# Applications of Cumulants to Array Processing: Direction-Finding in Coherent Signal Environment

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## Abstract

*Doğan and Mendel [2] have developed the virtual-ESPRIT Algorithm (VESPA) for direction-finding and recovery of independent sources. VESPA can calibrate an array of unknown configuration and arbitrary response by using just one additional pair of identical sensors (instead of a copy of the entire array or storage of the entire array response for every possible scenario, which is required by existing alternatives). In this work we present an approach that generalizes VESPA to handle the case of highly correlated or coherent sources. Unlike existing methods, our method is not restricted to linear arrays, and no search procedure is needed. Just as in VESPA, it is still possible to detect more sources than sensors, and suppress both Gaussian as well as non-Gaussian noise. A simulation experiment supporting our conclusions is provided.*

## 1 Introduction

Highly correlated or coherent sources is often the case in multipath propagation environments or in military scenarios when there are smart jammers. Existing covariance-based high-resolution direction-finding methods for this case are limited in application due to the impractical assumptions about the array geometry or their computational requirements ( [4], [10], [11], [12], [1], [6], [5] as well as others). Our previous work provides hope that, by using cumulants, the coherent sources case can be handled without array calibration and aperture reduction or a multidimensional search procedure. In this paper, we generalize our earlier method VESPA for direction finding to the case of coherent sources. In our approach only a subarray is required to be linear. The other array elements may have *arbitrary* and *unknown* locations and responses. Unlike the approach in [4], [10], coherent signal powers are combined effectively instead of decorrelated.

## 2 Formulation of the Problem and Proposed Solution

Assume $P$ wavefronts from $n_g$ independent narrowband non-Gaussian sources with $p_i$ coherent wavefronts for each source $u_i(t)$ ($\sum_{i=1}^{n_g} p_i = P$). In the sequel, the collection of $p_i$ coherent wavefronts for the $i$th source will be referred to as the $i$th *group* (i.e., there are $n_g$ groups in our problem). Consider an $M$ element array composed of an $L$ element uniform linear subarray and an $M-L$ element arbitrary response and location subarray (see Fig. 1). The received signal is

$$\mathbf{r}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \qquad (1)$$

where $\mathbf{A}$ is an $M \times P$ *unknown* steering matrix; $\mathbf{s}(t)$ is a $P \times 1$ wavefront vector, and $\mathbf{n}(t)$ is the independent Gaussian [1] measurement noise vector. The coherence among the received wavefronts can be expressed by the following equation:

$$\mathbf{s}(t) = \begin{bmatrix} \mathbf{s}_1(t) \\ \mathbf{s}_2(t) \\ \vdots \\ \mathbf{s}_{n_g}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{c}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{c}_{n_g} \end{bmatrix}}_{\triangleq \mathbf{Q}} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n_g}(t) \end{bmatrix} \qquad (2)$$

where $\mathbf{s}_i(t)$ is a $p_i \times 1$ signal vector representing the coherent wavefronts from the $i$th independent source $u_i(t)$; $\mathbf{c}_i$ is $p_i \times 1$ complex attenuation vector for the $i$th source ($1 \leq i \leq n_g$). Partially-correlated sources are handled similarly since they are linear combinations of the independent sources. The received signal vector, written in terms of independent sources, is:

$$\mathbf{r}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) = \mathbf{A}\mathbf{Q}\mathbf{u}(t) + \mathbf{n}(t) = \mathbf{B}\mathbf{u}(t) + \mathbf{n}(t) \qquad (3)$$

where $\mathbf{B} \triangleq \mathbf{A}\mathbf{Q}$.

[1] We show in Section 3 that our method can suppress non-Gaussian as well as Gaussian noise.



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Columns of $\mathbf{B}$, called the *generalized steering vectors*, can be estimated, as explained next. Using $r_1(t)$ twice, the following cumulant can be estimated:

$$\text{cum}(r_1^*(t), r_1(t), r_k^*(t), r_l(t)) = \sum_{i=1}^{n_g} \gamma_{4,u_i} |\mathbf{B}(1,i)|^2 \mathbf{B}^*(k,i)\mathbf{B}(l,i) \quad (4)$$

where $\mathbf{B}(m,n)$ denotes the $(m,n)$th element of $\mathbf{B}$; $\{\gamma_{4,u_i}\}_{i=1}^{n_g}$ are the fourth-order cumulants of the sources, and $1 \leq k,l \leq M$. (4) is derived using cumulant properties [CP1], [CP3],[CP5], [CP6] in [7], and independence of the source signals. Note that the cumulant of the additive Gaussian measurement noise is zero. Next, by choosing $r_1(t)$ and $r_2(t)$ as the guiding sensor pair measurements, we compute:

$$\text{cum}(r_1^*(t), r_2(t), r_k^*(t), r_l(t)) = \sum_{i=1}^{n_g} \gamma_{4,u_i} \mathbf{B}^*(1,i)\mathbf{B}(2,i)\mathbf{B}^*(k,i)\mathbf{B}(l,i) \quad (5)$$

Defining $\text{cum}(r_1^*(t), r_1(t), \mathbf{r}(t), \mathbf{r}^H(t)))$ as the matrix whose $(l,k)$th entry is $\text{cum}(r_1^*(t),\ r_1(t),\ r_k^*(t),\ r_l(t))$, (4) can be expressed as $(1 \leq k,l \leq M)$:

$$\text{cum}(r_1^*(t), r_1(t), \mathbf{r}(t), \mathbf{r}^H(t)) = \mathbf{B}\mathbf{\Lambda}\mathbf{B}^H \quad (6)$$

where we define $\mathbf{\Lambda} \triangleq diag\{\gamma_{4,u_1}|\mathbf{B}(1,1)|^2,\ \cdots,\ \gamma_{4,u_{n_g}} |\mathbf{B}(1,n_g)|^2\}$. Using a similar definition, (5) becomes:

$$\text{cum}(r_1^*(t), r_2(t), \mathbf{r}(t), \mathbf{r}^H(t)) = \mathbf{B}\mathbf{D}\mathbf{\Lambda}\mathbf{B}^H \quad (7)$$

where $\mathbf{D} \triangleq diag\{\frac{\mathbf{B}(2,1)}{\mathbf{B}(1,1)}, \cdots, \frac{\mathbf{B}(2,n_g)}{\mathbf{B}(1,n_g)}\}$. Equations (6) and (7) are now in the form required by the generalized eigenvalue problem associated with ESPRIT [9]; hence, the columns of $\mathbf{B}$ can be estimated. Note that coherent signal powers are effectively combined instead of decorrelated which increases the SNR virtually.

Once we have estimated $\mathbf{B}$, the next step is to identify the steering vectors and DOAs. We can do this, as we describe next. The general form of coherence between the received wavefronts lets us express each generalized steering vector as a linear combination of steering vectors from *one* coherent group, *independent* of the other steering vectors, where the combination coefficients are the elements of the *unknown* complex attenuation vector for that group. To see this, partition the matrices $\mathbf{B}$ and $\mathbf{A}$ as

$$\mathbf{B} = [\mathbf{b}_1, \cdots, \mathbf{b}_{n_g}]\,; \mathbf{A} = [\mathbf{A}_1, \cdots, \mathbf{A}_{n_g}], \quad (8)$$

where $\mathbf{b}_i$ is $M \times 1$, and the steering matrix for the $i$th group, $\mathbf{A}_i \triangleq [\mathbf{a}(\theta_{i,1}), \cdots, \mathbf{a}(\theta_{i,p_i})]$, is $M \times p_i$. Additionally, $\theta_{i,m}$ represents angle-of-arrival of the $m$th source in the $i$th coherent group with $1 \leq m \leq p_i$. Consequently, $\mathbf{B}$ can be expressed as

$$\mathbf{B} = \mathbf{A}\mathbf{Q} = [\mathbf{A}_1\mathbf{c}_1, \cdots, \mathbf{A}_{n_g}\mathbf{c}_{n_g}]\,; \quad (9)$$

and hence, its $i$th column as

$$\mathbf{b}_i = \mathbf{A}_i\mathbf{c}_i \quad (10)$$

where $i = 1, \cdots, n_g$. Now, the problem of estimating the steering vectors and direction-of-arrivals (DOAs) of groups of coherent sources is reduced to $n_g$ independent sub-problems, namely estimating the steering vectors and DOAs of wavefronts in each group separately. To solve the sub-problems we propose keeping the part of each estimated generalized steering vector that corresponds to the linear part of the main array. By doing so, we will be able to incorporate spatial smoothing [4], [10] which, in turn, will *restore* the rank of the rank-one source correlation matrix. For this purpose, let $\mathbf{b}_{L,i}$ be the portion of $\mathbf{b}_i$ corresponding to the linear array of sensors (the first $L$ elements of $\mathbf{b}_i$ for the array configuration in Fig.1). Now, $\mathbf{b}_{L,i}$ can be interpreted as a received signal for an $L$ element uniform linear array that is illuminated by coherent wavefronts from a *single* source represented by $\mathbf{c}_i$, whose steering matrix is $\mathbf{A}_{L,i}$, the first $L$ rows of $\mathbf{A}_i$. Treating $\mathbf{b}_{L,i}\mathbf{b}_{L,i}{}^H$ as the covariance matrix of the received signal, spatial smoothing can be applied to $\mathbf{b}_{L,i}$ where $i = 1, \cdots, n_g$.

Forward spatial smoothing [4], [10] for the $i$th group $(i = 1, \cdots, n_g)$ starts by dividing the $L$-vector $\mathbf{b}_{L,i}$ into $K = L - S + 1$ overlapping subvectors $\mathbf{b}_{S,i}^k = [\mathbf{b}_{L,i}(k), \cdots, \mathbf{b}_{L,i}(k+S-1)]^T$ of size $S$, where $k = 1, \cdots, K$. Let $\mathbf{A}_{L,i}$ be the first $S \times p_i$ part of $\mathbf{A}_{S,i}$. We can express $\mathbf{b}_{S,i}^k$ as

$$\mathbf{b}_{S,i}^k = \mathbf{A}_{S,i}\mathbf{\Phi}_i{}^{(k-1)}\mathbf{c}_i, \quad (11)$$

where $\mathbf{\Phi}_i{}^{(k-1)}$ denotes the $(k-1)$ power of the $p_i \times p_i$ diagonal matrix

$$\mathbf{\Phi}_i = \text{diag}\{e^{-jw_c\Delta sin\theta_{i,1}/c}, \cdots, e^{-jw_c\Delta sin\theta_{i,p_i}/c}\}, \quad (12)$$

in which $\theta_{i,m}$ represents the angle-of-arrival of the $m$th source in the $i$th coherent group, where $1 \leq m \leq p_i$, and $\Delta$ is the separation between the elements of the uniform linear part of the array under consideration. Consequently, for the $k$th subvector

$$\mathbf{b}_{S,i}^k \mathbf{b}_{S,i}^k{}^H = \mathbf{A}_{S,i}\mathbf{\Phi}_i{}^{(k-1)}\mathbf{c}_i\mathbf{c}_i{}^H\mathbf{\Phi}_i{}^{(k-1)^H}\mathbf{A}_{S,i}{}^H. \quad (13)$$

Define the spatially smoothed matrix $\overline{\mathbf{b}_{S,i}\mathbf{b}_{S,i}{}^H}$ as the average of $\mathbf{b}_{S,i}^k \mathbf{b}_{S,i}^k{}^H$ over $k$ , $k = 1, \cdots, K$ , i.e.,

$$\overline{\mathbf{b}_{S,i}\mathbf{b}_{S,i}{}^H} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{b}_{S,i}^k \mathbf{b}_{S,i}^k{}^H \tag{14}$$

$$= \mathbf{A}_{S,i} \underbrace{\left( \frac{1}{K}\sum_{k=1}^{K} \mathbf{\Phi}_i{}^{(k-1)} \mathbf{c}_i \mathbf{c}_i{}^H \mathbf{\Phi}_i{}^{(k-1)^H} \right)}_{\triangleq \mathbf{R}_i} \mathbf{A}_{S,i}{}^H$$

The matrix $\mathbf{R}_i$ can be expressed explicitly, as

$$\mathbf{R}_i \triangleq \frac{1}{K}\mathbf{F}_i \mathbf{F}_i{}^H; \qquad \mathbf{F}_i \triangleq \mathbf{C}_i \mathbf{V}_i \tag{15}$$

where $\mathbf{C}_i = diag\{\mathbf{c}_i(1), \cdots, \mathbf{c}_i(p_i)\}$, and $\mathbf{V}_i$ is a $p_i \times K$ Vandermonde matrix with the $(m,n)$th element as $\mathbf{V}_i(m,n) = e^{-jw_c \Delta sin\theta_{i,m}(n-1)/c}$ $(1 \leq m \leq p_i,\ 1 \leq n \leq K)$.

The rank of $\mathbf{R}_i$ is equal to the rank of $\mathbf{F}_i$, and, the rank of $\mathbf{F}_i$ is the same as that of $\mathbf{V}_i$. For the Vandermonde matrix $\mathbf{V}_i$, $\text{rank}(\mathbf{V}_i) = \min(p_i, K)$; and hence, $\text{rank}(\mathbf{V}_i) = p_i$ if and only if $K \geq p_i$, i.e., if the number of subvectors used in smoothing is greater than or equal to the number of coherent signals in the $i$th group [8]. Thus, if $K = L - S + 1 \geq p_i$, or equivalently $L \geq p_i + S - 1$, then $\text{rank}(\mathbf{R}_i) = p_i$. Additionally, if $S \geq p_i + 1$, the columns of $\mathbf{A}_{S,i}$ are linearly independent; and hence, $\text{rank}(\overline{\mathbf{b}_{S,i}\mathbf{b}_{S,i}{}^H}) = p_i$. Combining both constraints on $L$, we get the following result: If $L \geq 2p_i$, the rank of $\overline{\mathbf{b}_{S,i}\mathbf{b}_{S,i}{}^H}$ is restored to $p_i$, hence by applying any subspace algorithm repetitively to each group, it is possible to estimate DOAs of up to $L/2$ coherent signals in each group.

In the preceding discussion we considered only forward-spatial smoothing. Extension of these results to using both forward and backward smoothing is possible. In this case, the number of resolvable sources in each group becomes $2L/3$ [8]. Since the columns of $\mathbf{B}$ are linearly independent, the maximum number of groups that can be resolved is $M - 1$. Consequently, the maximum total number of coherent sources that can be resolved is equal to $2(M-1)L/3$. If all sensors are in a uniform linear array, $(M = L)$, then a maximum of $2(L-1)L/3$ coherent sources can be resolved.

Note that in deriving this two-step method we assumed the array manifold is unknown, and we did not restrict the entire array to be linear. Only an L-element subarray must be linear, where $L$ determines the maximum number of resolvable targets in a group. In contrast, when the sources are coherent, existing covariance-based high-resolution direction-finding methods [4], [10] are limited to uniform linear arrays. In addition, while our method can resolve $2(L-1)L/3$ coherent sources, these methods can resolve only $2L/3$ sources using an $L$ element linear array. This result clearly demonstrates the aperture extension gained by just replacing existing covariance-based signal processing with our cumulant-based method in a direction finding system. Another point to note is that, if a coherent group contains more than $2L/3$ wavefronts, only the DOA estimates of that group is affected, because, each group is treated *independently* in the second step of our method. Finally, even if there is no coherence, our method works, because it reduces to our original VESPA; therefore, our method is a general one that is applicable to any linearly correlated or coherent signals scenario.

## 3 Non-Gaussian Noise Suppression

In order to demonstrate the link with VESPA, we used the first two sensors which have identical responses as the guiding sensor pair in the formulation of our method in (4)-(7); however, inspection of these equations reveals that the guiding sensor pair elements do not have to have identical responses in our method, and that the choice of the guiding sensor pair is not limited to the first two sensors as opposed to VESPA. Therefore, if it is known that a particular sensor in the array has independent noise from the others, it can be used with any other two sensors to form the guiding sensor pairs thereby making non-Gaussian suppression possible. Since it is most likely for the sensors separated by the greatest distance to have independent noises, these sensors constitute natural choices for the guiding sensor pairs for non-Gaussian noise suppression purpose. See [3] for a detailed discussion on non-Gaussian noise suppression.

## 4 Simulation Experiment

In this experiment we consider the fourteen-element dipole array depicted in Fig. 2. Each dipole has a response $\cos(\phi)$ where $\phi$ is the angle with respect to the normal of the dipole (see Fig. 3). If the dipole has an orientation of $\alpha$, its response becomes $\cos(\phi - \alpha)$. The dipoles have orientations $\{95^o, 85^o, 87^o, 92^o, 90^o, 90^o, 90^o, 90^o, 90^o, 90^o, 90^o, 90^o, 90^o, 90^o\}$. We assume four independent groups of sources. Each group contains a direct-path signal and four scaled and delayed replicas of the direct-path signal which represent the multipaths and "smart" jammers. In this experiment there are 20 sources. Circularly symmetric White Gaussian noise and a different $SNR$ is assumed for each direct-path signal. The directions of arrival and propagation constants of the wavefronts within each group relative to the direct-path and direct-path $SNR$s are chosen as follows, where unity propagation constants correspond to direct-path signals: Group 1:, DOAs: $\{40^o, 68^o, 80^o, 115^o, 130^o\}$, Propagation Constants: {(0.2+0.8i), 1, (0.8-0.5i), (0.75+0.65i), (0.8-0.2i)}, Direct-Path SNR: 20dB; Group 2:, DOAs: $\{50^o, 70^o, 90^o, 120^o, 135^o\}$, Propagation Constants: {(0.9+0.3i), 1, (0.9-0.3i), (0.8+0.7i), 0.95}, Direct-Path SNR: 18dB; Group 3: DOAs: $\{45^o, 65^o, 85^o, 110^o, 125^o\}$, Propagation Constants: {1, (0.8-0.7i), (0.7+0.7i), (0.65-0.8i), (0.9+0.1i)}, Direct-Path SNR: 18dB; Group 4: DOAs: $\{60^o, 85^o, 105^o, 118^o, 140^o\}$, Propagation Constants: {(0.3-0.8i), (0.4+0.9i), (0.8+0.6i), (0.9+0.7i), 1}, Direct-Path SNR: 19dB. Assuming perfect knowledge of the number of sources, and taking the rightmost two dipoles in Fig. 2 as the guiding sensor pair, we applied our method. 3000 snapshots were used to estimate the cumulant matrices. Both forward and backward spatial smoothing with MUSIC were used in the second step of our method. The experiment was repeated 100 times. Figure 4 shows the MUSIC spectrum for each coherent group obtained with our method. The actual arrival angles are also marked in Fig. 4. It is seen that we are able to estimate all arrival angles, and our estimates are consistent. In addition, it can be observed from Fig. 4 that our method is able to successfully separate fairly close arriving angles belonging to signals in different groups. Note that we have a total of 20 sources and a 10 sensor linear subarray; therefore, covariance-based spatial smoothing fails, since it can only estimate angles for a maximum of $P = 7$ ($P = 2L/3, L = 10$) sources. Even if the sources were independent, covariance-based methods would fail, because the number of sources is larger than the number of sensors.

## 5 Conclusions

We have shown that cumulant-based array processing can handle the source coherency problem without restricting the array to be completely linear or using any search procedure; only a subarray is required to be linear. Other array elements may have arbitrary and unknown locations and responses. No calibration is needed. We have also shown that both aperture extension and non-Gaussian noise suppression are possible with our method. A simulation experiment supporting some of our theoretical developments was provided.

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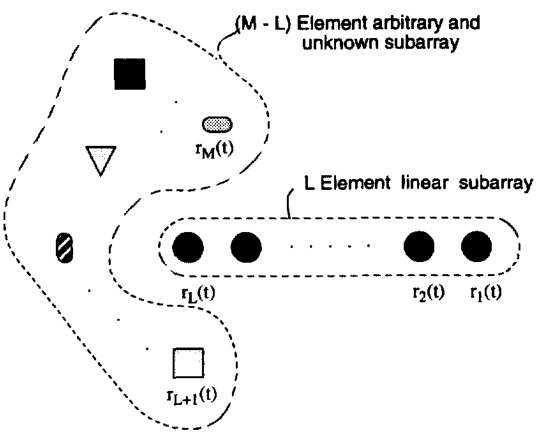


Figure 1: An example array configuration. There are $M$ sensors, $L$ of which are uniform linearly positioned; $r_1(t)$ and $r_2(t)$ are guiding sensors. Linear subarray elements are separated by $\Delta$.

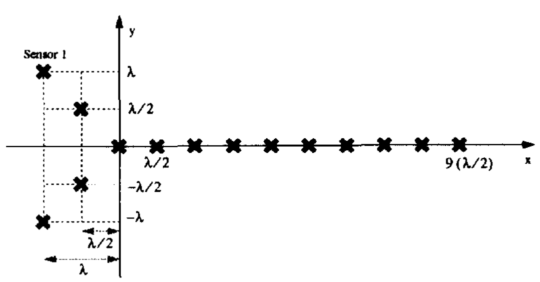


Figure 2: The array configuration used in the experiment. The antenna elements are dipoles oriented by the angles given in the text.

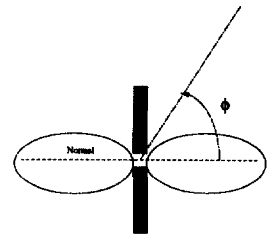


Figure 3: The response of the dipole antennas used in the experimental array.

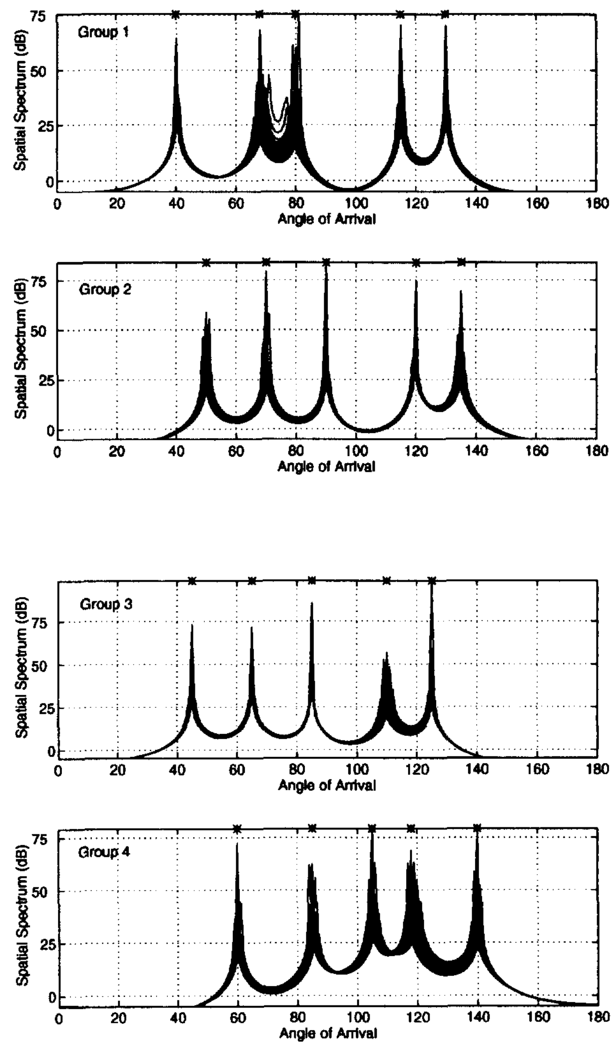


Figure 4: MUSIC spectrum estimates for each coherent group, obtained with our method for 100 runs of the experiment. The actual arrival angles are marked with the symbol "*".