Optimal Direction Finding with Partly Calibrated Arrays in Spatially Correlated Noise Fields

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Abstract

The problem of using a partly calibrated array for maximum likelihood (ML) bearing estimation of possibly coherent signals buried in unknown correlated noise fields is shown to admit a neat solution under fairly general conditions. The ML estimator introduced in this paper (and referred to as MLE) is shown to be asymptotically equivalent to a recently proposed subspace-based bearing estimator (called UNCLE and re-derived herein by a simpler approach than in the original work). A statistical analysis is performed, proving that the MLE and UNCLE methods are asymptotically equivalent and statistically efficient. In a simulation study, the methods are also found to possess very similar finite-sample properties.

1 Introduction and Preliminaries

Many practical arrays contain sensors with uncertain or unknown characteristics (such as gains, phase shifts and locations), which operate in unknown correlated noise fields. Using a *partly calibrated array* for maximum-likelihood (ML) bearing estimation of *possibly coherent signals* buried in *spatially correlated noise* may appear as a formidable problem. Nevertheless, the present paper shows that this problem admits a rather neat solution under fairly general conditions.

Consider an array comprising m sensors, m_1 of which are calibrated. The output vector of the m_1 calibrated sensors (after pre-processing) is denoted $\boldsymbol{y}_1(t)$. Assume that the array is planar and that the n sources which generate the signals impinging on the array are far-field point emitters. Additionally, either the signals or the receiving equipment are assumed to be narrowband. The output vector $\boldsymbol{y}_1(t)$ of the calibrated subarray can then be described by the equation:

$$\boldsymbol{y}_1(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{e}_1(t) \quad t = 1, 2, \cdots \tag{1}$$

where $\boldsymbol{x}(t)$ is the $n \times 1$ vector of the (complex envelopes of the) impinging signals, $\boldsymbol{e}_1(t)$ is a noise term and $\boldsymbol{A}(\boldsymbol{\theta})$ is the array transfer matrix. For the case of narrowband processing assumed herein, $\boldsymbol{A}(\boldsymbol{\theta})$ is given by

$$\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1) \cdots \boldsymbol{a}(\theta_n)] \tag{2}$$

where $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_n]^T$ and θ_k is the bearing of the kth source (also called the kth direction of arrival (DOA)). The $m_1 \times 1$-vector $\boldsymbol{a}(\theta_k)$ denotes the array response to a plane wave with DOA equal to θ_k. It is assumed that $\boldsymbol{a}(\theta_k)$ is a known function of the unknown parameter θ_k.

The array is assumed to also comprise m_2 uncalibrated sensors ($m_2 \leq m - m_1$), which are enough separated from the calibrated sensors so that the noises in the two "subarrays" are uncorrelated. We thus assume that

$$\mathrm{E}\,[\boldsymbol{e}_1(t)\boldsymbol{e}_2^*(s)] = \mathrm{E}\,[\boldsymbol{e}_1(t)\boldsymbol{e}_2^T(s)] = \boldsymbol{O} \quad \text{for all t, s} \tag{3}$$

holds, where $\boldsymbol{e}_2(t)$ denotes the noise in the uncalibrated sensors, and the superscripts $^{T/*}$ are the transpose / conjugate transpose operators. Throughout this paper, we use the symbol $\boldsymbol{O}$ for a zero matrix of suitable dimension. The output of the uncalibrated sensors is denoted $\boldsymbol{y}_2(t)$,

$$\boldsymbol{y}_2(t) = \text{noise-free signal term} + \boldsymbol{e}_2(t) \tag{4}$$

No model is assumed available for the signal term in (4).

The following assumptions are further imposed on the array data:

A1. The array output snapshots

$$\boldsymbol{y}(t) = \begin{bmatrix} \boldsymbol{y}_1(t) \\ \boldsymbol{y}_2(t) \end{bmatrix} \in \mathcal{C}^{m_1+m_2} \text{ for } t = 1, 2, 3, \cdots \tag{5}$$

form a sequence of independent and identically distributed complex Gaussian random variables with zero mean and the following covariances,

$$\mathrm{E}[\boldsymbol{y}(t)\boldsymbol{y}^*(t)] \triangleq \boldsymbol{R} \triangleq \begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{R}_{12}^* & \boldsymbol{R}_{22} \end{bmatrix} \quad (6)$$

$$\mathrm{E}[\boldsymbol{y}(t)\boldsymbol{y}^T(t)] = \boldsymbol{O} \quad (7)$$

A2. The signal $\{\boldsymbol{x}(t)\}$ and the noise $\{\boldsymbol{e}_k(t)\}$ are statistically independent of one another. Additionally, the noise terms satisfy (3). However, no constraint is imposed on the (spatial) covariance matrices of the noise subvectors,

$$\boldsymbol{\Sigma}_k = \mathrm{E}[\boldsymbol{e}_k(t)\boldsymbol{e}_k^*(t)] \quad k = 1, 2 \quad (8)$$

which, therefore, are arbitrary Hermitian and positive definite matrices.

A3. Let $\boldsymbol{P}$ denote the signal covariance matrix,

$$\boldsymbol{P} = \mathrm{E}[\boldsymbol{x}(t)\boldsymbol{x}^*(t)] \quad (9)$$

and let $\bar{n} \leq n$ be its rank,

$$\bar{n} = \mathrm{rank}\,(\boldsymbol{P}) \quad (10)$$

(if $\bar{n} < n$ then some signals are fully correlated, or *coherent*). According to (3), the $\boldsymbol{R}_{12}$ block of $\boldsymbol{R}$ can be written as (we often omit the explicit dependence of $\boldsymbol{A}(\boldsymbol{\theta})$ on $\boldsymbol{\theta}$ when there is no risk of confusion):

$$\boldsymbol{R}_{12} = \boldsymbol{A}\boldsymbol{C}^*; \quad \boldsymbol{A} \in \mathcal{C}^{m_1 \times n}, \quad \boldsymbol{C}^* \in \mathcal{C}^{n \times m_2} \quad (11)$$

The matrix $\boldsymbol{C}$ above is assumed to have the same rank as $\boldsymbol{P}$,

$$\mathrm{rank}\,(\boldsymbol{C}) = \bar{n} \quad (12)$$

Additionally, it is assumed that the array is *unambiguous of order* $n+1$, i.e.,

$$\mathrm{rank}\left([\boldsymbol{a}(\theta_1), \ldots, \boldsymbol{a}(\theta_{n+1})]\right) = n+1 \quad (13)$$

for all distinct DOAs $\theta_1, \ldots, \theta_{n+1}$ within the interesting field-of-view.

The following sections focus on ML estimation of the DOAs $\{\theta_k\}$ from N snapshots of the array output, $\{\boldsymbol{y}(t)\}_{t=1}^N$. The number of signals, n, and the rank of the signal covariance matrix, $\bar{n}$, are assumed to be given. Methods for estimating these integer-valued parameters can be found, e.g. in [1]. It should be noted that other existing algorithms designed to operate in noise fields with block-diagonal spatial covariances (see, e.g., [2]–[5]) are more or less *ad-hoc* compared to the ML-based DOA estimation algorithms introduced in the present paper.

2 Exact ML Estimation

The problem dealt with in this section is the estimation of $\{\theta_k\}_{k=1}^n$ from $\{\boldsymbol{y}(t)\}_{t=1}^N$ by the ML method. Under the assumptions made, the negative log-likelihood function of the observed data sequence is given by

$$l \sim N \ln|\boldsymbol{R}| + \sum_{t=1}^N \boldsymbol{y}^*(t)\boldsymbol{R}^{-1}\boldsymbol{y}(t) \quad (14)$$

where the symbol $\sim$ denotes an equality from which all uninteresting additive constants have been eliminated, and where $|\cdot|$ denotes the determinant. Let

$$\hat{\boldsymbol{R}} = \frac{1}{N}\sum_{t=1}^N \boldsymbol{y}(t)\boldsymbol{y}^*(t) = \begin{bmatrix} \hat{\boldsymbol{R}}_{11} & \hat{\boldsymbol{R}}_{12} \\ \hat{\boldsymbol{R}}_{12}^* & \hat{\boldsymbol{R}}_{22} \end{bmatrix} \quad (15)$$

denote the sample covariance matrix. By using this notation, one can rewrite (14) as

$$l \sim N\left\{\ln|\boldsymbol{R}| + \mathrm{tr}\,(\boldsymbol{R}^{-1}\hat{\boldsymbol{R}})\right\} \quad (16)$$

A most natural parameterization of the function l in (16) is via

$$\{\theta_k\}_{k=1}^n\,,\ \boldsymbol{R}_{11}\,,\ \boldsymbol{R}_{22} \text{ and } \boldsymbol{C} \quad (17)$$

where $\boldsymbol{R}_{11}$ and $\boldsymbol{R}_{22}$ are arbitrary Hermitian and positive definite matrices (since $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ are assumed to be so) and $\mathrm{rank}\,(\boldsymbol{C}) = \bar{n}$. The exact ML estimate is obtained by minimizing (16) with respect to all unknowns in (17). At a first glance, this seems like a formidable task, due to the large number of parameters involved in the optimization. However, some tedious calculations presented in [6] show that the log-likelihood function is separable with respect to all nuisance parameters, leading to a non-linear search of standard (n in this case) dimension:

Theorem 1 *The DOA estimate obtained by minimizing (16) with respect to all unknown parameters (17) can be found by solving the following n-dimensional problem*

$$\hat{\boldsymbol{\theta}}_{ML} = \arg\min_{\boldsymbol{\theta}} f(\boldsymbol{\theta}) \quad (18)$$

$$f(\boldsymbol{\theta}) = \ln\{\Pi_{k=1}^{\bar{n}}[1 - \lambda_k(\boldsymbol{\theta})]\}\,, \quad (19)$$

where $\{\lambda_k(\boldsymbol{\theta})\}$ are the ordered eigenvalues ($\lambda_1(\boldsymbol{\theta}) \geq \lambda_2(\boldsymbol{\theta}) \geq \cdots \geq \lambda_n(\boldsymbol{\theta})$) of the $(n \times n)$ matrix

$$\boldsymbol{K}(\boldsymbol{\theta}) = (\boldsymbol{A}^*\hat{\boldsymbol{Q}}^{-1}\boldsymbol{A})^{-1/2}\boldsymbol{A}^*\hat{\boldsymbol{Q}}^{-1} \times \hat{\boldsymbol{R}}_{12}\hat{\boldsymbol{\Omega}}^{-1}\hat{\boldsymbol{R}}_{12}^*\hat{\boldsymbol{Q}}^{-1}\boldsymbol{A}(\boldsymbol{A}^*\hat{\boldsymbol{Q}}^{-1}\boldsymbol{A})^{-1/2}\,. \quad (20)$$

Here, the matrices $\hat{\boldsymbol{Q}}$ ($m_1 \times m_1$) and $\hat{\boldsymbol{\Omega}}$ ($m_2 \times m_2$) are defined by the relations

$$\hat{\boldsymbol{Q}} = \hat{\boldsymbol{R}}_{11} - \hat{\boldsymbol{R}}_{12}\hat{\boldsymbol{R}}_{22}^{-1}\hat{\boldsymbol{R}}_{12}^{*} \quad (21)$$

$$\hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{R}}_{22} + \hat{\boldsymbol{R}}_{12}^{*}\hat{\boldsymbol{Q}}^{-1}\hat{\boldsymbol{R}}_{12} \, . \quad (22)$$

3 Large Sample MLE

The large-sample ML estimator derived in what follows is shown to coincide with a bearing estimator recently introduced in [7] and [8] by a more complicated approach. This large-sample ML estimator is termed UNCLE, following the terminology in the original work. As explained and illustrated later on (and further in [6]), UNCLE offers the same statistical performance as the exact MLE, even in small samples, but at a reduced computational cost.

We begin the derivation by observing that $\boldsymbol{K}(\boldsymbol{\theta})$ in (20) is a Grammian matrix, whose rank is asymptotically equal to $\bar{n}$. Then, it follows from Lemma A.1. in [6] that the smallest $(n-\bar{n})$ eigenvalues of $\boldsymbol{K}(\boldsymbol{\theta})$ differ from zero by terms of order $O_p(N^{-1})$ (uniformly in $\boldsymbol{\theta}$). Consequently, the MLE and the bearing estimates obtained by minimizing the function

$$\ln|\boldsymbol{I} - \boldsymbol{K}(\boldsymbol{\theta})| \quad (23)$$

have the same asymptotics. A number of further large-sample approximations [6] lead to the following result:

Theorem 2 *Let*

$$\hat{\boldsymbol{R}}_{11}^{-1/2}\hat{\boldsymbol{R}}_{12}\hat{\boldsymbol{R}}_{22}^{-1}\hat{\boldsymbol{R}}_{12}^{*}\hat{\boldsymbol{R}}_{11}^{-1/2} = [\hat{\boldsymbol{U}}_s \;\; \hat{\boldsymbol{U}}_p] \times \begin{bmatrix} \hat{\boldsymbol{\Gamma}}_s & \boldsymbol{O} \\ \boldsymbol{O} & \hat{\boldsymbol{\Gamma}}_p \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{U}}_s^{*} \\ \hat{\boldsymbol{U}}_p^{*} \end{bmatrix} \quad (24)$$

denote the EVD of the left-hand-side matrix, where $\hat{\boldsymbol{U}}_s$ contains the $\bar{n}$ principal eigenvectors. Then, a large sample ML bearing estimate can be obtained as follows

$$\hat{\boldsymbol{\theta}}_{UNCLE} = \arg\min_{\boldsymbol{\theta}} h(\boldsymbol{\theta}) \quad (25)$$

$$h(\boldsymbol{\theta}) = tr\left[\hat{\boldsymbol{U}}_s\hat{\boldsymbol{\Gamma}}_s\hat{\boldsymbol{U}}_s^{*}\,\mathring{\boldsymbol{\Pi}}_{\hat{\boldsymbol{R}}_{11}^{-1/2}\boldsymbol{A}}\right] , \quad (26)$$

where

$$\mathring{\boldsymbol{\Pi}}_{\hat{\boldsymbol{R}}_{11}^{-1/2}\boldsymbol{A}} = \boldsymbol{I} - \hat{\boldsymbol{R}}_{11}^{-1/2}\boldsymbol{A}(\boldsymbol{A}^{*}\hat{\boldsymbol{R}}_{11}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}^{*}\hat{\boldsymbol{R}}_{11}^{-1/2} \, . \quad (27)$$

The criterion function in (26) is recognized as the UNCLE criterion of [7] and [8].

An interesting observation is that *overestimation of $\bar{n}$ does not affect the asymptotics of MLE or UNCLE.* In particular, this means that minimization of either of the criteria (19) or (26) with $\bar{n} = n$, yields a large-sample realization of the exact MLE (which uses the true value of $\bar{n}$).

4 Statistical Analysis

It is straightforward to verify the consistency of the MLE and UNCLE methods, under the additional assumption that

$$m_1 > 2n - \bar{n} \, . \quad (28)$$

The latter requirement is necessary for the DOAs to be identifiable (by similar arguments as in [9]). Once it is established that the DOA estimates converge to their true values (with probability one) as $N \to \infty$, the asymptotic normality and statistical efficiency of the ML estimates follow from the general theory of ML estimation. Hence, we have the following result:

Theorem 3 *Let $\hat{\boldsymbol{\theta}}$ be obtained by minimizing either of the functions (19) or (26). Then,*

$$\sqrt{N}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \in \mathrm{As}\mathcal{N}(\boldsymbol{O}, \mathbf{CRB}) \, ,$$

where **CRB** *denotes N times the Cramér-Rao lower bound on the covariance matrix of $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ for any (asymptotically) unbiased estimator.*

The usefulness of the previous general result depends on the availability of an explicit expression for the CRB. In [6] it is shown that

$$\mathbf{CRB} = \frac{1}{2}\left\{\mathrm{Re}\left[\left(\boldsymbol{D}^{*}\boldsymbol{R}_{11}^{-1/2}\,\mathring{\boldsymbol{\Pi}}_{\boldsymbol{R}_{11}^{-1/2}\boldsymbol{A}}\,\boldsymbol{R}_{11}^{-1/2}\boldsymbol{D}\right) \odot \left(\boldsymbol{C}^{*}\boldsymbol{R}_{22}^{-1}\boldsymbol{C}\right)^{T}\right]\right\}^{-1} \, . \quad (29)$$

In the equation above, $\odot$ denotes the Hadamard matrix product and $\boldsymbol{D}$ is the matrix made up from the first-order derivatives of the steering vectors

$$\boldsymbol{D} = [\boldsymbol{d}_1 \cdots \boldsymbol{d}_n] \;\; , \;\; \boldsymbol{d}_j = \left.\frac{d\,\boldsymbol{a}(\theta)}{d\,\theta}\right|_{\theta=\theta_j} \, . \quad (30)$$

Let us compare the CRB matrix (29) derived under the conditions of this paper with the CRB corresponding to a somewhat different scenario. Let $\boldsymbol{A}_u$ denote the transfer matrix of the uncalibrated subarray. Then, the matrix $\boldsymbol{C}^{*}$ in (29) can typically be written as

$$\boldsymbol{C}^{*} = \boldsymbol{P}\boldsymbol{A}_u^{*} \quad (31)$$

where $\boldsymbol{P}$ is the signal covariance matrix (see (9) and (11)). Assume that $\underline{\lambda}\boldsymbol{I} \leq \boldsymbol{\Sigma}_2 \leq \bar{\lambda}\boldsymbol{I}$ holds for some $0 < \underline{\lambda} \leq \bar{\lambda} < \infty$ and for all m_2, and additionally that $\|\boldsymbol{A}_u\|$ increases without bound as $m_2 \to \infty$. Assuming a large number of uncalibrated sensors, the CRB matrix can then be approximated using the limiting CRB as $m_2 \to \infty$, given by the expression [6]

$$\frac{1}{2}\left\{\mathrm{Re}\left[\left(\boldsymbol{D}^*\boldsymbol{\Sigma}_1^{-1/2}\,\mathring{\boldsymbol{\Pi}}_{\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{A}}\,\boldsymbol{\Sigma}_1^{-1/2}\boldsymbol{D}\right)\odot\boldsymbol{P}^T\right]\right\}^{-1}$$

The above matrix is recognized as the deterministic CRB associated with the first subarray, under the condition that $\boldsymbol{\Sigma}_1$ is known and that the signals are arbitrary deterministic quantities. The latter CRB is known not to be reachable by any algorithm using data from only the first subarray, (see e.g. [10, 11]). We conclude that for large enough m_2, *the methods introduced herein will yield more accurate DOA estimates than any method which does not employ the uncalibrated array, even if the latter method uses knowledge of* $\boldsymbol{\Sigma}_1$.

5 Numerical Examples and Simulations

This section reports the results of a comparative performance study based on Monte-Carlo simulations. In the examples, a uniform linear array of 16 elements with half-wavelength separation is employed. The first $m_1 = 8$ contiguous sensors form the calibrated subarray, whereas the last $m_2 = 8$ sensors define the uncalibrated subarray. Impinging on the array are two plane waves from DOAs 0° and 5° relative the array broadside. The outputs of both subarrays are perturbed by additive zero-mean temporally white Gaussian noise, with identical covariance matrices, $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$, having klth elements

$$\sigma^2\, 0.9^{|k-l|}e^{j\frac{\pi}{2}(k-l)}\,,$$

where σ^2 is adjusted to give a signal-to-noise ratio (SNR) varying from -4 to 10 dB. This noise is reminiscent of a strong signal cluster at the location $\theta = 30°$. The empirical results presented below are based on 512 independent noise and signal realizations, each using a batch of $N = 100$ samples. The MLE and UNCLE methods are implemented using a Newton method based on criterion function evaluations only. The iterative search is initialized at the true DOAs, since we are only interested in the quality of the global minimum. An important question not addressed herein is the probability of reaching the global minimum using a realistic initialization scheme.

In the first example, the emitter signals are zero-mean Gaussian with covariance matrix $\boldsymbol{P} = \boldsymbol{I}$. Figure 1 shows the theoretical root-mean-square (RMS)

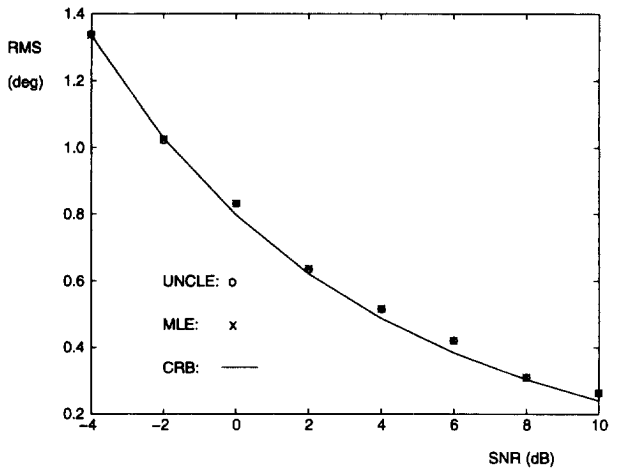

Figure 1: *RMS error of DOA estimate vs SNR. Uncorrelated signals.*

error, as predicted by the CRB (29), versus the SNR. The empirical results for the MLE and UNCLE methods are also included in the plot. Only the results for $\theta_1 = 0°$ are shown, the ones for θ_2 being similar. As seen from the figure, the two methods perform virtually identically even at low SNR. The performance prediction based the CRB is excellent in this scenario.

Next, the effect of coherent signals is investigated. The signal covariance is chosen as

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

implying that the signal waveforms from the two emitters are identical at the first sensor. Figure 2 displays the results obtained with the MLE and UNCLE methods. Also in this relatively difficult scenario, the two methods perform virtually identically and they both attain the CRB throughout the studied SNR region. The methods are not insensitive to multipath, though, as the RMS values in Figure 2 are somewhat higher than those in Figure 1.

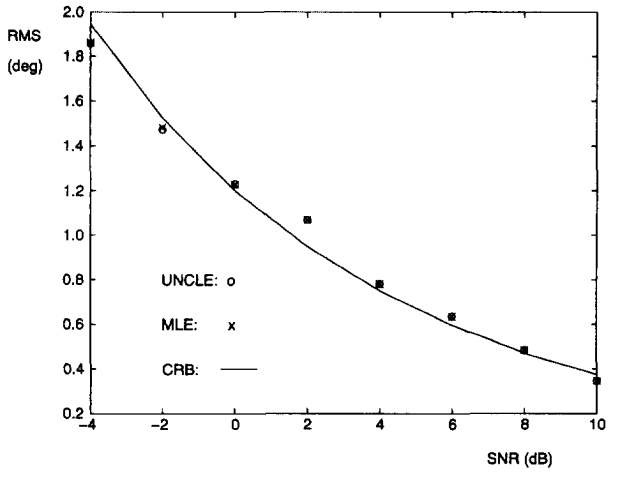

Figure 2: *RMS error of DOA estimate vs SNR. Coherent signals.*

6 Conclusions

Signal parameter estimation using measurements from an array of sensors has been considered. The array was assumed to contain some calibrated sensors as well as sensors whose gain and phase characteristics or positions are unknown. The array receives the wavefronts of narrowband signals corrupted by additive noise. The noises in the calibrated and uncalibrated subarrays are uncorrelated, but otherwise arbitrary. The standard assumption of spatially white noise is thus avoided. The exact ML estimator for this problem has been presented, assuming temporally white Gaussian signals and noise. Explicit solutions for all nuisance parameters were found, leading to a non-linear search problem over the parameters of interest – the directions-of-arrival – only. An approximate ML method has also been presented, and shown to coincide with the exact MLE in large samples. The approximate method is in fact the UNCLE technique, originally introduced in [7] using a more complicated approach. Both methods provide statistically efficient estimates, i.e., the covariance matrix of their estimation errors coincide with the CRB in large samples. An explicit expression for the CRB matrix was presented. Computer simulations indicated that the MLE and UNCLE methods have identical statistical properties also in fairly small samples, at low SNR or for closely spaced signals. Hence, it was concluded that the UNCLE method is preferable owing to its lower computational complexity.

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