Inter-Space and Intra-Space Transformations for Sensor Array Processing*

J. Scott Goldstein,[1,2,3] Douglas B. Williams,[2] Russell M. Mersereau,[2] and E. Jeff Holder[3]

[1] USAF Rome Laboratory
Space Communications Branch
525 Brooks Rd.
Griffiss AFB, NY 13441-4505
USA

[2] Digital Signal Processing Laboratory
School of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0250
USA

[3] Georgia Tech Research Institute
Sensors and Electomagnetic Applications Lab
Georgia Institute of Technology
Atlanta, GA 30332-0800
USA

Abstract— **The use of inter-space and intra-space transformations are considered for improving the transient performance of the LMS processor and reducing the dimension of the weight vector, respectively. Within the context of sensor array processing, a subband LMS algorithm is described for inter-space processing and a cross-spectral subspace selection metric is presented for intra-space transformations. The transform operators may be combined with the Generalized Sidelobe Canceller signal blocking matrix.**

I. Introduction

Technology has advanced to the state where arrays composed of a large number of elements may be realized on airborne and space-segment platforms. However, these platforms impose computational and power constraints which make fully adaptive array processors computationally unrealizeable. This problem is further compounded when wideband signal environments require space-time processing.

The computational complexity of an adaptive processor is a function of the rank of the filter and the requirements of the adaptive algorithm. The filter rank is conventionally chosen to satisfy the desired Wiener solution and the resulting steady state minimum mean-square error (MMSE). The complexity of the algorithm determines how quickly the filter coefficients converge to the Wiener solution.

This paper examines two subspace-based techniques for reducing the overall computational complexity of the sensor array. The methodology to be developed begins by forming an orthogonal basis for the space spanned by the columns of the received data correlation matrix. The inter-space transformation defined by those basis vectors improves the convergence behavior of the simple LMS algorithm. A cross-spectral metric is then introduced to select a lower dimensional subspace prior to adaptive filtering. This intra-space transformation reduces the rank of the filter, resulting in a special case of a partially adaptive sensor array. When used with the Generalized Sidelobe Canceller (GSC) [1], these transformations and the signal blocking matrix may be combined.

The inter-space and intra-space operators will be addressed in sections II and III, respectively. Section IV will consider an example with a narrowband minimum variance, distortionless response (MVDR) sensor array. The conclusions will be discussed in section V.

II. Inter-Space Transformations

The LMS algorithm is computationally simple, but has an undesirable speed of convergence dependency on the signal environment. This convergence dependency may be alleviated via a unitary transformation of the received signals. The LMS algorithm can be generalized to have the form

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \boldsymbol{\mu}(k)\mathbf{x}(k)\varepsilon^*(k), \qquad (1)$$

where the time-varying step size $\boldsymbol{\mu}(k)$ is a diagonal matrix, $\mathbf{x}(k)$ is the received signal vector, and $\varepsilon(k)$ is the error between some desired reference signal $d(k)$ and the adaptive filter output $\hat{d}(k) = \mathbf{w}^H(k)\mathbf{x}(k)$. The LMS weight update is the sum of the previous value and a step to be taken in the direction of the negative gradient from that last value on the mean-square error (MSE) performance surface. Assuming the impinging signals are second-order stationary (and zero-mean, with no loss in generality), the eigenvalue spread of the received data correlation matrix $\mathbf{R} = \mathbf{E}[\mathbf{x}(k)\mathbf{x}^H(k)]$ determines the ellipticity of the contours of constant MSE. The degree of ellipticity affects the speed of convergence of the LMS algorithm. If the eigenvalue spread is unity, then the contours are circular and the negative gradient $\mathbf{x}(k)\varepsilon^*(k)$ from any point on the performance surface points in the direction of MMSE. As the eigenvalue spread increases from unity, the ellipticity increases and, unless the weight vector is initialized on

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a principal axis (given by the eigenvectors of $\mathbf{R}$), the negative gradient no longer points towards the MMSE. Hence, the weight vector updates must walk down the negative gradient until they reach a principal axis before it is possible to reach the bottom of the performance surface.

Consider a transformation of the received data by the unitary operator $\mathcal{E}$ composed of the eigenvectors of $\mathbf{R} = \mathcal{E}\Lambda\mathcal{E}^H$. Defining the transformed observation data $\mathbf{z}(k) = \mathcal{E}^H\mathbf{x}(k)$, the LMS algorithm becomes

$$\mathbf{w}_{\mathcal{E}}(k+1) = \mathbf{w}_{\mathcal{E}}(k) + \mu(k)\mathbf{z}(k)\alpha^*(k), \quad (2)$$

where $\alpha(k) = d(k) - \mathbf{w}_{\mathcal{E}}^H(k)\mathbf{z}(k)$. If the ii-th step size is chosen to be the inverse of the i-th eigenvalue, then

$$\mathbf{E}[\mathbf{w}_{\mathcal{E}}(k+1)] = \mathbf{E}[\mathbf{w}_{\mathcal{E}}(k)] + \Lambda^{-1}\mathbf{E}[\left(d(k) - \mathbf{w}_{\mathcal{E}}^H(k)\mathbf{z}(k)\right)^* \mathbf{z}(k)]. \quad (3)$$

Defining $\mathbf{w}(k) = \mathcal{E}\mathbf{w}_{\mathcal{E}}(k)$ and evaluating equation (3) without the expectation operator, we find

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mathbf{R}^{-1}\mathbf{x}(k)\varepsilon^*(k), \quad (4)$$

which is Newton's method if the statistics are known exactly. If the correlation matrix is estimated from the data, then the dynamic performance of this LMS algorithm would exhibit the same properties as recursive least squares.

The unitary transformation of the received data by the eigenvector operator corresponds to a rotation of the space spanned by the columns of the array correlation matrix. However, the inter-space transformation that decorrelates the data and reduces the eigenvalue spread is not unique. One interpretation is that the eigenvectors are filters which process the received data into uncorrelated frequency bins, and the eigenvalues are representative of the signal power in those bins. The Gram-Schmidt (GS) lower diagonal matrix operator corresponds to another filterbank having the same decorrelating properties. In this case, the step size matrix would then have diagonal values equal to the inverses of the powers estimated in each GS subband. Note that this subband LMS algorithm is an extension of transform domain techniques in that the implemented filterbank may be uniform (DFT or DCT) or nonuniform (KLT, GS, or arbitrary paraunitary filterbank) [2].

III. Intra-Space Transformations

We now discuss the desire to reduce the rank of the Wiener filter. The goal is to reduce the number of coefficients that must be computed for the noise-cancelling filter without suffering a great degredation in MMSE performance. This reduction may be achieved by considering the problem from a subspace selection viewpoint. If the unit vector pointing in the direction of MMSE were known, then the solution of a large N-weight array could be found exactly at every lower dimension down to rank one via the transformation of the data to a lower dimensional subspace whose span includes that vector. The difficulty lies in choosing such a subspace with less computation than what is necessary to find the optimal weight vector.

It is a common misconception that the Wiener solution within the eigen-subspace Ψ, formed by the $M < N$ eigenvectors corresponding to the M largest eigenvalues, will provide the lowest MMSE. Denoting the $N \times M$ unitary matrix composed of the M largest eigenvectors as $\mathcal{E}_{max}$, Ψ is the space spanned by the columns of $\mathcal{E}_{max}^H\mathbf{R}\mathcal{E}_{max}$. While the M-dimensional eigen-subspace is the best rank M representation of the full rank space, the Wiener filter in the eigen-subspace <u>is not</u> the Wiener filter of rank M which provides the lowest MMSE. The best subspace in terms of MMSE performance is selected by a cross-spectral (C-S) metric which is a function of both the basis representation of the space spanned by the columns of $\mathbf{R}$ and the space spanning the reference signal. This C-S metric may be found by extending the work in [3] and is a generalization of the work done in [4].

Define the joint process vector $\mathbf{v}$ as

$$\mathbf{v} = \begin{bmatrix} \varepsilon_r \\ y_r \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{w}^H - \mathbf{w}_r^H\mathbf{Q}^H \\ 0 & \mathbf{w}_r^H\mathbf{Q}^H \end{bmatrix} \begin{bmatrix} \varepsilon \\ \mathbf{x} \end{bmatrix}, \quad (5)$$

where we have denoted the full rank Wiener-Hopf weight vector as $\mathbf{w}$, the reduced rank weight vector as $\mathbf{w}_r$, the observation vector rank-reducing $M \times N$ transformation matrix as $\mathbf{Q}^H$, the reduced degree of freedom error ε_r, and the reduced rank filter output as y_r. Defining the vectors $\mathbf{g} = \mathbf{Q}\mathbf{w}_r$ and $\tilde{\mathbf{w}} = \mathbf{w}^H - \mathbf{g}^H$, we find that the joint covariance matrix $\mathbf{R}_v$ is

$$\mathbf{R}_v = \mathbf{E}[\mathbf{v}\mathbf{v}^H] = \begin{bmatrix} P + \tilde{\mathbf{w}}\mathbf{R}\tilde{\mathbf{w}}^H & 0 \\ 0 & \mathbf{g}\mathbf{R}\mathbf{g}^H \end{bmatrix}, \quad (6)$$

where P is the error covariance which equals the MMSE. It is now desired to choose $\mathbf{Q}$ such that $\mathbf{g}$ minimizes the trace of the extra covariance:

$$\min_{\mathbf{g}^H} tr\left[\left(\mathbf{w}^H - \mathbf{g}^H\right)\mathbf{R}\left(\mathbf{w}^H - \mathbf{g}^H\right)^H\right]. \quad (7)$$

The solution is to choose $\mathbf{Q}$ such that $\mathbf{g}^H\mathbf{R}^{1/2}$ is the best low rank approximation to $\mathbf{w}^H\mathbf{R}^{1/2}$. This solution is equivalent to choosing $\mathbf{g}$ such that it is the closest vector to the cross correlation vector $\mathbf{r} = \mathbf{E}[\mathbf{x}(k)d^*(k)]$ in the weighted Euclidean norm. With this selection, the columns of the transform domain correlation matrix

$\mathbf{R}_z = \mathbf{Q}^H\mathbf{R}\mathbf{Q}$ will span the M-dimensional subspace which provides the lowest MMSE. Hence, we choose the M eigenvectors of $\mathbf{R}$ corresponding to the largest M of

$$\cos^2\theta(i) = \left|\frac{\boldsymbol{\mathcal{E}}_i^H\mathbf{r}}{\sqrt{\lambda_i}}\right|^2 \tag{8}$$

to form the matrix $\mathbf{Q}$. This choice of $\mathbf{Q}$ selects the C-S subspace $\boldsymbol{\Omega}$ which is the optimal subspace of dimension M with respect to the columns of $\boldsymbol{\mathcal{E}}$ for the Wiener filter.

It is noted that in general, the matrix $\mathbf{Q} \neq \boldsymbol{\mathcal{E}}_{max}$ and the subspace $\boldsymbol{\Omega} \neq \boldsymbol{\Psi}$. We now consider the MMSE with the unitary decomposition of the correlation matrix performed by the matrix of eigenvectors $\boldsymbol{\mathcal{E}}$. The MMSE may be written as

$$\text{MMSE} = \sigma_d^2 - \sum_{i=1}^{N}\frac{\left|\boldsymbol{\mathcal{E}}_i^H\mathbf{r}\right|^2}{\lambda_i}, \tag{9}$$

from which it is obvious that selecting the subspace which provides the largest C-S contribution results in the lowest MMSE. Further, since $\mathbf{Q} \neq \boldsymbol{\mathcal{E}}_{max}$, the subspace which provides the best low rank approximation to the matrix $\mathbf{R}$ is not the subspace which provides the best low rank Wiener filter.

The C-S metric of equation (8) may also be used with any unitary transformation or paraunitary filterbank. In this case, the vector $\boldsymbol{\mathcal{E}}_i$ is replaced by the i-th filter impulse response and the power of the i-th filtered signal takes the place of λ_i. Thus, if the inter-space transformation decorrelates the received data, the basis vectors defined by the transformation may be used directly in equation (8) to reduce the rank prior to LMS filtering. If the inter-space transformation does not completely decorrelate the data, then the C-S metric applied to the basis vectors defined by the inter-space operator may increase the eigenvalue spread of the reduced rank correlation matrix.

IV. Simulation Results

We will now consider a narrowband GSC using the MVDR constraint as shown in Figure 1. We will assume that the array is linear, composed of $K = 16$ elements spaced at half-wavelength, signal aligned, and that there is only one desired signal present. The conventional beamforming matrix $\mathbf{w}_c$ enforces the look-direction constraint. For the MVDR constraint, this beamforming matrix takes the form $\mathbf{w}_c = \frac{1}{K}\mathbf{1}$, where $\mathbf{1}$ represents a vector whose elements are all unity. The desired signal is blocked from the adaptive processor through the signal blocking matrix $\mathbf{W}_s$, which is of dimension $N \times K$, where $N = K - 1$. The full row rank

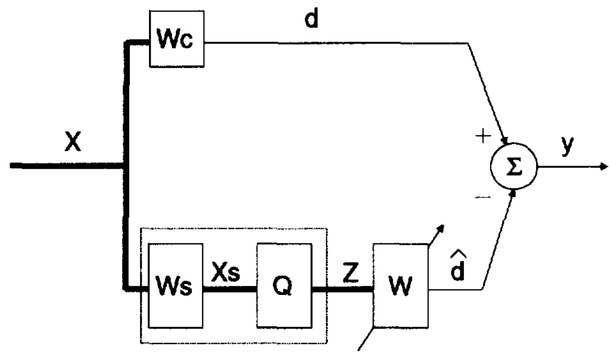

Fig. 1. GSC Form MVDR Array Processor

matrix $\mathbf{W}_s$ is composed of rows $\mathbf{r}_i$ such that $\mathbf{r}_i\mathbf{1} = 0\ \forall i$. This matrix filter will simply take the difference of adjacent elements, so that the implementation is multiplier free.

The K-dimensional received data vector is denoted $\mathbf{x}(k)$. The N-dimensional noise subspace data vector $\mathbf{x}_s(k)$, the M-dimensional transformed observation data vector $\mathbf{z}(k)$, the beamformed output $d(k)$, and the beamformed noise estimator $\hat{d}(k)$ are then given by

$$\begin{array}{ll} \mathbf{x}_s(k) = \mathbf{W}_s\mathbf{x}(k) & \mathbf{z}(k) = \mathbf{Q}^H\mathbf{x}_s(k) \\ d(k) = \mathbf{w}_c{}^H\mathbf{x}(k) & \hat{d}(k) = \mathbf{w}^H(k)\mathbf{z}(k). \end{array} \tag{10}$$

where $\mathbf{w}(k)$ is the rank reduced M-dimensional adaptive weight vector. The array output, which is the error signal in conventional adaptive signal processing, is $y(k) = \left(\mathbf{w}_c{}^H - \mathbf{w}^H(k)\mathbf{Q}^H\mathbf{W}_s\right)\mathbf{x}(k)$.

In many applications, such as space communications, prespecified interference scenarios may be defined. In this example, we will assume that eight jammers from angles $\theta_i = -61.18°, -45.00°, -30.69°, -10.24°, 10.24°, 21.93°, 40.00°$, and $60.00°$ are prespecified and that these interference signals have a power level of 40 dB above the noise floor. An array correlation matrix $\mathbf{R}_a$ is then formed assuming the presence of these signals, and its its eigen-decomposition $\mathbf{R}_a = \boldsymbol{\mathcal{E}}\boldsymbol{\Lambda}\boldsymbol{\mathcal{E}}^H$ performed. The inter-space operator to

Table 1. Signal Geometry

SIGNAL	DOA	POWER
desired	0.00 deg	0.00 dB
jammer 1	−61.18 deg	40.00 dB
jammer 2	−30.69 deg	43.52 dB
jammer 3	−10.24 deg	33.98 dB
jammer 4	10.24 deg	38.06 dB
jammer 5	21.93 deg	40.00 dB

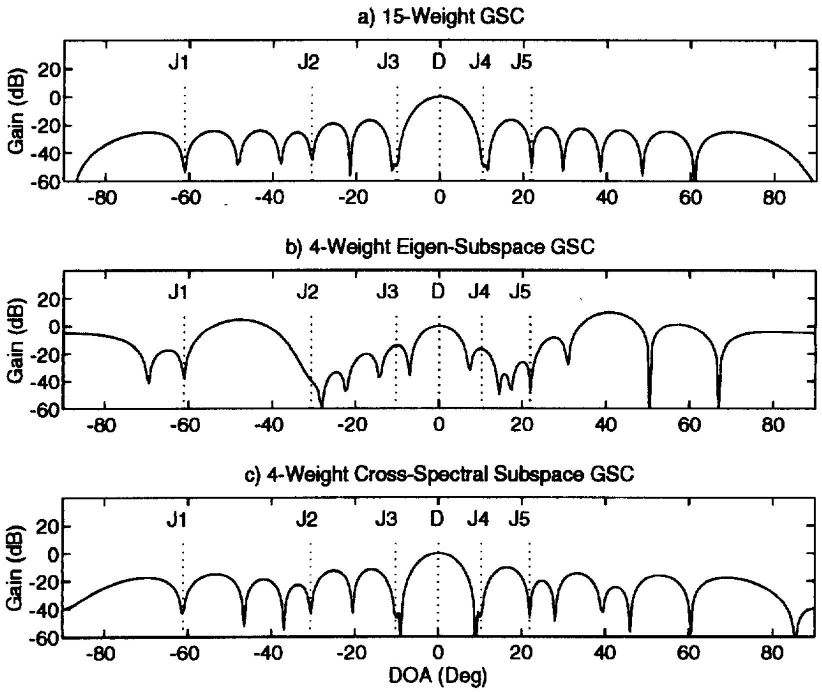

Fig. 2. Array power gain patterns as a function of direction of arrival are shown for three 16-sensor GSC realizations using the optimal Wiener-Hopf weight vector for each: (a) the fully adaptive 15-weight GSC, (b) a partially adaptive 4-weight eigen-subspace GSC, and (c) a partially adaptive 4-weight cross-spectral subspace GSC.

be considered is the matrix $\mathcal{E}$. The intra-space transformation $\mathbf{Q}$ will be found via the C-S metric, where the basis vectors defined by the inter-space operator are utilized. The full rank 16-element array has a weight vector whose rank is 15. We will consider reducing the rank of the weight vector to 4. The signal environment, described in Table 1, is composed of 5 interference signals and 1 desired signal, where the powers are relative to the noise floor. The true interference scenario is a subset of the prespecified signals with varying jammer powers. It is noted that the adaptive filter rank has been reduced below the eigenstructure of the noise field. Thus, the Wiener filter in the 4-dimensional eigen-subspace is expected to yield a poor solution, while the Wiener solution in the cross-spectral subspace is anticipated to provide reasonable results.

Figure 2a presents the full rank (15-weight) Wiener solution. It can be seen that all interference signals are attenuated, and the resulting MMSE is -11.10 dB. Figure 2b depicts the Wiener solution in the 4-dimensional eigen-subspace, formed by the 4-largest eigenvectors of $\mathbf{R}_a$. The MMSE in the eigen-subspace is -3.82 dB, reflecting a loss of 7.28 dB in reducing the rank of the Wiener filter from 15 to 4. From Figure 2b, it is clear that the Wiener filter in the eigen-subspace is not capable of attenuating the jammers from $-10.24°$ and $10.24°$. Figure 2c displays the performance of the 4-weight Wiener solution in the cross-spectral subspace of $\mathbf{R}_a$. It may be seen that all jammers are attenuated, and the resulting array pattern is very similar to that provided by the full rank Wiener solution. The MMSE in the cross-spectral subspace is -11.07 dB, indicating a loss of only 0.03 dB in performance from the full rank Wiener solution while decreasing the number of weights from 15 to 4.

Figure 3 depicts the beam patterns for the full rank, eigen-subspace, and cross-spectral subspace GSC processors after 100 adaptations of the LMS algorithm. The weight vectors used to generate Figure 3 were ensemble averaged over 180 independent runs. The eigenvalue spread of the full rank correlation matrix, the eigen-subspace correlation matrix, and the C-S subspace correlation matrix are $1\text{x}10^5$, 979.5, and 9.30, respectively. These values reflect the performance of the LMS beam patterns in Figure 3. After 100 adaptations, the 4-weight eigen-subspace and cross-spectral GSC processors have nearly converged; conversely, the 15-weight GSC has just started to attenuate the strongest interference signals. The MSE after 100 adaptations is 34.92 dB, -2.24 dB, and -9.58 dB for the full rank, eigen-subspace, and C-S subspace processors, respectively. Therefore, the full rank GSC is 46.02 dB above

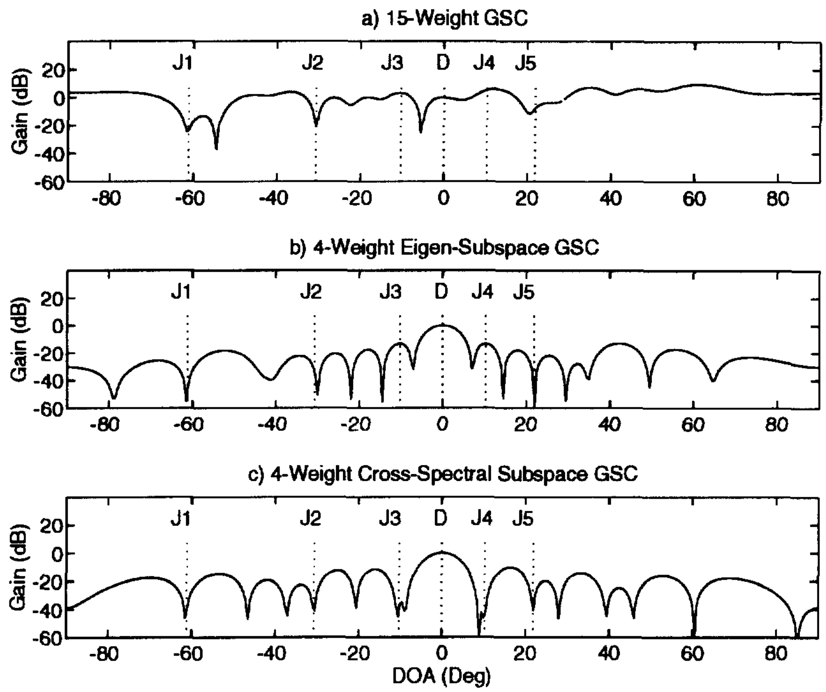

Fig. 3. Array power gain patterns as a function of direction of arrival are shown for three 16-sensor GSC realizations using the LMS weight vector after 100 adaptations averaged over 180 runs for each: (a) the fully adaptive 15-weight GSC, (b) a partially adaptive 4-weight eigen-subspace GSC, and (c) a partially adaptive 4-weight cross-spectral subspace GSC.

its MMSE, the eigen-subspace GSC is 1.58 dB above its MMSE and the C-S GSC is 1.49 dB above its MMSE.

The speed of convergence for both subspace LMS processors is greatly improved as compared to the standard full rank GSC. It is noted that if least squares techniques were used for all three processors, the two with reduced rank would still converge quicker, since the transient behavior is then a function of the number of adaptive coefficients.

V. Conclusions

The use of inter-space and intra-space transformations for sensor array processing have been demonstrated to reduce the computational complexity requirements of the adaptive processor. Both the transient behavior of the sensor array and the steady state low rank Wiener solutions have been examined. A subband LMS algorithm was introduced for generalized subband filtering of the received data, and a cross-spectral metric was derived for reduced rank Wiener filtering.

The effective signal blocking matrix $\mathbf{B} = \mathbf{Q}^H \mathbf{W}_s$ is of dimension $M \times K$. This partially adaptive GSC may be interpreted as reducing the rank of the data observed by the adaptive filter through the implementation of a higher-order constraint in the optimization process. Thus, by implementing the GSC with the matrix filter $\mathbf{B}$ directly, the number of required operations to form the observation data vector $\mathbf{z}(k)$ may be greatly reduced.

The example generated in this paper was a narrowband array, but all of the analysis and results extend trivially to the wideband case via a change in the definition of the data vectors and filters.

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