D locus of the ridge described by (7) and (8).

4. Proposed Fast Searching Algorithm

Step 1: Collect $\underline{x}(nt_s)$, where $n = 0, 1, \ldots, N-1$.

Step 2: Construct steering vector

$$\underline{a}(\alpha,\tau,n) = [1, e^{j[\omega\tau+\alpha\tau^2+2\alpha\tau n]}, \ldots, e^{j[(M-1)\omega\tau+(M-1)^2\alpha\tau^2+2(M-1)\alpha\tau n]}]^T,$$

where ω is assumed to be known *a priori*, $\tau = \frac{d}{C}\sin(\theta)$ is the time delay, d is the distance between two sensors, C is the propagation speed of the signal and θ is the direction of arrival shown in the Figure 1.

Step 3: Select an arbitrary α_r inside the searching range and find τ_r (or θ_r) by 1-D MLE:

$$\tau_r = \arg\{\max_{\tau} [F(\alpha_r,\tau)]\}.$$

Step 4: Locate the locus of $\alpha\sim\tau$ (or $\alpha\sim\theta$) by (7):

Locus:
$$\tau = \frac{1}{N}\sum_{n=0}^{N-1} \frac{-[\omega+2\alpha n] \pm \sqrt{[\omega+2\alpha n]^2+4\alpha[\omega\tau_r+\alpha_r\tau_r^2+2\alpha_r\tau_r n]}}{2\alpha},$$

where $\tau_k = \frac{d}{C}\sin(\theta_k)$.

Step 5: Perform a 1-D search for the global maximum along the locus of the $\alpha\sim\tau$ (or $\alpha\sim\theta$) from Step 4:

$$[\hat{\alpha},\hat{\theta}] = \arg\{\max_{\tilde{\alpha},\tilde{\theta}} [F(\tilde{\alpha},\tilde{\theta})]\},$$

where $\tilde{\alpha}$ and $\tilde{\theta}$ are the estimators along the locus of the proposed ridge.

5. The Cramer-Rao Bound of DOA & FSR

The performance of the estimation can be compared with the Cramer-Rao Lower Bound (CRLB). The CRLB of α and τ are shown in the follows:

$CRLB(\alpha)$

$$= \frac{\sigma^2}{2}\{N\tau^4\{\frac{M(M-1)(2M-1)}{30}[3(M-1)^2+3(M-1)-1] -[\frac{M(M-1)^2(2M-1)^2}{36}]\} + \frac{N(N-1)}{2}\tau^3\{[M^2(M-1)^2] -[\frac{M(M-1)^2(2M-1)}{3}]\} + \frac{N(N-1)(2N-1)}{6}\tau^2\{[\frac{2}{3}M(M-1)(2M-1)] -[M(M-1)^2]\}\}^{-1}. \quad (9)$$

$CRLB(\tau)$

$$= \frac{\sigma^2}{2}\{\frac{N(N-1)(2N-1)}{6}[(M-1)\alpha^2[\frac{2}{3}M(2M-1)-(M-1)]] +\frac{N(N-1)}{2}(M-1)[\frac{2}{3}M(2M-1)\omega\alpha+2M^2(M-1)\alpha^2\tau -(M-1)\omega\alpha-\frac{2}{3}\alpha(M-1)(2M-1)] +N[M(M-1)\omega^2[\frac{2M-1}{6}-\frac{M-1}{4}] +M(M-1)^2\omega\alpha\tau[M-\frac{2M-1}{3}] +N(M-1)(2M-1)\alpha^2\tau^2[\frac{2}{5}(M-1)^2+\frac{2}{5}(M-1)-\frac{2}{15} -\frac{(M-1)(2M-1)}{9}]]\}^{-1}. \quad (10)$$

6. Simulation & Results

In this simulation, before we estimate the DOA and FSR, we generate our example as follows: the chirp signal, $s(t)=\exp\{j[2\pi(1)t+2\pi(10)t^2]\}$ impinges the ULA with $\theta=10^o$. Let N=50 and M=4. Assume the initial angular frequency ω be known and $w(n)$ is white Gaussian noise. To demonstrate the estimation accuracy, we present the mean-square-error (MSE) of θ and α with varying signal-to-noise ratios (SNR). The computation saving for the proposed FMLE is compared with 2-D MLE by the number of MATLAB flops and is shown in the Table 1. Each simulation consists of 100 independent trials. In the Figure 3(a) and 3(b), we show the MSE of θ and α for the 2-D MLE and the proposed FMLE compared with the CRLB. When the SNR is high, the 2-D MLE and 1-D FMLE both approach the CRLB. When the SNR is low, the MSE of 1-D FMLE will be larger than 2-D MLE because the global maximum is deviated from the locus slightly. In the Figure 4(a) and 4(b), we show the MSE of θ and α with varied α_r ranging from 2 to 14. From the Step 3 of the FMLE algorithm, we select an arbitrary α_r inside the searching range. With the different selection of α_r, the estimation results will be varied slightly.

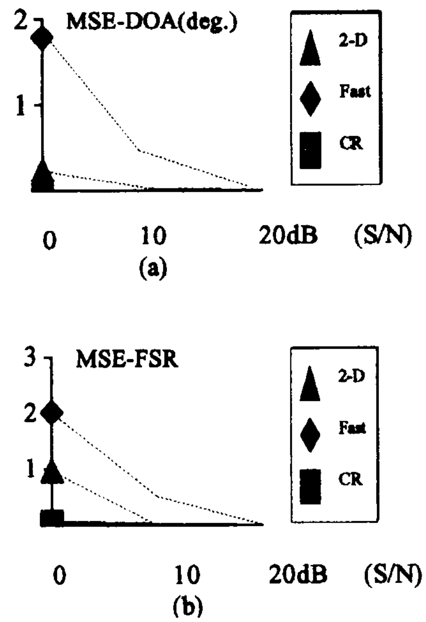

Figure 3: (a) The MSE of θ, and (b) the MSE of α for 2-D and proposed ML methods compared with CRLB.

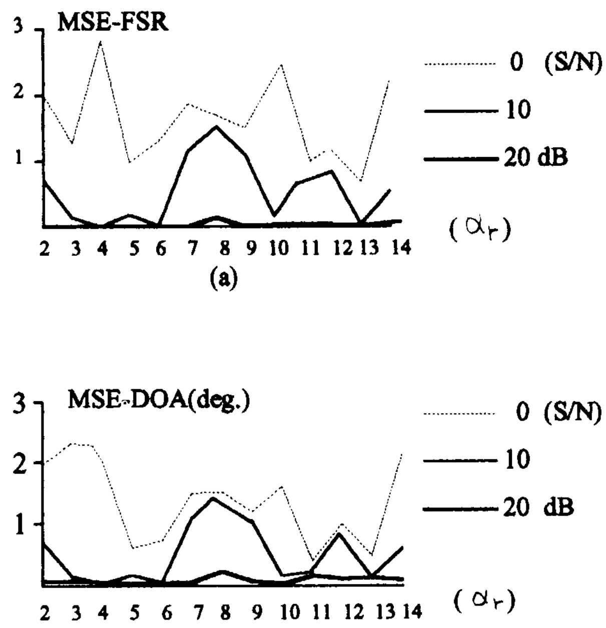

Figure 4: (a) The MSE of α, (b) the MSE of θ with different reference α_r.

TABLE 1

	Number of flops
2-D MLE	698,253
Fast MLE	3,020

6. Conclusion

Estimating the DOA [1], [2] or FSR [4] has been discussed in many papers. The estimation for both of the DOA and FSR jointly may not yet be discussed. The 2-D MLE can be used to estimate DOA and FSR but it is computationally exhaustive. In this paper, a fast MLE algorithm is proposed to reduce the heavy computation burden. The FMLE can search for the global maximum along the proposed ridge by doing only 1-D instead of 2-D maximization. The performance of 1-D FMLE is similar to the 2-D MLE when the SNR is equal or larger than 20 dB. The performance of the FMLE is slightly worse than the 2-D MLE when the SNR is less than 20 dB. However, the FMLE is computationally efficient. When considering two or more coherent DOAs, the similar FMLE with searching along reduced subspace can locate the DOAs and FSR with less computation load.

7. Acknowledgment

The authors would like to thank Dr. Victor E. DeBrunner and Dr. Gerald M. Pitstick for their helpful comments and suggestions.

Appendix A:

The relationship of α and τ of (7) can be given when M=2 and the signal is noiseless. From (5) the cost function is:

$$F(\alpha,\tau)=\sum_{n=0}^{N-1} s(n)s^*(n)\{1+\cos[\phi+\psi+2(\alpha_0\tau_0-\alpha\tau]\}, \tag{A.1}$$

where $\phi=\omega(\tau_0-\tau)$, $\psi=\alpha_0\tau_0^2-\alpha\tau^2$ and α_0, τ_0 are true parameters. Maximizing the cost function of (A.1) is equal to let $\phi+\psi+2(\alpha_0\tau_0-\alpha\tau)n=0$. Therefore, we get:

$$\alpha\tau^2+(\omega+2\alpha n)\tau-(\omega\tau_0+2\alpha_0\tau_0 n+\alpha_0\tau_0^2)=0\ . \tag{A.2}$$

$$\tau=\frac{-[\omega+2\alpha n]\pm\sqrt{[\omega+2\alpha n]^2+4\alpha\left[\omega\tau_0+2\alpha_0\tau_0 n+\alpha_0\tau_0^2\right]}}{2\alpha}\ . \tag{A.3}$$

References

[1] Simon Haykin, 'Advances in Spectrum Analysis and Array Processing', Prentice Hall, 1991.

[2] Ilan Ziskind & Mati Wax, 'Maximum Likelihood Localization of Multiple sources by alternating projection', IEEE tran. on ASSP, Vol. 36, pp 1553~ 1560, No. 1, Oct. 1988.

[3] Donald W. Tufts, Hongya Ge and S. Umesh, ' Fast Maximum Likelihood Estimation of Signal Parameters Using the Shape of the Compressed Likelihood Function' pp. 388 ~400, IEEE Journal of Ocean Engineering October 1993 .

[4] R. M. Liang, K. S. Arun, 'Parameter Estimation For Superimposed Chirp Signals', Proc. ICASSP-92, pp. V-273~276.