On Spatial Smoothing for Two-Dimensional Direction-of-Arrival Estimation of Coherent Signals

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Abstract

In this paper, we present an analysis of the spatial smoothing schemes extended for the estimation of two-dimensional directions of arrival of coherent signals using a uniform rectangular array. The scheme extended from the plain spatial smoothing technique is shown requiring at least $(2K-1)$ *specifically chosen square subarrays, each with size* $(K+1)\times(K+1)$ *at least, in order to guarantee the "decorrelation" of* K *coherent signals in all possible situations. The the minimum size of the total array is shown to be* $2K\times 2K$.

1 Introduction

In recent years, there has been a growing interest in developing high resolution techniques for estimating the directions of arrival (DOA's) of coherent signals using multiple sensors. Several techniques have been proposed to take care of this situation [2, 3], of which the spatial smoothing scheme for the case of a uniform linear array is specially noteworthy [4, 5]. Their solution is based on a preprocessing scheme that groups the total array of sensors into overlapping subarrays and then averages the subarray ouput covariance matrices to form the spatially smoothed covariance matrix. It has been shown that when the spatially smoothed covariance matrix is used in conjunction with an eigenstructure-based technique, *e.g.*, MUSIC [1], at least $2K$ sensor elements are required in order to "decorrelate" and resolve K coherent signals. Excluding certain rare situations, the required number of sensor elements can be reduced to $\lceil 3K/2\rceil$ ($\lceil x\rceil$ stands for the integer no less than x) using an improved smoothing scheme referred to as the forward/backward smoothing scheme [6].

The spatial smoothing scheme is first extended by Yeh *et al.* [7] for the two-dimensional (2-D) case of using a uniform planar array for estimating 2-D DOA's of coherent signals. They have shown that using two rectangular subarrays to form the spatially smoothed covariance matrix is not sufficient to "decorrelate" two coherent sources in certain situations. They have thus suggested using an additional specific subarray to form the spatially smoothed covariance matrix, which is proved being able to decorrelate two coherent sources in all situations. However no general solution for the problem of resolving 2-D DOA's of K coherent sources is given.

In this paper, we present a more complete analysis of the extended spatial smoothing scheme for the 2-D case based on the forward smoothing scheme, which is referred to as the 2-D spatial smoothing scheme. We shall prove that at least $(2K-1)$ specific chosen square subarrays must be used for forming the spatially smoothed covariance matrix, with K subarrays displaced in each direction of two independent directions, and the size of the subarrays must be $(K+1)\times(K+1)$ at least in order to guarantee the "decorrelation" of K coherent signals in all possible situations. The minimum size of the total square array is thus shown to be $2K\times 2K$.

2 Problem Formulation

Consider K narrow-band, far-field, and coherent radiating sources observed by a passive uniform rectangular array of $M\times N$ identical sensors with interelement spacing d_x and d_y, respectively. Using analytic signal respresentation, the received signal at the (m,n)th sensor can be expressed by

$$v_{mn}(t)=\sum_{k=1}^{K}s_k(t)\beta_k^{(m-1)}\gamma_k^{(n-1)}+n_{mn}(t) \qquad (1)$$

where $s_k(t)=\alpha_k s_1(t)$ is the signal of the kth wavefront, $(\beta_k,\gamma_k)=(e^{j\kappa d_x\cos\phi_k\cos\theta_k}, e^{j\kappa d_y\cos\phi_k\sin\theta_k})$, θ_k and ϕ_k are the azimuth and elevation angles of the kth source, respectively, κ denotes the wavenumber corresponding to the signal center frequency, and $n_{mn}(t)$ is the additive spatially white noise with variance σ^2. Rewritting (1) in vector notation, we have

$$\vec{v}(t) = A\vec{s}(t) + \vec{n}(t) \quad (2)$$

where $\vec{v}(t)$ is the $MN \times 1$ vector

$$\begin{aligned}\vec{v}(t) &= [v_{11}(t), v_{21}(t), \cdots, v_{M1}(t), \\ &\quad v_{12}(t), \cdots, v_{MN}(t)]^T \end{aligned} \quad (3)$$

$\vec{s}(t)$ is the $K \times 1$ vector

$$\begin{aligned}\vec{s}(t) &= [s_1(t), s_2(t), \cdots, s_K(t)]^T \\ &= s_1(t)[\alpha_1, \alpha_2, \cdots, \alpha_K]^T \\ &= s_1(t)\vec{\alpha} \end{aligned} \quad (4)$$

A is the $MN \times K$ phase vector matrix

$$A = [\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_K] \quad (5)$$

and $\vec{a}_k$ is the $MN \times 1$ phase vector of the kth source:

$$\vec{a}_k = [\vec{a_{xk}}^T, \gamma_k \vec{a_{xk}}^T, \cdots, \gamma_k^{(N-1)} \vec{a_{xk}}^T]^T \quad (6)$$

$$\vec{a_{xk}} = [1, \beta_k, \cdots, \beta_k^{(M-1)}]^T \quad (7)$$

From (5), it follows that the covariance matrix of $\vec{v}(t)$ is given by

$$\begin{aligned}R &= E[\vec{v}(t)\vec{v}^\dagger(t)] \\ &= ASA^\dagger + \sigma^2 I \end{aligned} \quad (8)$$

where S is the $K \times K$ signal covariance matrix

$$\begin{aligned}S &= E[\vec{s}(t)\vec{s}^\dagger(t)] \\ &= E[|s_1(t)|^2]\vec{\alpha}\vec{\alpha}^\dagger \end{aligned} \quad (9)$$

and "$\dagger$" denotes the complex conjugate transpose operation. It is noted that the "signal" subspace of R is of rank one instead of K and the "noise" subspace is orthogonal to $A\vec{\alpha}$ instead of the columns of A, which implies the failure of eigenstructure-based techniques when the covariance matrix R is used.

3 The 2-D Spatial Smoothing Preprocessing Scheme

As pointed out in [1], the nonsingularity of the signal covariance matrix S and the linear independence of the columns of the phase vector matrix A are the keys to a successful application of the eigenstructure-based techniques. In this section, we present an analysis of a 2-D spatial smoothing scheme that guarantees these two properties in all possible situations.

3.1 The Linear Independence of Phase Vectors

Rewrite (5) more explicitly, we have

$$A = \begin{bmatrix} \vec{a_{x1}} & \vec{a_{x2}} & \cdots & \vec{a_{xK}} \\ \gamma_1 \vec{a_{x1}} & \gamma_2 \vec{a_{x2}} & & \gamma_K \vec{a_{xK}} \\ \vdots & & \ddots & \\ \gamma_1^{(N-1)} \vec{a_{x1}} & \gamma_2^{(N-1)} \vec{a_{x2}} & & \gamma_K^{(N-1)} \vec{a_{xK}} \end{bmatrix} \quad (10)$$

where $\vec{a_{xk}}$ is the $M \times 1$ phase vector of the kth source observed from the first column of the sensor array as given in (7). To guarantee the independence of the columns of A, all phase vectors must first be different for sources located in the field of view, *e.g*, $-\pi \leq \theta_k \leq \pi$ and $0 \leq \phi_k \leq \frac{\pi}{2}$. This can be achieved by letting the interelement spacings d_x and d_y be less than half a wavelength. In this case, $(\beta_k, \gamma_k) \neq (\beta_l, \gamma_l)$ for $k \neq l$. However, the condition of $MN > K$ is not sufficient for the independence of the phase vectors as illustrated in the following statements. Alghthough $(\beta_k, \gamma_k) \neq (\beta_l, \gamma_l)$ for $k \neq l$, it is possible that $\beta_k = \beta_l$ or $\gamma_k = \gamma_l$ for $k \neq l$. In an extreme case that $\gamma_k = \gamma_c$ for all k, A can be rewritten as

$$A = \begin{bmatrix} A_x \\ \gamma_c A_x \\ \vdots \\ \gamma_c^{(N-1)} A_x \end{bmatrix} \quad (11)$$

where A_x is an $M \times K$ Vandermonde matrix:

$$A_x = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & & \beta_K \\ \vdots & & \ddots & \\ \beta_1^{(M-1)} & \beta_2^{(M-1)} & & \beta_K^{(M-1)} \end{bmatrix} \quad (12)$$

Thus the rank of A (denoted as $\rho(A)$) is equal to that of A_x, which is $min(M, K)$ since all β_k must be different. While in the other extreme case that $\beta_k = \beta_c$ for all k, A can be rewritten as

$$A = \begin{bmatrix} \vec{a_{xc}} & \vec{a_{xc}} & \cdots & \vec{a_{xc}} \\ \gamma_1 \vec{a_{xc}} & \gamma_2 \vec{a_{xc}} & & \gamma_K \vec{a_{xc}} \\ \vdots & & \ddots & \\ \gamma_1^{(N-1)} \vec{a_{xc}} & \gamma_2^{(N-1)} \vec{a_{xc}} & & \gamma_K^{(N-1)} \vec{a_{xc}} \end{bmatrix} \quad (13)$$

where

$$\vec{a_{xc}} = [1, \beta_c, \cdots, \beta_c^{(M-1)}]^T \quad (14)$$

After a suitable permutation, we have

$$per(A) = \begin{bmatrix} A_y \\ \beta_c A_y \\ \vdots \\ \beta_c^{(M-1)} A_y \end{bmatrix} \quad (15)$$

where $per(\cdot)$ denotes a permutation operation, and A_y is an $N \times K$ Vandermonde matrix:

$$A_y = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & & \gamma_K \\ \vdots & & \ddots & \\ \gamma_1^{(N-1)} & \gamma_2^{(N-1)} & & \gamma_K^{(N-1)} \end{bmatrix} \quad (16)$$

Thus $\rho(A) = \rho(A_y)$, which is $min(N, K)$ since all γ_k must be different. Consequently, we need $N \geq K$ and $M \geq K$ to guarantee the independence of the phase vectors. Additionally, the "signal" and "noise" subspaces of R can be found when the signal covariance matrix is nonsingular, where the former has same rank and range as A and the later is orthogonal to the columns of A. However, $N = K$ or $M = K$ is not sufficient to guarantee the "resolution" of K sources in all possible situations when the signal covariance matrix is nonsigular. For example, in the extreme case that $\gamma_k = \gamma_c$ for all k, any phase vector (given in (6)) that has $\gamma = \gamma_c$ will lie on the range of A, *i.e.*, the "signal" subspace, regardless of the value of β since the matrix A_x is of full rank. This indicates there will be no resolution in β. A similar phenomenon is also noticed for the other extreme case. This problem can be easily solved by letting $M > K$ and $N > K$, which indicates the smallest size of the subarrays is $(K+1) \times (K+1)$.

3.2 The Nonsingularity of the Signal Covariance Matrix

In this subsection, we present an analysis of a 2-D spatial smoothing scheme, introduced by Yeh *et al.*[7], to derive the sufficient conditions that guarantees the nonsingularity of the signal covariance matrix. Let a uniform rectangular array with $L_x \times L_y$ sensors be divided into overlapping rectangular subarrays of size $M \times N$, as shown in Figure 1. Following the notation of (2), we have the vector of received signals at the (m, n)th subarray given by

$$\vec{v}_{mn}(t) = AD_x^{(m-1)} D_y^{(n-1)} \vec{s}(t) + \vec{n}(t) \quad (17)$$

where $D_x^{(n)}$ and $D_y^{(n)}$ denote the nth power of the $K \times K$ diagonal matrices:

$$D_x = diag(\beta_1, \beta_2, \cdots, \beta_K) \quad (18)$$

$$D_y = diag(\gamma_1, \gamma_2, \cdots, \gamma_K) \quad (19)$$

The covariance matrix of the (m, n)th subarray is therefore given by

$$R_{mn} = AD_x^{(m-1)} D_y^{(n-1)} S D_x^{\dagger(m-1)} D_y^{\dagger(n-1)} A^\dagger + \sigma^2 I \quad (20)$$

The 2-D spatially smoothed covariance matrix is defined as the sample means of the subarray covariance matrices:

$$\bar{R} = \frac{1}{M_s N_s} \sum_{m=1}^{M_s} \sum_{n=1}^{N_s} R_{mn} = A\bar{S}A^\dagger + \sigma^2 I \quad (21)$$

where $M_s = L_x - M + 1$, $N_s = L_y - N + 1$, $\bar{S}$ is the modified covariance matrix of the signals, given by

$$\bar{S} = \frac{1}{M_s N_s} \sum_{m=1}^{M_s} \sum_{n=1}^{N_s} D_x^{(m-1)} D_y^{(n-1)} S D_x^{\dagger(m-1)} D_y^{\dagger(n-1)} = \frac{1}{M_s N_s} GG^\dagger \quad (22)$$

and G is the $K \times M_s N_s$ block matrix

$$G = [G_x, D_y G_x, \cdots, D_y^{(N_s-1)} G_x] \quad (23)$$

$$G_x = [\vec{\alpha}, D_x \vec{\alpha}, \cdots, D_x^{(M_s-1)} \vec{\alpha}] = \begin{bmatrix} \alpha_1 & \alpha_1\beta_1 & \cdots & \alpha_1\beta_1^{(M_s-1)} \\ \alpha_2 & \alpha_2\beta_2 & \cdots & \alpha_2\beta_2^{(M_s-1)} \\ \vdots & & \cdots & \\ \alpha_K & \alpha_K\beta_K & \cdots & \alpha_K\beta_K^{(M_s-1)} \end{bmatrix} \quad (24)$$

It is noted that $\rho(\bar{S}) = \rho(G)$. Following the statements given in the previous subsection, consider the extreme case that $\gamma_k = \gamma_c$ for all k, *i.e.*, $D_y = \gamma_c I$. Then G can be rewritten as

$$G = [G_x, \gamma_c G_x, \cdots, \gamma_c^{(N_s-1)} G_x] \quad (25)$$

Thus

$$\rho(G) = \rho(G_x) = min(M_s, K), \quad (26)$$

since G_x is a $K \times M_s$ Vandermonde matrix and all β_k must be different. It implicitly means that there is no spatial smoothing effect in the direction of the y axis since $range(D_y^{(n)} G_x) = range(G_x)$ for all n. Similarly, in the other extreme case that $\beta_k = \beta_c$ for all k, *i.e.*, $D_x = \beta_c I$, a permutation of G is shown as

$$per(G) = [G_y, D_x G_y, \cdots, D_x^{(M_s-1)} G_y] = [G_y, \beta_c G_y, \cdots, \beta_c^{(M_s-1)} G_y] \quad (27)$$

$$G_y = [\vec{\alpha}, D_y\vec{\alpha}, \cdots, D_y^{(N_s-1)}\vec{\alpha}]$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1\gamma_1 & \cdots & \alpha_1\gamma_1^{(N_s-1)} \\ \alpha_2 & \alpha_2\gamma_2 & \cdots & \alpha_2\gamma_2^{(N_s-1)} \\ \vdots & & \cdots & \\ \alpha_K & \alpha_K\gamma_K & \cdots & \alpha_K\gamma_K^{(N_s-1)} \end{bmatrix} \quad (28)$$

Consequently, we have

$$\rho(G) = \rho(per(G)) = \rho(G_y) = min(N_s, K) \quad (29)$$

since G_y is a $K \times N_s$ Vandermonde matrix and all γ_k must be different. Again, it implies that there is no smoothing effect in the direction of the x axis, since $range(D_x^{(m)}G_y) = range(G_y)$ for all m. In summary, we need $M_s \geq K$ and $N_s \geq K$ to guarantee the nonsingularity of the signal covariance matrix. It follows that the needed minimum size of the total rectangular array is $2K \times 2K$, *i.e.*, $M = N = K + 1$ and $M_s = N_s = K$, for a successful application of the 2-D spatial smoothing scheme incorporating with the eigenstructure-based techniques in all possible situations.

As noted from previous statements, the "decorrelation" of K coherent sources, *i.e.*, $\rho(\bar{S}) = K$, in all possible situations can be assured as long as we use at least K subarrays displaced in each direction of the x and y axes. Therefore, the least number of the subarrays required is $(2K - 1)$. For example, we can use the $(m, 1)$th, $m = 1, 2, \cdots, K$, and the $(1, n)$th, $n = 2, 3, \cdots, K$, subarrays to form the spatially smoothed covarince matrix:

$$\bar{R} = \frac{1}{2K-1}[\sum_{m=1}^{K} R_{m1} + \sum_{n=2}^{K} R_{1n}]$$

$$= A\bar{S}A^{\dagger} + \sigma^2 I \quad (30)$$

where $\bar{S}$ is then given by

$$\bar{S} = \frac{1}{2K-1}[\sum_{m=1}^{K} D_x^{(m-1)} S D_x^{\dagger(m-1)} + \sum_{n=2}^{K} D_y^{(n-1)} S D_y^{\dagger(n-1)}]$$

$$= \frac{1}{2K-1} G G^{\dagger} \quad (31)$$

and

$$G = [\vec{\alpha}, G'_x, G'_y] \quad (32)$$

$$G'_x = \begin{bmatrix} \alpha_1\beta_1 & \alpha_1\beta_1^{(2)} & \cdots & \alpha_1\beta_1^{(K-1)} \\ \alpha_2\beta_2 & \alpha_2\beta_2^{(2)} & \cdots & \alpha_2\beta_2^{(K-1)} \\ \vdots & & \ddots & \\ \alpha_K\beta_K & \alpha_K\beta_K^{(2)} & \cdots & \alpha_K\beta_K^{(K-1)} \end{bmatrix} \quad (33)$$

$$G'_y = \begin{bmatrix} \alpha_1\gamma_1 & \alpha_1\gamma_1^{(2)} & \cdots & \alpha_1\gamma_1^{(K-1)} \\ \alpha_2\gamma_2 & \alpha_2\gamma_2^{(2)} & \cdots & \alpha_2\gamma_2^{(K-1)} \\ \vdots & & \ddots & \\ \alpha_K\gamma_K & \alpha_K\gamma_K^{(2)} & \cdots & \alpha_K\gamma_K^{(K-1)} \end{bmatrix} \quad (34)$$

When $\gamma_k = \gamma_c$ for all k, we have

$$G = [\vec{\alpha}, G'_x, \gamma_c\vec{\alpha}, \cdots, \gamma_c^{(K-1)}\vec{\alpha}] \quad (35)$$

$\rho(G) = \rho([\vec{\alpha}, G'_x]) = \rho(G_x)$, and when $\beta_k = \beta_c$ for all k, we have

$$per(G) = [\vec{\alpha}, G'_y, \beta_c\vec{\alpha}, \cdots, \beta_c^{(K-1)}\vec{\alpha}] \quad (36)$$

and $\rho(G) = \rho(per(G)) = \rho(G_y)$. Therefore, the "decorrelation" is guaranteed.

4 Simulation Results

In this section, we present simulation results that demonstrate the analysis of the 2-D spatial smoothing scheme. Consider three narrow-band, far-field, and coherent sources observed by a 6×6 uniform square array with interelement spacings $d_x = d_y = \frac{\lambda}{2}$. The three sources are with common $SNR = 10dB$ and located at $(\theta, \phi) = (0°, 60°), (45°, 45°)$, and $(-45°, 45°)$, respectively, *i.e.*, $\beta_1 = \beta_2 = \beta_3 = e^{j\frac{\pi}{2}}$ and $\gamma_1 = e^{j0}$, $\gamma_2 = e^{j\frac{\pi}{2}}$, $\gamma_3 = e^{-j\frac{\pi}{2}}$. 500 snapshots are taken to form the estimate of the covariance matrix of the array outputs. The results of applying the 2-D spatial smoothing scheme in conjunction with the eigenstructure technique of Schmidt [1] are shown in Figures 2-5. Figure 2 shows the result for the subarray configuration of $M = N = 3$ and $M_s = N_s = 3$, which demonstrates the effect of using sufficient subarrays of insufficient size. Figure 3 is for the case that $M = N = 4$, $M_s = 3$, and $N_s = 2$, which shows the result of using insufficient subarrays of sufficient size. The result of sufficient subarrays of sufficient size ($M = N = 4$ and $M_s = N_s = 3$) is shown in Figure 4. The result for the case that $M = N = 4$, $M_s = 1$, and $N_s = 3$, shown in Figure 5, is very similar to that shown in Figure 4, which reveals that the smoothing effect is achieved simply by using subarrays displaced in the y axis, and thus indicates there is no smoothing effect by using subarrays displaced in the x axis when the β's of all sources are the same.

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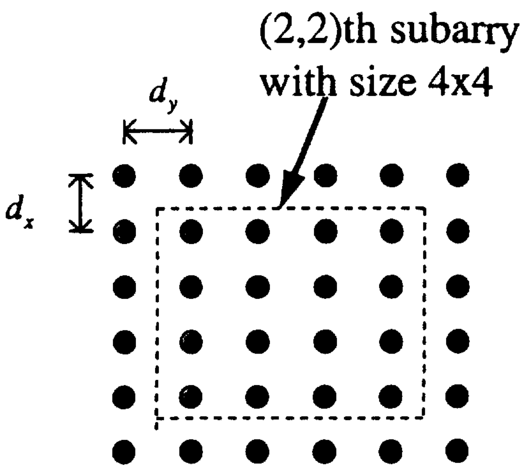

Figure 1. Subarray grouping

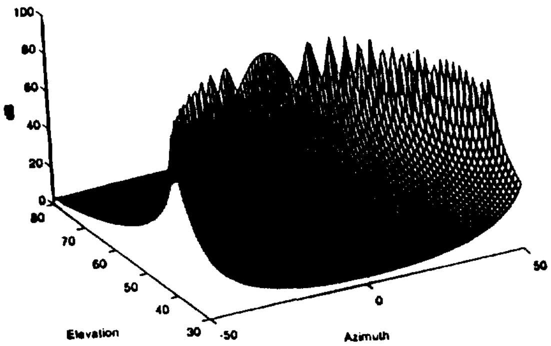

Figure 2. Angular spectrum for M=N=3 and M_s=N_s=3

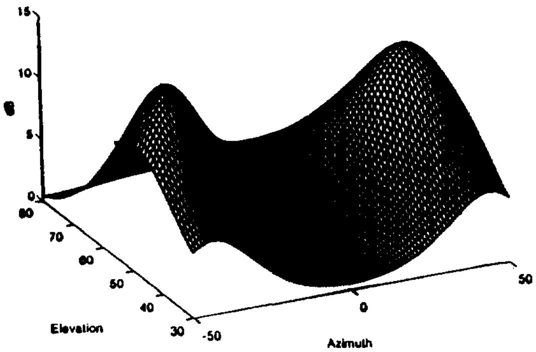

Figure 3. Angular spectrum for M=N=4, M_s=3, and N_s=2

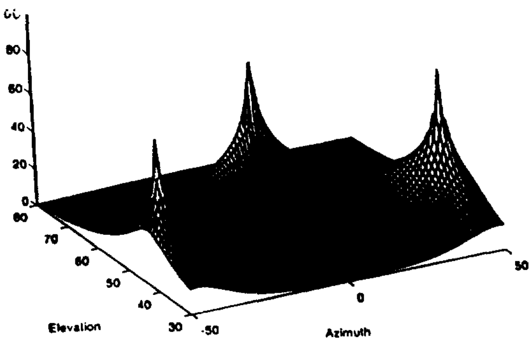

Figure 4. Angular spectrum for M=N=4 and M_s=N_s=3

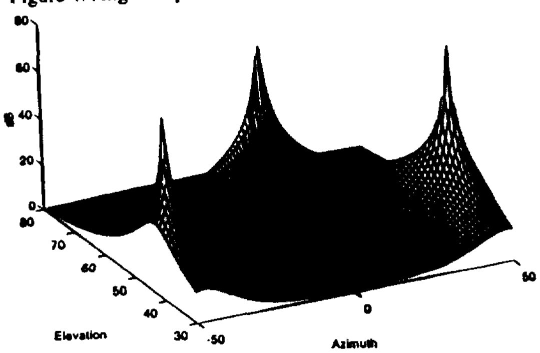

Figure 5. Angular spectrum for M=N=4, M_s=1, and N_s=3