A Modified QR Adaptive Filtering Algorithm - Elementary Approach -

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Abstract

This paper presents a modified form of the conventional QR algorithm for adaptive filtering. By exploiting the displacement structure of the correlation matrix of the data, the paper suggests a fast method of constructing a unitary rotation that can be used to propagate an estimate in the QR algorithm. As a result, the paper provides a numerically better version of the so-called "Hybrid QR/LLS" adaptive filtering algorithm discovered by Regalia and Bellanger; in their original version, a so-called likelihood variable computed by using variance-normalized backwards a posteriori *prediction errors may not be positive semi-definite because of round-off errors. In our approach, this variable is guaranteed to be positive semi-definite.*

I Introduction

Lattice adaptive filtering algorithms can be divided into two classes, depending on whether they use a feedforward (or transmission) lattice filter or a feedback (or scattering) lattice filter. Most algorithms in the literature are based on a feedforward lattice filter, *e.g.*, the Lattice Least-Squares algorithm [1, 2], the Givens-based least-squares algorithm [3], and the fast QR algorithm [4]. Recently, Regalia and Bellanger [5] suggested a new adaptive filtering algorithm, called "Hybrid QR/LLS", which is based on a feedback lattice filter. While Regalia and Bellanger's algorithm has several attractive numerical features [6], it can (as noted in [7]) have problems with the updating method used for a critical so-called likelihood variable, which may lose positivity (or positive definiteness in the multi-channel case) because of round-off errors.

In this paper, we shall show how to avoid this critical problem by fully exploiting the QR array (or square-root) point of view to derive adaptive filtering algorithms.

The structure of the paper is as follows: Section 2 briefly describes the multi-channel least-squares adaptive filtering problem and addresses the square-root (array) algorithm called the QR algorithm and adds a useful new row to the arrays of the square-root algorithm, which will be called a modified QR algorithm in our paper. Section 3 investigates an elementary version of the modified QR algorithm. Section 4 exploits the displacement properties of the correlation matrix of the data so as to find elementary operators used in array-updates. Section 5 presents a new hybrid QR/LLS algorithm. The appendix summarizes Regalia and Bellanger's hybrid QR/LLS algorithm and explains the difference between their algorithm and our algorithm.

II Adaptive Filtering

The basic **transversal** adaptive filtering problem is the following: Given $N+1$ observed data M-component matrices $\mathbf{u}_{M,i} = [\ u_i \quad u_{i-1} \quad \cdots \quad u_{i-M+1}\ ]$ and a desired sequence, $d_i \in \mathcal{C}^{p\times 1}$, where $i = 0 : N$ and $u_j = 0$ for $j < 0$, let us find an $M \times 1$ weight vector $\mathbf{w}_M$ that will minimize the exponentially weighted sum of squares

$$\begin{aligned}\mathcal{J}_N &= \sum_{i=0}^{N} \lambda^{N-i}(d_i - \mathbf{u}_{M,i}\mathbf{w}_M)^* W^{-1}(d_i - \mathbf{u}_{M,i}\mathbf{w}_M) \\ &+ \lambda^{N+1}\mathbf{w}_M^* \Phi_{M,-1}\mathbf{w}_M,\end{aligned}$$

where $0 < \lambda \leq 1$ and the $W \in \mathcal{C}^{p\times p}$ is a given "channel weighting" matrix. Without loss of generality, assume that $W \equiv I$. (For any W, defining $\bar{d}_i = W^{-1/2}d_i$ and $\bar{u}_i = W^{-1/2}u_i$ has the same effect as assuming $W \equiv I$.) The now well-known (see, *e.g.*, [8] for a single channel case) solution to this problem is the following: Let

$$\Phi_{M,N} = \sum_{i=0}^{N} \lambda^{N-i}\mathbf{u}_{M,i}^*\mathbf{u}_{M,i} + \lambda^{N+1}\Phi_{M,-1}, \quad (1)$$

$$\Omega_{M,N} = \sum_{i=0}^{N} \lambda^{N-i}\mathbf{u}_{M,i}^* d_i. \quad (2)$$

Then the minimizing weight vector $\mathbf{w}_{M,N}$ is the solution to the so-called **normal** equations $\Phi_{M,N}\mathbf{w}_{M,N} = \Omega_{M,N}$.

The QR Algorithm

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From Eq. (1) and Eq. (2), the following relations hold:

$$\Phi_{M,N} = \mathbf{u}^*_{M,N}\mathbf{u}_{M,N} + \lambda\Phi_{M,N-1},$$
$$\Omega_{M,N} = \mathbf{u}^*_{M,N}d_N + \lambda\Omega_{M,N-1}.$$

Since, using the normal equation, we can represent the above second equation with

$$\Phi^{1/2}_{M,N}\bar{\mathbf{w}}_{M,N} = \mathbf{u}^*_{M,N}d_N + \lambda\Phi^{1/2}_{M,N-1}\bar{\mathbf{w}}_{M,N-1},$$

where $\bar{\mathbf{w}}_{M,N} \stackrel{\triangle}{=} \Phi^{*/2}_{M,N}\mathbf{w}_{M,N}$, we can rewrite the above equations in a square-root form by using a unitary operator Θ^M_N to zero out the (1,1) entry of the pre-array. The resulting post-array is as shown below, where $(*)$ denotes a currently unidentified entry. This array form is known as the QR algorithm in least-squares adaptive filtering.

Algorithm II.1 The QR Algorithm
Given $\Phi_{M,-1} \geq 0$, $\Phi^{1/2}_{M,i}$ and $\bar{\mathbf{w}}_{M,i}$ can be propagated via

$$\left[\begin{array}{cc} \mathbf{u}^*_{M,i} & \sqrt{\lambda}\Phi^{1/2}_{M,i-1} \\ \hline d^*_i & \sqrt{\lambda}\bar{\mathbf{w}}^*_{M,i-1} \end{array}\right]\Theta^M_i = \left[\begin{array}{cc} 0 & \Phi^{1/2}_{M,i} \\ \hline (*) & \bar{\mathbf{w}}^*_{M,i} \end{array}\right],$$

where Θ^M_i is any unitary operator that zeros out the (1,1) entry of the pre-array. The estimate $\mathbf{w}_{M,i}$ can be found via a backsubstitution in the equation: $(\Phi^{*/2}_{M,i})\mathbf{w}_{M,i} = \bar{\mathbf{w}}_{M,i}$. ∎

Now, we shall add one row to the pre-array in the QR algorithm, which will allow us to more easily see the meaning of the entry $(*)$ in the post-array:

$$\left[\begin{array}{cc} \mathbf{u}^*_{M,i} & \sqrt{\lambda}\Phi^{1/2}_{M,i-1} \\ \hline I & 0 \\ d^*_i & \sqrt{\lambda}\bar{\mathbf{w}}^*_{M,i-1} \end{array}\right]\Theta^M_i = \left[\begin{array}{cc} 0 & \Phi^{1/2}_{M,i} \\ \hline X & Y \\ z^* & \bar{\mathbf{w}}^*_{M,i} \end{array}\right],$$

A cross-product between the first row and the second row provides $\mathbf{u}_{M,i} = Y\Phi^{*/2}_{M,i}$, and an inner-product of the second row with itself provides $I = XX^* + YY^*$. In addition, a cross-product between the second row and the third row gives $Xz = d_i - \mathbf{u}_{M,i}\mathbf{w}_{M,i} (\stackrel{\triangle}{=} \epsilon_{M,i})$, which shows that Xz is a unnormalized *a posteriori* estimation error.

Let us assume that u_0 is not zero; then, in general, we can choose Y as $\mathbf{u}_{M,i}(\Phi^{*/2}_{M,i})^\dagger$ and $XX^* = I - \mathbf{u}_{M,i}\Phi^\dagger_{M,i}\mathbf{u}^*_{M,i}$, where $A^\dagger$ denotes the pseudo inverse of A. Let us denote XX^* as $\Gamma_{M,i}$, which is known as a likelihood function in the literature. Since X is, in fact, a part of the unitary rotation $\Theta_{M,i}$, X is called an angle, and thus z is called an angle-normalized *a posteriori* estimation error.

Therefore, we have the following modified version of the QR algorithm.

Algorithm II.2 A Modified QR Algorithm (I)

$$\left[\begin{array}{cc} \mathbf{u}^*_{M,i} & \sqrt{\lambda}\Phi^{1/2}_{M,i-1} \\ \hline I & 0 \\ d^*_i & \sqrt{\lambda}\bar{\mathbf{w}}^*_{M,i-1} \end{array}\right]\Theta^M_i = \left[\begin{array}{cc} 0 & \Phi^{1/2}_{M,i} \\ \hline \Gamma^{1/2}_{M,i} & \mathbf{u}_{M,i}(\Phi^{*/2}_{M,i})^\dagger \\ \epsilon^*_{M,i}\Gamma^{-*/2}_{M,i} & \bar{\mathbf{w}}^*_{M,i} \end{array}\right].$$

III Parameter Identification

From the definition of the $\mathbf{u}_{M,i-1}$, it is clear that the $\Phi_{M,i-1}$ satisfies the following displacement property:

$$\Phi_{M,i-1} = \left[\begin{array}{cc} \Phi_{M-1,i-1} & (*) \\ (*) & (*) \end{array}\right].$$

Therefore, we can represent the lower triangular square-root factor $\Phi^{1/2}_{M,i-1}$ of the $\Phi_{M,i-1}$ with the lower triangular square-root factor $\Phi^{1/2}_{M-1,i-1}$ of the $\Phi_{M-1,i-1}$ such that

$$\Phi^{1/2}_{M,i-1} = \left[\begin{array}{cc} \Phi^{1/2}_{M-1,i-1} & 0 \\ x & y \end{array}\right], \tag{3}$$

where x and y are values we shall first identify in this section. Let us represent x and y with $\bar{\mathbf{w}}^*_{b,M-1,i-1}$ and $E^{1/2}_{b,M-1,i-1}$, respectyively. In this case, we can rewrite the pre-array in Algorithm II.2 into the following:

$$\left[\begin{array}{cc} \mathbf{u}^*_{M,i} & \sqrt{\lambda}\Phi^{1/2}_{M,i-1} \\ \hline I & 0 \\ d^*_i & \sqrt{\lambda}\bar{\mathbf{w}}^*_{M,i-1} \end{array}\right] \equiv$$
$$\left[\begin{array}{ccc} \mathbf{u}^*_{M-1,i} & \sqrt{\lambda}\Phi^{1/2}_{M-1,i-1} & 0 \\ u^*_{i-M+1} & \sqrt{\lambda}\bar{\mathbf{w}}^*_{b,M-1,i-1} & \sqrt{\lambda}E^{1/2}_{b,M-1,i-1} \\ \hline I & 0 & 0 \\ d^*_i & \sqrt{\lambda}\bar{\mathbf{w}}^*_{M,i-1(1:M-1)} & \sqrt{\lambda}\bar{\mathbf{w}}^*_{M,i-1(M)} \end{array}\right],$$

where $\bar{\mathbf{w}}^*_{M,i-1(1:M-1)}$ and $\bar{\mathbf{w}}^*_{M,i-1(M)}$ are defined as

$$\left[\begin{array}{cc} \bar{\mathbf{w}}^*_{M,i-1(1:M-1)} & \bar{\mathbf{w}}^*_{M,i-1(M)} \end{array}\right] \stackrel{\triangle}{=} \bar{\mathbf{w}}^*_{M,i-1}.$$

Let us apply unitary operators $\Theta^M_{m,i}$ from m=1 to M-1 to the above pre-array; then we can obtain the following intermediate-array (which can be obtained via Algorithm II.2 but $\bar{\epsilon}^*_{b,M-1,i}$, $\bar{\mathbf{w}}^*_{b,M-1,i}$, and $\bar{\epsilon}^*_{M,i(M-1)}$):

$$\left[\begin{array}{ccc} 0 & \Phi^{1/2}_{M-1,i} & 0 \\ \bar{\epsilon}^*_{b,M-1,i} & \bar{\mathbf{w}}^*_{b,M-1,i} & \sqrt{\lambda}E^{1/2}_{b,M-1,i-1} \\ \hline \Gamma^{1/2}_{M-1,i} & \mathbf{u}_{M-1,i}(\Phi^{*/2}_{M-1,i})^\dagger & 0 \\ \bar{\epsilon}^*_{M,i(M-1)} & \bar{\mathbf{w}}^*_{M,i(1:M-1)} & \sqrt{\lambda}\bar{\mathbf{w}}^*_{M,i-1(M)} \end{array}\right], \tag{4}$$

where, from comparisons between the pre-array and the intermediate-array, $\bar{\epsilon}_{b,M-1,i}$, $\bar{\mathbf{w}}_{b,M-1,i}$, and $\bar{\epsilon}^*_{M,i(M-1)}$ can be defined as

$$\Gamma^{-1/2}_{M-1,i}\{u_{i-M+1} - \mathbf{u}_{M-1,i}(\Phi^{*/2}_{M-1,i})^\dagger\bar{\mathbf{w}}_{b,M-1,i}\}, \tag{5}$$
$$\Phi^{-1/2}_{M-1,i}\{\lambda(\Phi^{1/2}_{M-1,i-1})(\bar{\mathbf{w}}_{b,M-1,i-1}) + \mathbf{u}^*_{M-1,i}u_{i-M+1}\},$$
$$\Gamma^{-1/2}_{M-1,i}\{d_i - \mathbf{u}_{M-1,i}(\Phi^{*/2}_{M-1,i})^\dagger\bar{\mathbf{w}}_{M,i(1:M-1)}\}.$$

A comparison of the above second equation and Eq. (3) allows us to claim that that $\mathbf{w}_{b,M-1,i}$, where $\bar{\mathbf{w}}_{b,M-1,i} \equiv \Phi^{*/2}_{M-1,i}\mathbf{w}_{b,M-1,i}$ and $\epsilon_{b,M-1,i} \equiv u_{i-M+1} -$

$\mathbf{u}_{M-1,i}\mathbf{w}_{b,M-1,i}$, is the optimal solution for minimizing the exponentially weighted sum of squares

$$\mathcal{J}_{b,M-1,i} = \sum_{j=0}^{i} \lambda^{i-j}\Gamma_{M-1,i}\epsilon^*_{b,M-1,j}\epsilon_{b,M-1,j} + \lambda^{i+1}\mathbf{w}^*_{b,M-1}\Phi_{M-1,-1}\mathbf{w}_{b,M-1}. \quad (6)$$

In this case, the $\bar{\epsilon}_{b,M-1,i}$ in Eq. (5) can be called **an angle-normalized backwards** *a posteriori* **prediction error**, because the $(\Gamma^{1/2}_{M-1,i}\bar{\epsilon}_{b,M-1,i})$ is a unnormalized backwards *a posteriori* prediction error and the $\Gamma^{1/2}_{M-1,i}$ is a part of the rotation $\Theta^M_{1,i}\cdots\Theta^M_{M-1,i}$, *i.e.*, an angle.

Now, let us apply the last elementary rotation $\Theta^M_{M,i}$ to the above intermediate-array in Eq. (4) in order to zero out the (2,1) entry of the intermediate-array by using the (2,3) entry of the intermediate-array. Then, we can obtain the post-array in Algorithm II.2.

$$\left[\begin{array}{ccc} 0 & \Phi^{1/2}_{M-1,i} & 0 \\ 0 & \bar{\mathbf{w}}^*_{b,M-1,i} & E^{1/2}_{b,M-1,i} \\ \hline \Gamma^{1/2}_{M,i} & \mathbf{u}_{M-1,i}(\Phi^{*/2}_{M-1,i})^{\dagger} & \bar{\beta}_{M-1,i} \\ \bar{\epsilon}^*_{M,i(M)} & \bar{\mathbf{w}}^*_{M,i(1:M-1)} & \bar{\mathbf{w}}^*_{M,i(M)} \end{array}\right], \quad (7)$$

where, from comparisons of the intermediate-array and the post-array, $E^{1/2}_{b,M-1,i}$ and $\bar{\beta}_{M-1,i}$ can be represented by

$$\{\bar{\epsilon}^*_{b,M-1,i}\bar{\epsilon}_{b,M-1,i} + \lambda E^{1/2}_{b,M-1,i-1}E^{*/2}_{b,M-1,i-1}\}^{1/2}, \quad (8)$$

$$\{\Gamma^{1/2}_{M-1,i}\bar{\epsilon}_{b,M-1,i}\}E^{-*/2}_{b,M-1,i}. \quad (9)$$

Since $E_{b,M-1,i}$ in Eq. (8) can rewritten with $\sum_{j=0}^{i}\lambda^{i-j}\bar{\epsilon}^*_{b,M-1,j}\bar{\epsilon}_{b,M-1,j} + \lambda^{i+1}E_{b,M-1,-1}$, it is called a backwards prediction error residual or a backwards prediction error variance and, therefore, $\bar{\beta}_{M-1,i}$ in Eq. (9) can be called a **variance-normalized backwards** *a posteriori* **prediction error**.

It is interesting to note that we can find the following displacement structure of the variance-normalized backwards *a posteriori* prediction errors $\mathbf{u}_{M,i}(\Phi^{*/2}_{M,i})^{\dagger}(\triangleq \bar{\mathbf{b}}_{M,i})$: $\left[\ \bar{\mathbf{b}}_{M-1,i} \quad \bar{\beta}_{M-1,i}\ \right] = \left[\ \bar{\beta}_{0,i} \ \cdots \ \bar{\beta}_{M-1,i}\ \right]$.

Without loss of generality, we can replace M with $m < M$ in Eq. (4), and thus obtain the following elementary expression of the modified QR algorithm:

Algorithm III.1 A Modified QR Algorithm – Elementary Expression *For $m = 1$ to M, repeat the following procedures:*

$$\left[\begin{array}{cc} \bar{\epsilon}^*_{b,m-1,i} & \sqrt{\lambda}E^{1/2}_{b,m-1,i-1} \\ \hline \Gamma^{1/2}_{m-1,i} & 0 \\ \bar{\epsilon}^*_{M,i(m-1)} & \sqrt{\lambda}\bar{\mathbf{w}}^*_{M,i-1(m)} \end{array}\right]\Theta_{b,m,i} = \left[\begin{array}{cc} 0 & E^{1/2}_{b,m-1,i} \\ \hline \Gamma^{1/2}_{m,i} & \bar{\beta}_{m-1,i} \\ \bar{\epsilon}^*_{M,i(m)} & \bar{\mathbf{w}}^*_{M,i(m)} \end{array}\right],$$

where $\Theta_{b,m,i}$ is a unitary operator that zeros out (1,1) entry of the pre-array, $\bar{\epsilon}_{M,i(0)} = d_i$, and $\bar{\epsilon}_{M,i(M)} = \bar{\epsilon}_{M,i}$.

IV Rotational Basis Rows

To propagate the last row of the pre-array of the modified QR adaptive filtering algorithm, one should find the rotation Θ^M_i. One can find this rotation by using either of the following row vectors:

Array 1: $\left[\ \mathbf{u}^*_{M,i} \quad \sqrt{\lambda}\Phi^{1/2}_{M,i-1}\ \right]$, **Array 2:** $\left[\ \Gamma^{1/2}_{M,i} \quad \bar{\mathbf{b}}_{M,i}\ \right]$,

which we shall, henceforth, call *rotational basis rows.*

It is interesting to note that, by using the structure of the correlation matrix $\Phi_{M,i}$ under some constraint about the initial $\Phi_{M,-1}$, we can improve computational speed for constructing the rotational basis rows, given information at time (i-1). In this section, let us consider which constraint about the initial $\Phi_{M,-1}$ one should assume for fast computation.

First, let us temporarily assume that $\Phi_{M,-1} = 0$; then the $\Phi_{M,i}$ satisfies two displacement properties:

$$\Phi_{M,i} = \left[\begin{array}{cc} \Phi_{M-1,i} & (*) \\ (*) & (*) \end{array}\right] = \left[\begin{array}{cc} (*) & (*) \\ (*) & \Phi_{M-1,i-1} \end{array}\right].$$

However, in general, $\Phi_{M,-1} \neq 0$ and the second displacement property is destroyed. Now we shall put one constraint on the initial $\Phi_{M,-1}$, which maintains the two displacement properties of the $\Phi_{M,i}$ consistent with the shift property of the data stream $\mathbf{u}_{M,i}$.

Assumption: *We shall henceforth assume*

$$\Phi_{M,-1} = \left[\begin{array}{cc} (*) & (*) \\ (*) & \lambda^{-1}\Phi_{M-1,-1} \end{array}\right]. \quad (10)$$

For the sake of simplicity, we shall additionally assume that $\Phi_{M,-1} = \xi^2\text{diag}\{I, \lambda^{-1}I, \cdots, \lambda^{-(M-1)}I\}$. To recursively update the rotatinal basis rows, we shall first show a fast way to find the $\Phi_{M,i}$ from the $\Phi_{M,i-1}$, which we can perform through the following two procedures:

P1. **Downdating Procedure:**

$$\Phi_{M,i-1} = \left[\begin{array}{cc} \Phi_{M-1,i-1} & (*) \\ (*) & (*) \end{array}\right] \Longrightarrow \Phi_{M-1,i-1}.$$

P2. **Updating Procedure:**

$$\Phi_{M-1,i-1} \Longrightarrow \Phi_{M,i} = \left[\begin{array}{cc} (*) & (*) \\ (*) & \Phi_{M-1,i-1} \end{array}\right].$$

As far as the square-root factor of the $\Phi_{M,i}$ is concerned, the *lower triangular* square-root factor is natural for the downdating procedure and the *block upper triangular* square-root factor is natural for the up-dating procedure, where we represent the latter with $\tilde{\Phi}^{1/2}_{M,i}$:

$$\tilde{\Phi}^{1/2}_{M,i} = \left[\begin{array}{cc} x & y \\ 0 & \Phi^{1/2}_{M-1,i-1} \end{array}\right], \quad (11)$$

where $\Phi^{1/2}_{M-1,i-1}$ is assumed to be still lower triangular. Let us represent x and y with $E^{1/2}_{f,M-1,i}$ and $\bar{\mathbf{w}}^*_{f,M-1,i-1}$.

Since this procedure is closely related to the upper triangular QR algorithm, we shall write the resulting elementary expression without detailed proofs.

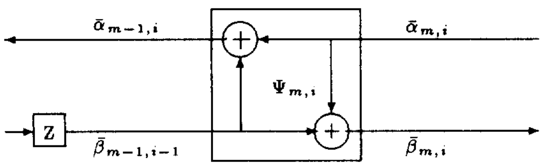

Figure 1: A Lattice Filter

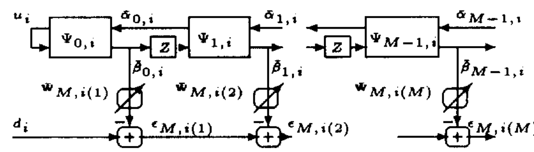

Figure 2: Feedback Lattice

Algorithm IV.1 (Elementary Upper Triangular Modified QR Algorithm) *For $m = 1$ to M, repeat the following procedures:*

$$\begin{bmatrix} \bar{\epsilon}^*_{f,m-1,i} & \lambda^{1/2}E^{1/2}_{f,m-1,i-1} \\ \Gamma^{1/2}_{m-1,i-1} & 0 \\ \bar{\epsilon}^*_{M,i+M-m(m-1)} & \lambda^{1/2}\tilde{\mathbf{w}}^*_{M,i-1+M-m(m)} \end{bmatrix} \Theta_{f,m,i} = \begin{bmatrix} 0 & E^{1/2}_{f,m-1,i} \\ \Gamma^{1/2}_{m,i} & \bar{\alpha}_{m-1,i} \\ \bar{\epsilon}^*_{M,i+M-m(m)} & \tilde{\mathbf{w}}^*_{M,i+M-m(m)} \end{bmatrix},$$

where $\bar{\epsilon}_{f,m-1,i}$ and $\bar{\alpha}_{m-1,i}$ can be interpreted as an angle-normalized forwards a posteriori *prediction error and a variance-normalized forwards* a posteriori *prediction error, respectively, and $\bar{\epsilon}_{0,i(0)} = d_i$, $\bar{\epsilon}_{M,i(M)} = \bar{\epsilon}_{M,i}$.*

In addition, we should transform the block upper triangular $\tilde{\Phi}^{1/2}_{M,i}$ to the lower triangular $\Phi^{1/2}_{M,i}$. Since $\tilde{\Phi}^{1/2}_{M,i}\tilde{\Phi}^{*/2}_{M,i} = \Phi^{1/2}_{M,i}\Phi^{*/2}_{M,i} = \Phi_{M,i}$, it is clear that there exists a unitary operator Ψ_i^{M-1} satisfying below.

P3. **Lower Triangularization:**

$$\begin{bmatrix} E^{1/2}_{f,M-1,i} & \tilde{\mathbf{w}}^*_{f,M-1,i-1} \\ 0 & \Phi^{1/2}_{M-1,i-1} \end{bmatrix} \Psi_i^{M-1} = \begin{bmatrix} \Phi^{1/2}_{M-1,i} & 0 \\ \tilde{\mathbf{w}}^*_{b,M-1,i} & E^{1/2}_{b,M-1,i} \end{bmatrix}.$$

We can also obtain the following elementary expression for this lower triangularization:

$$\begin{bmatrix} E^{1/2}_{f,m,i} & \tilde{\mathbf{w}}^*_{f,M-1,i-1(m)} \\ \bar{\alpha}_{m,i} & \bar{\beta}_{m-1,i-1} \\ 0 & E^{1/2}_{b,m-1,i-1} \end{bmatrix} \Psi_{m,i} = \begin{bmatrix} E^{1/2}_{f,m-1,i} & 0 \\ \bar{\alpha}_{m-1,i} & \bar{\beta}_{m,i} \\ \tilde{\mathbf{w}}^*_{b,m,i(m)} & E^{1/2}_{b,m,i} \end{bmatrix},$$

where $\Psi_{m,i}$ is a unitary operator that zeros out the (1,2) entry of the pre-array and $\tilde{\mathbf{w}}^*_{b,m,i(m)}$ is the first entry of $\mathbf{w}^*_{b,m,i}\tilde{\Phi}^{1/2}_{m,i}$.

It is very interesting to note that a corresponding filter structure to the above expression is a (variance-)normalized lattice filter – see Figure 1. Therefore, the third procedure described in the last section is equivalent to the Lattice Least-Squares (LLS) algorithm [1, 2].

V Hybrid QR/LLS Algorithm

Since the unnormalized *a posteriori* estimation errors can be represented as

$$\epsilon_{M,i} \triangleq d_i - \mathbf{u}_{M,i}\mathbf{w}_{M,i} = d_i - \sum_{m=1}^{M} \bar{\beta}_{m-1,i}\bar{\mathbf{w}}_{M,i(m)},$$

we can use a lattice filter instead of a transversal filter if we want to minimize $\mathcal{J}_{M,i}$ which is related to $\{\epsilon_{M,i}\}$.

Because Regalia and Bellanger [5] showed the duality between the fast QR algorithm [4] and the QR-decomposition-based Lattice Least-Squares algorithm, which is related to a method of employing Array 1 as a rotational basis row, we shall consider a method of employing Array 2 as a rotational basis row. If we employ

$$\begin{bmatrix} \Gamma^{1/2}_{M,i} & \bar{\mathbf{b}}_{M,i} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \Gamma^{1/2}_{m,i} & \bar{\beta}_{m-1,i} \end{bmatrix}$$

in order to find the $[\Theta_i^M]^{-1}$ in Algorithm II.2, we need to perform the three procedures mentioned in the previous section. Here, we note that the last procedure is related to the Lattice Least-Squares algorithm. Therefore, these procedures are called the hybrid QR/LLS algorithm [5].

- Downdating
  $\begin{bmatrix} \Gamma^{1/2}_{M,i-1} & \bar{\mathbf{b}}_{M-1,i-1} & \bar{\beta}_{M-1,i-1} \end{bmatrix} \Theta^{-1}_{M,M,i-1} =$
  $\begin{bmatrix} \Gamma^{1/2}_{M-1,i-1} & \bar{\mathbf{b}}_{M-1,i-1} & 0 \end{bmatrix}.$
- Updating
  $\begin{bmatrix} \Gamma^{1/2}_{M-1,i-1} & 0 & \bar{\mathbf{b}}_{M-1,i-1} \end{bmatrix} \begin{bmatrix} \Theta_{f,M,i} & 0 \\ 0 & I_{M-1} \end{bmatrix} =$
  $\begin{bmatrix} \Gamma^{1/2}_{M,i} & \bar{\alpha}_{M-1,i} & \bar{\mathbf{b}}_{M-1,i-1} \end{bmatrix}.$
- Lower-Triangularization
  $\begin{bmatrix} \Gamma^{1/2}_{M,i} & \bar{\alpha}_{M-1,i} & \bar{\mathbf{b}}_{M-1,i-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \Psi_i^{M-1} \end{bmatrix} =$
  $\begin{bmatrix} \Gamma^{1/2}_{M,i} & \bar{\mathbf{b}}_{M,i} \end{bmatrix}.$

If we represent these rotational basis row updates using elementary rotations, we can construct the following algorithm:

Algorithm V.1 (The Hybrid QR/LLS algorithm based on Variance-Normalized Backwards *A Posteriori* Prediction Errors)
Assume that $\Phi_{M,-1} = \xi^2\mathrm{diag}(I, \lambda^{-1}I, \cdots, \lambda^{1-M}I)$,

where $\xi = 0$ for exact start or $0 < \xi \ll 1$ for soft start. In order to find the weights $\bar{\mathbf{w}}_{M,i(m)}$ and the lattice filter sections $\Psi_{m-1,i}$ for $m = 1 : M$ and $i = 0 : N$, run the following procedures:

- **Initialization:** *Put* $\Gamma^{1/2}_{M-1,-1} = I$, $E^{1/2}_{f,M-1,-1} = \xi I$, $\bar{\epsilon}_{f,M-1,0(M-1)} = u_0$. *For m=1 to M*, $\bar{\beta}_{m-1,-1} = 0$, $\bar{\mathbf{w}}_{f,M-1,-1(m)} = \bar{\mathbf{w}}_{M,-1(m)} = 0$.

For $i = 0$ to n, repeat the following procedures:

- **Rotational Basis Updates and Lattice Filter Section Compositions:**

$$\begin{bmatrix} 0 & E^{1/2}_{f,M-1,i} \\ \Gamma^{1/2}_{M,i} & \bar{\alpha}_{M-1,i} \end{bmatrix} := \begin{bmatrix} \bar{\epsilon}^*_{f,M-1,i(M-1)} & \sqrt{\lambda}E^{1/2}_{f,M-1,i-1} \\ \Gamma^{1/2}_{M-1,i-1} & 0 \end{bmatrix} \Theta_{f,M,i}.$$

For m=M-1 : 1, construct the filter sections $\Psi_{m,i}$:

$$\begin{bmatrix} E^{1/2}_{f,m-1,i} & 0 \\ \bar{\alpha}_{m-1,i} & \bar{\beta}_{m,i} \end{bmatrix} := \begin{bmatrix} E^{1/2}_{f,m,i} & \bar{\mathbf{w}}^*_{f,M-1,i-1(m)} \\ \bar{\alpha}_{m,i} & \bar{\beta}_{m-1,i-1} \end{bmatrix} \Psi_{m,i}.$$

- **QR Array Updates:** *Put* $\bar{\beta}_{0,i} = \bar{\alpha}_{0,i}$. *For $m = M$ to 1, run the procedures to compose the rotations* $[\Theta^{-1}_{b,m,i}]$ *and the* $\Gamma^{1/2}_{M-1,i}$:

$$\begin{bmatrix} \Gamma^{1/2}_{m-1,i} & 0 \end{bmatrix} := \begin{bmatrix} \Gamma^{1/2}_{m,i} & \bar{\beta}_{m-1,i} \end{bmatrix} [\Theta^{-1}_{b,m,i}.]$$

Put $\bar{\epsilon}_{f,M-1,i+1(0)} = u_{i+1}$, $\bar{\epsilon}_{M,i(0)} = d_i$. *For m = 1 to M-1, run the procedures to propagate the estimates* $\bar{\mathbf{w}}_{M,i(m)}$ *and* $\bar{\mathbf{w}}_{f,M-1,i(m)}$:

$$\begin{bmatrix} \bar{\epsilon}^*_{M,i(m)} & \bar{\mathbf{w}}^*_{M,i(m)} \\ \bar{\epsilon}^*_{f,M-1,i+1(m)} & \bar{\mathbf{w}}^*_{f,M-1,i(m)} \end{bmatrix} := \begin{bmatrix} \bar{\epsilon}^*_{M,i(m-1)} & \sqrt{\lambda}\bar{\mathbf{w}}^*_{M,i-1(m)} \\ \bar{\epsilon}^*_{f,M-1,i+1(m-1)} & \sqrt{\lambda}\bar{\mathbf{w}}^*_{f,M-1,i-1(m)} \end{bmatrix} [\Theta^{-1}_{b,m,i}]^*,$$

and perform the last rotation for m=M:

$$[\bar{\epsilon}^*_{M,i(m)}\ \bar{\mathbf{w}}^*_{M,i(m)}] := [\bar{\epsilon}^*_{M,i(m-1)}\ \sqrt{\lambda}\bar{\mathbf{w}}^*_{M,i-1(m)}][\Theta^{-1}_{b,m,i}]^*.$$

Remark: Since $E_{b,0,i} = E_{f,0,i}$, $\bar{\beta}_{0,i} = \bar{\alpha}_{0,i}$. Therefore, we do not need the first filter section in the above algorithm.

VI Concluding Remarks

We provided a modified version of the QR algorithm so as to derive the hybrid QR/LLS algorithm. From the QR algorithm point of view, we considered a method of updating the rotations which were used to propagate the estimates in the QR algorithm. By investigating this method step by step, we discovered that we could use a lattice filter instead of a transversal filter, and finally supplied a hybrid QR/LLS algorithm, which was different from the original version of the hybrid QR/LLS algorithm by Regalia and Bellanger [5].

VII Appendix

Recently, Regalia and Bellanger [5] developed a so-called "Hybrid QR/LLS" algorithm, which uses variance-normalized backwards *a posteriori* prediction errors to compute the Θ^M_i in Equation (3). Their algorithm follows the following procedures that are different from ours:

- $$\begin{bmatrix} \bar{\epsilon}^*_{f,M,i(M)} & \lambda^{1/2}E^{1/2}_{f,M,i-1} \\ \Gamma^{1/2}_{M,i-1} & 0 \end{bmatrix} \Theta_{f,M+1,i-1} = \begin{bmatrix} 0 & E^{1/2}_{f,M,i} \\ \Gamma^{1/2}_{M+1,i} & \bar{\alpha}_{M,i} \end{bmatrix}.$$

- $$\begin{bmatrix} 0 & E^{1/2}_{f,M,i} & \bar{\mathbf{w}}^*_{f,M,i-1} \\ \Gamma^{1/2}_{M+1,i} & \bar{\alpha}_{M,i} & \bar{\mathbf{b}}_{M,i-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \Psi^M_i \end{bmatrix} = \begin{bmatrix} 0 & \begin{bmatrix} E^{1/2}_{b,0,i} & 0 \end{bmatrix} \\ \Gamma^{1/2}_{M+1,i} & \begin{bmatrix} \bar{\mathbf{b}}_{M,i} & \bar{\beta}_{M,i} \end{bmatrix} \end{bmatrix}.$$

- $\Gamma^{1/2}_{M,i} = \{I - (\bar{\mathbf{b}}_{M,i}\bar{\mathbf{b}}^*_{M,i})\}^{1/2}$.

As noted by Regalia and others [5, 7], a weak link in their algorithm was a way to compute the $\Gamma^{1/2}_{M,i}$, which does not guarantee that the $\Gamma_{M,i}$ will always be positive semi-definite under round-off errors.

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