Global Convergence of Fractionally Spaced Godard Equalizers

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ABSTRACT: *In this paper we present a convergence analysis of blind fractionally spaced equalizers (FSE) utilizing the Godard algorithm. The FSE can in fact be represented as a special* vector equalizer *which exploits the spectral diversity (cyclostationarity) of digital channel output and also the spatial diversity (antenna array) when available. It is shown that for channels satisfying a mild* length and zero condition, *the Godard FSE always converges to a global minimum point. Computer simulation demonstrates the performance improvement by the adaptive Godard FSE.*

I Introduction

Adaptive channel equalization is an important tool for eliminating inter-symbol interference (ISI) caused by linear channel distortion or multipath. It is useful when training signals for conventional equalization are not available or very costly.

Among various adaptive algorithms, the Godard algorithm (GA) [5], also known as the constant modulus algorithm (CMA), is one of the best known and simplest adaptive schemes. The CMA was designed for linear equalization using a T-spaced FIR equalizer (TSE). It has two major weaknesses. First, it has been shown [2] that for finitely parametrized equalizers, the Godard algorithm has local minima that do not correspond to acceptable equalizer setting. Local convergence of GA or CMA was also reported in experimental studies [3]. Second, noise enhancement can be severe for channels with zeros on or near the unit circle. Longer equalizer filter is typically required which needs more measured data for convergence.

CMA has often been implemented in fractionally-spaced equalizers (FSE). However, the main advantages of FSE have been thought as suppressing timing phase sensitivity [11][12]. In fact, CMA implemented with FSE also has other important advantages. We shall show in this paper that applied with FSE, Godard algorithm or CMA can also overcome the two major weaknesses of the TSE as described above. It can even be applied to channels with zeros on the unit circle. For channels with deep spectral nulls, the CMA-FSE does not require a large number of parameters and therefore can converge faster.

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The main content of this paper is contained in the following sections. In Section II, the problem of blind equalization is formulated and the Godard blind fractionally spaced equalizer (FSE) is introduced. The global convergence condition for Godard FSE is derived in Section III. Finally, Simulation results are given in Section IV.

II Blind Adaptive FSE

The channel output of a quadratic-amplitude-modulation (QAM) communication system can be described using the baseband representation as

$$x(t) = \sum_{n=0}^{+\infty} a_n h(t - nT - t_0) + w(t). \qquad (2.1)$$

In this formulation, a sequence of independent identically distributed (i.i.d.) complex data $\{a_n\}$ is sent by the transmitter through a linear time invariant (LTI) channel with impulse response $h(t)$. The receiver attempts to recover the input data sequence $\{a_n\}$ from the measurable channel output $x(t)$ in which T is the symbol period. The channel output may be corrupted by channel noise $w(t)$. which is zero-mean stationary, white and complex Gaussian with variance σ^2, and is independent of the channel input a_n. In this paper, we will assume that the complex data and noise satisfy $E\{a_n^2\} = E\{w_t^2\} = 0$ and $E\{|a_n|^4\} - 2E^2\{|a_n|^2\} < 0$ (i.e. the kurtosis of a_n, $K(a_n) < 0$), as is often the case in QAM systems.

From (2.1), it is apparent that $x(t)$ is a continuous-time cyclostationary process with period T as long as the channel bandwidth is greater than the minimum bandwidth $1/(2T)$ [8]. Let the sampling interval be

$$\Delta = \frac{T}{p}, \quad p \in \mathbb{Z}^+. \qquad (2.2)$$

Then the sampled channel output becomes

$$x(k\Delta) = \sum_{n=0}^{\infty} a_n h(k\Delta - np\Delta - t_0). \qquad (2.3)$$

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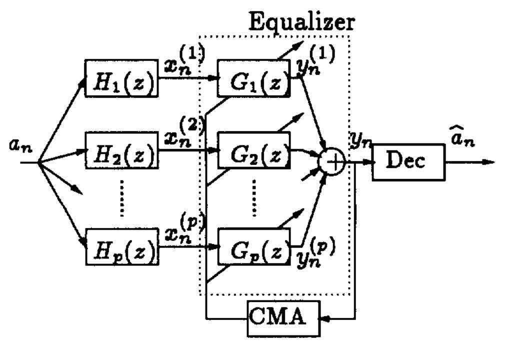

Figure 1: The vector structure of blind adaptive FSE.

For $p > 1$, the oversampled channel output $x(k\Delta)$ can be divided into p subsequences

$$x_k^{(i)} \triangleq x[(kp+i)\Delta] = x(kT + i\Delta), \quad i = 1, \ldots, p. \tag{2.4}$$

By defining subchannel impulse response as

$$h_k^{(i)} \triangleq h(pk\Delta + i\Delta - t_0) = h(kT + i\Delta - t_0), \tag{2.5}$$

the p subsequences can be written as

$$x_k^{(i)} = \sum_{n=0}^{+\infty} a_n h_{k-n}^{(i)} + w_k^{(i)}, \qquad i = 1, \ldots, p. \tag{2.6}$$

These p subsequences can be viewed as stationary outputs of p discrete FIR channels

$$H_i(z) = \sum_{k=0}^{K} h_k^{(i)} z^{-k},$$

with a common input sequence a_k.

The vector representation of the fractionally-spaced equalizer (FSE) is shown in Figure 1. One adjustable filter is provided for each subsequence $x_k^{(i)}$. Thus, the actual equalizer is a vector of filters

$$G_i(z) = \sum_{k=0}^{N} \theta_{i,k} z^{-k}, \qquad i = 1, \ldots p. \tag{2.7}$$

The p filter outputs $\{y_n^{(i)}\}$ are summed to form the equalizer output y_n that is stationary.

Define the FSE parameter vector as

$$\theta \triangleq [\theta_{1,0} \ \ldots \ \theta_{1,N} \ \ldots \theta_{p,0} \ \ldots \ \theta_{p,N}]'$$

To adaptively adjust θ without training sequence, CMA can be implemented to jointly update the p filters to minimize the Godard cost function

$$J_{cma}(\theta) = E\{(|y_n|^2 - r)^2\}, \quad r = \frac{E\{|a_n|^4\}}{E\{|a_n|^2\}}. \tag{2.8}$$

where $y_n = \sum_{i=1}^{p} y_n^{(i)}$. More recently, the Shalvi-Weinstein algorithm (SWA) [7] was developed to maximize

$$\left|K(y_n)\right| \ \text{subject to} \ E\{|y_n|^2|\} = E\{|a_n|^2\}, \tag{2.9}$$

where $K(y_n)$ is the kurtosis of y_n.

III Convergence of Godard FSE

Denote s_n as the impulse response of the combined channel-equalizer system, i.e.

$$s_n = \sum_{i=1}^{p}\sum_{k=0}^{N} \theta_{i,k} h_{n-k}^{(i)}, \qquad n = 0, 1, \ldots, N+K. \tag{3.1}$$

$$\mathrm{s} \triangleq [s_0 \quad s_1 \quad \ldots \quad s_{N+K}]'$$

Suppose that the channel and the equalizer are both bounded-input-bounded-output (BIBO) systems, then s_n is also the impulse response of a BIBO system. The equalizer output can be expressed as

$$y_n = \sum_{k=0}^{N+K} s_k a_{n-k} + \sum_{i=1}^{p}\sum_{k=0}^{N} \theta_{i,k} w_{n-k}^{(i)}. \tag{3.2}$$

From (2.8) and (3.2), Godard cost function can be written as

$$J_{cma}(\theta) = K(a_n)\sum_k |s_k|^4 + 2m_2^2 v^2(\theta) - 2m_4 v(\theta) + r^2 \tag{3.3}$$

where $m_2 = E\{|a_n|^2\}$, $m_4 = E\{|a_n|^4\}$, and

$$v(\theta) = \sum_k |s_k|^2 + \frac{\sigma^2}{m_2}\sum_{i=1}^{p}\sum_k |\theta_{i,k}|^2. \tag{3.4}$$

Similarly, Shalvi-Weinstein cost function can be expressed as

$$J_{sw}(\theta) \triangleq -|K(y_n)| = K(a_n)\sum_{k=0}^{N+K} |s_k|^4 \tag{3.5}$$

subject to $v(\theta) = 1$. Before discussing the convergence of CMA and SWA, some useful definitions and their properties need to be introduced.

The Attainable Set S_A is the set of all s *that the finite length equalizer θ can attain*

$$S_A \triangleq \{\mathrm{s}: \ s_n = \sum_{i=1}^{p}\sum_{k=0}^{N} \theta_{i,k} h_{n-k}^{(i)}, \theta_{i,k} \in C\}, \tag{3.6}$$

A Cone S_n is defined as

$$S_n \triangleq \{\mathrm{s}: \ |s_n| > |s_k| \ \textit{for all} \ k \neq n\} \tag{3.7}$$

Let $\{\mathrm{e_n}\}_0^{N+K}$ be the unit basis vectors in $\mathbb{R}^{N+K+1}$. The definitions of S_n and S_A yield the following result.

Lemma 3.1 *S_n has the following properties:*

1. $\mathbf{E}_n = \{\mathbf{s} = e^{j\phi}\mathbf{e}_n : \phi \in [0,\ 2\pi]\}$ *is the only set of minima for Godard and Shalvi-Weinstein cost functions in the cone S_n,*
2. *if* $\mathbf{s} \in S_A \bigcap S_n$*, then* $\alpha\mathbf{s} \in S_n$ *for any* $\alpha \in C$*, and*
3. *If* $\mathbf{s}^{(1)}$ *and* $\mathbf{s}^{(2)} \in S_n$*, then* $\alpha_1 \mathbf{s}^{(1)} + \alpha_2 \mathbf{s}^{(2)} \in S_n$ *for all* $\alpha_1,\ \alpha_2 \in C$ *such that* $\arg(\alpha_1) + \arg(s_n^{(1)}) = \arg(\alpha_2) + \arg(s_n^{(2)})$.

3.1 Local Minima of Godard and Shalvi-Weinstein FIR Equalizers

Theorem 3.1 *For the QAM communication system with FIR equalizers, there is a one-to-one correspondence between minimum sets of the Shalvi-Weinstein equalizer and those of the Godard equalizer.*

Proof: Based on the equation (3.3) and (3.5), we have that

$$J_{cma}(\theta) = J_{sw}(\theta) + 2m_2^2 v^2(\theta) - 2m_4 v(\theta) + r^2 \quad (3.8)$$

Let θ_0 be a stable minimum point of Shalvi-Weinstein equalizer. Then it is the minimum point of $J_{sw}(\theta)$ subject to $v(\theta) = 1$. Hence, $\alpha\theta_0$ ($\alpha \in \mathcal{R}^+$) is a minimum point of $J_{sw}(\theta)$ subject to $v(\theta) = \alpha^2$ with minimum value

$$J_{sw}(\alpha\theta_0) = \alpha^4 J_{sw}(\theta_0). \quad (3.9)$$

Thus, $\alpha_0\theta_0$ is a stable minimum of $J_{cma}(\theta)$, where

$$\alpha_0 = \left(\frac{m_4}{2m_2^2 + J_{sw}(\theta)}\right)^{1/2}. \quad (3.10)$$

Conversely, direct calculation yields that if θ_0 is a minimum point of $J_{cma}(\theta)$, then $\frac{1}{v^{1/2}(\theta_0)}\theta_0$ is a minimum of $J_{sw}(\theta)$ subject to $v(\theta) = 1$.

Hence, there is a one-to-one correspondence between the minimum points of Shalvi-Weinstein FIR equalizer and those of Godard equalizer.

□

3.2 Uniqueness of Minimum in A Cone

The CMA or SWA will be globally convergent with zero ISI if $\{\mathbf{E}_n\}_0^{N+K}$ are the only sets of minima. For arbitrary FIR equalizers, this is generally not true. The following theorem reveals the location of these minimum sets.

Theorem 3.2 *Let S_A be the attainable set of a finite length FSE equalizer in noiseless QAM communication systems. If $\mathbf{E}_n \subset S_A \bigcap S_n$, then $\mathbf{E}_n$ is the only minimum set in $S_A \bigcap S_n$ while there is no minimum points on its boundary $S_A \bigcap B_n$.*

Proof: From Theorem 3.1, we only need to prove the theorem for the SWA equalizer. Without loss of generality, we will prove the $n = 0$ case.

Since there is no channel noise, $v(\theta) = ||s||^2 = 1$, For any $\mathbf{s} \in S_A \bigcap S_0 \bigcap \Phi(1)$, by definition we have

$$|s_0| > |s_n| \text{ for all } n \neq 0. \quad (3.11)$$

In above expression,

$$\Phi(1) = \{\mathbf{s} : \ ||\mathbf{s}|| = 1\}. \quad (3.12)$$

Let

$$\epsilon_0 = e^{j\phi}\mathbf{e}_0, \quad (\phi = arg(s_0)) \quad (3.13)$$

Based on Lemma 3.1(3), for any $t \in [0, 1]$

$$\mathbf{s}(t) \triangleq \frac{\mathbf{s} + t(\epsilon_0 - \mathbf{s})}{||\mathbf{s} + t(\epsilon_0 - \mathbf{s})||} \in S_A \bigcap S_0^+ \bigcap \Phi(1). \quad (3.14)$$

Direct calculation yields

$$\left.\frac{d\ F_2(\mathbf{s}(t))}{d\ t}\right|_{t=0} = -4(2m_2^2 - m_4)|s_0| \sum (|s_0|^2 - |s_n|^2)|s_n|^2. \quad (3.15)$$

Since the necessary condition for s to be a minimum point is $\left.\frac{d\ F_2(\mathbf{s}(t))}{d\ t}\right|_{t=0} = 0$, which means

$$|s_n|^2 = \frac{1}{M}\sum_{i \in I_M} \delta(n - i), \qquad I_M = \text{set of M integers}. \quad (3.16)$$

Hence, ϵ_0 is a candidate stable minimum point inside $S_A \bigcap S_0$. The other candidates are on the boundary $S_A \bigcap B_0$ and are not stable since

$$\left.\frac{d^2\ F_2(\mathbf{s}(t))}{d\ t^2}\right|_{t=0} = -8(3m_2^2 - m_4)\frac{1}{M}(1 - \frac{1}{M}) < 0. \quad (3.17)$$

Since $\epsilon_0 \in S_A \bigcap S_n$, by Lemma 3.1 (2) $\mathbf{E}_0 \subset S_A \bigcap S_n$. Hence, $\mathbf{E}_0$ is the only minimum sets in $S_A \bigcap S_n$ for all elements in $\mathbf{E}_0$ having the same cost for the Shalvi-Weinstein cost function.

□

3.3 Global Convergence of CMA-FSE

In general, most channels can be approximated by an impulse response $h(t)$ with finite time support $[0, T_d)$. Thus the subchannels after sampling can be considered as FIR. The following condition is critical to the convergence result of CMA-FSE.

Length and Zero Condition: *The p discrete subchannels are such that*

$$\begin{aligned} &h_0^{(i)} \neq 0, \quad \textit{for some } 1 \leq i \leq p \\ &h_K^{(i)} \neq 0, \quad \textit{for some } 1 \leq i \leq p \\ &\{H_i(z)\}_0^p \quad \textit{have no common zeros.} \end{aligned} \quad (3.18)$$

As shown in [10], this condition is equivalent to the channel identifiability condition based on second cyclostationarity. It depends on the availability of excess bandwidth (spectral diversity) as well as the physical composition of the composite channel $h(t)$.

Our analysis of CMA-FSE is based on the noiseless assumption as in other studies of blind equalization algorithms [1] [2][4][6][7]. When modest amount of white channel noise is present, the stable equilibria of blind equalizers will be perturbed but remain in the neighborhood of the ideal equilibria of the noiseless channel.

Theorem 3.2 has shown that so long as the entire parameter space $\mathcal{C}^{N+K+1}$ for s is the attainable set for FSE, i.e.,

$$S_A = \mathcal{C}^{N+K+1}, \quad \text{or} \quad S_A \bigcap S_n = S_n,$$

then $\{\mathbf{E}_n\}_0^{N+K}$ are the only sets of minima under zero noise. By letting

$$\mathcal{H} \triangleq \begin{bmatrix} h_0^{(1)} & & & \cdots & h_0^{(p)} & & \\ \vdots & \ddots & & \vdots & \vdots & \ddots & \\ h_K^{(1)} & \ddots & h_0^{(1)} & \cdots & h_K^{(p)} & \ddots & h_0^{(p)} \\ & \ddots & \vdots & \vdots & & \ddots & \vdots \\ & & h_K^{(1)} & \cdots & & & h_K^{(p)} \end{bmatrix},$$

it is obvious that

$$\mathcal{H}\theta = \mathbf{s}. \tag{3.19}$$

Thus, the FSE-CMA will be globally convergent if the range of $\mathcal{H}$ is $\mathcal{C}^{N+K+1}$. Since $\mathcal{H}$ is an $(N+1)p \times (N+K+1)$ matrix, this would require

$$(N+1)p \geq (N+K+1) \quad \text{or} \quad (N+1)(p-1) \geq K.$$

In addition, $\mathcal{H}$ must have full row-rank, which is guaranteed by the Length and Zero condition [9]. To summarize the convergence result:

Theorem 3.3 *Suppose channel noise is zero. If the length of the equalizers are chosen such that $(N+1)(p-1) \geq K$, then the fractionally-spaced CMA (or SWA) equalizer is globally convergent if the p subchannels satisfy the Length and Zero condition.*

Since the total system parameter vector s has $N+K+1$ dimensions, all $(N+K+1)$ set of global minima are reachable by the vector equalizer. Thus the total system can have arbitrary delay between zero and $N+K$. Moreover, the lengths of the equalizer filters do not need to be excessively long even when subchannels have different zeros on or near the unit circle.

To set a simple initialization parameter vector for the equalizer, simply choose the initial FSE setting as

$$\begin{aligned} G_1(z) &= 1, \\ G_i(z) &= 0, \qquad i = 2, \ldots, p, \end{aligned} \tag{3.20}$$

then

$$s_n = h_n^{(0)}. \tag{3.21}$$

As long as the first subchannel has a peak at n^* in its impulse response $\{h_n^{(1)}\}$, our initialization will be acceptable and s_n will converge in cone S_n to

$$|s_n^{(op)}| = \delta[n - n^*]. \tag{3.22}$$

Hence the initial setting need not satisfy the center-tap criterion [6] as in TSE.

IV Simulation Results

Computer simulation examples are presented in this section to demonstrate the performance of the Godard vector equalizer. The CMA-FSE is tested on a three-ray multipath channel as in [10]. The channel is a raised cosine pulse in a three-ray multipath environment. The channel input $\{a_n\}$ is an independent, identically distributed 4-level PAM sequence and the channel noise is white Gaussian. The sampling frequency is chosen to be four times the baud rate, i.e $p = 4$ and $\Delta = T/4$.

The adaptive algorithm is used to update equalizer parameters

$$\theta_{i,k}^{(n+1)} = \theta_{i,k}^{(n)} - \mu y_{n-k}^{(i)} y_n (|y_n|^2 - r_2). \tag{4.1}$$

All initial tap coefficients are set to zero except for the lead tap coefficient of the first filter which is set to unity. In the FSE, there are 12 taps in each adjustable filter.

In Figures 2 and 3, we compare the performance of the CMA-FSE with the conventional CMA-TSE in terms of the probability of symbol error (PSE) and normalized ISI of the equalized system defined by

$$\text{ISI} = \frac{\sum_n |s_n|^2 - \max_n |s_n|^2}{\max_n |s_n|^2}. \tag{4.2}$$

The probability of symbol error (PSE) is averaged over 5,000 independent runs. To make fair comparisons, the original CMA-TSE was given the same number of tap coefficients (48) as the CMA vector equalizer.

In these figures, the dot-dashed lines are averaged ISI and PSE of CMA-TSE applied to each of four subchannels. From the simulation results, it is evident that the CMA-FSE generates much better performance than that of the conventional CMA-TSE scheme averaged over four different sampling phases.

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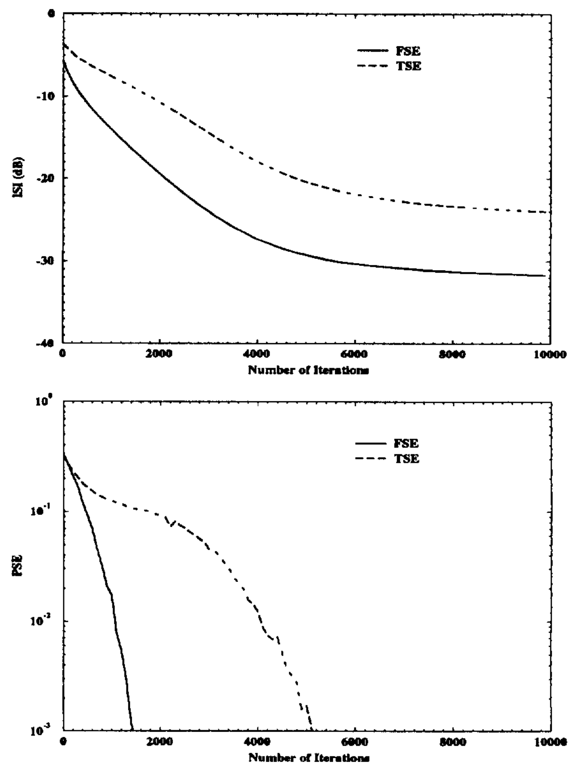

Figure 2: ISI and PSE of equalized system when SNR=20db.

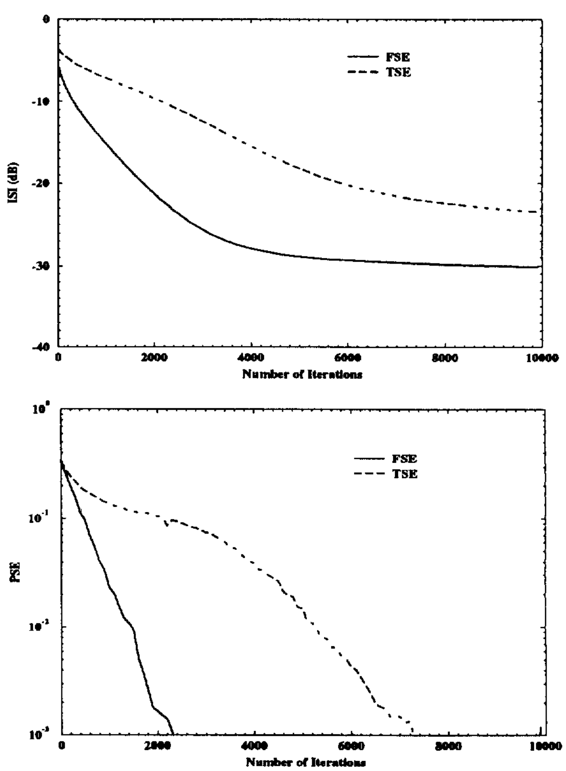

Figure 3: ISI and PSE of equalized system when SNR=10db.