# The Use of Doppler Tolerant Reference Signals in Time Synchronization Applications

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**Abstract**

*This paper describes a technique for achieving discrete time synchronization in systems which, in addition to additive gaussian noise and phase offsets, regularly experience the effects of Doppler frequency shifts. The approach, based on a maximum likelihood (ML) timing estimator, utilizes certain properties of special reference waveforms to provide a simplified, but robust, estimator implementation for receivers using the reference to recover timing. The reference signal properties are characterized and subsequently used in an implementation of an ML timing estimator. Analysis and simulation results indicate that in the presence of additive noise, phase, and Doppler errors, using special (Doppler tolerant) waveforms in ML estimator structures can provide significant performance improvements avoiding the complexity of phase and frequency compensation mechanisms.*

## 1 Introduction

The recovery of timing information is one of the crucial tasks that must be carried out when communicating in noisy environments. In the presence of additive noise, or random phase errors, the methods for accomplishing this are straight forward assuming that the ideal condition of zero or negligible Doppler frequency shifts holds [1]. In practical systems, this assumption is generally not valid. For example, considerable velocity between a signal source and the receiver (often the case in certain mobile communications or targeting radar applications) will guarantee a noticeable Doppler frequency shift in the signal. This results in time varying phase changes in the signal which, when not compensated for in timing estimator structures, results in errors that can have a major impact on system performance [1].

*Conventional (Coherent) Approaches:* A number of approaches can be taken to insure that the timing information is recovered properly under these stressed conditions. One method involves the use of a phase-locked loop (PLL) to precisely acquire the residual carrier resulting from a Doppler shift, and track its phase [1]. The recovered carrier and phase would subsequently be used to correct the signal distortion prior to estimating the timing error. One of the obvious drawbacks with this approach stems from the implementation complexity required of the PLL and compensation circuits. In addition, the acquisition time and the possibility of hang-ups, makes this closed-loop approach problematic in burst communication or pulsed applications where short acquisition time is required [4].

Other approaches developed to determine Doppler and phase errors independently include the use of the DFT for Doppler correction [5] and maximum search techniques for phase acquisition [4]. Unfortunately, the problem of implementation complexity is still present with these approaches.

*Noncoherent Approach:* In cases where the phase cannot be tracked or reduction in complexity is desired in the receiver, the timing estimator can average out the phase error. While there is often a price paid in terms of estimator performance, the receiver structure is simplified considerably by eliminating the phase recovery mechanisms. Since unknown Doppler shifts must still be dealt with in the estimator, the extent to which the receiver can be simplified is limited.

*Novel approach:* In the following an extension of the noncoherent approach is proposed where phase noise effects are similarly averaged out in the estimator structure. The innovation involves the use of special reference signals which provide robustness against the effects of Doppler shifts, thus eliminating the processing usually required to deal with these effects. In the remainder of this paper we derive a maximum likeli-



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hood (ML) timing estimator characterizing its performance with respect to the particular reference signal choice.

## 2 Signal Model and Estimator

Let $u(t)$ denote the complex envelope of a finite duration reference signal at the output of a transmitter. The baseband signal at the receiver input will be distorted by additive Gaussian noise in addition to such things as Doppler, timing, and phase offsets which are essentially constant over the observation interval. The resulting pulsed signal can be expressed as

$$r(t) = u(t-\tau')e^{j(2\pi f_d t+\phi)} + n(t)$$

where $n(t)$ is a narrowband Gaussian noise process and $\tau', \phi$, and $f_d$ represent unknown timing, phase, and Doppler offsets respectively. If $r(t)$ is bandlimited with a cutoff frequency of $\frac{1}{2\Delta T}$, no information is lost when the signal is sampled at the Nyquist rate $\frac{1}{\Delta T}$. Hence, the resulting discrete time representation has the form

$$r(m\Delta T) = u(m\Delta T-\tau')e^{j(2\pi f_d m\Delta T+\phi)} + n(m\Delta T)$$

or simply, $r(m) = u(m-\tau)e^{j(2\pi\nu m+\phi)} + n(m)$ where $\nu = f_d\Delta T$, $\nu \in [-\pi, \pi]$ and $\tau$ is an integer representing the discrete timing error, i.e. $\tau' = \tau\Delta T + \varepsilon$, $\varepsilon \in [0, \Delta T]$. The value of $\nu$, referred to here simply as the Doppler shift, represents the phase shift over each sample interval resulting from the Doppler frequency shift $f_d$.

At the receiver, the synchronization task is to process incoming blocks of signal samples in order to obtain an estimate of the discrete timing error $\tau$. For a block of $N$ signal samples, the received signal can be represented in vector notation as

$$\underline{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_{\tau+1} \\ \vdots \\ r_{\tau+M} \\ r_{\tau+M+1} \\ \vdots \\ r_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ u_1 e^{j2\pi\nu+\phi} \\ \vdots \\ u_M e^{j2\pi M\nu+\phi} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ \vdots \\ n_{\tau+1} \\ \vdots \\ n_{\tau+2} \\ n_{\tau+3} \\ \vdots \\ n_N \end{bmatrix}.$$

where $\tau \in [0, 1, \ldots, N-M]$.

The ML estimate $\hat{\tau}$ of the timing offset $\tau$ can be obtained by maximizing the probability of observing a particular sequence of received samples (assuming that a timing estimate is known and correct) over all possible estimates. When the transmit sequence $\underline{u} = (u_1, u_2, \ldots, u_M)$ and the Doppler and phase offsets are all known, the estimate may be found by maximizing the probability

$$\Lambda(\hat{\tau}) = P(r_1, r_2, \ldots, r_N|\hat{\tau}, \underline{u}) \quad (1)$$

over all values of $\hat{\tau}$.

Noting that nyquist sampling insures that the Gaussian noise samples $\{n_m\}$ are uncorrelated, the conditional distribution $P(r_1, r_2, \ldots, r_N|\hat{\tau}, \nu, \phi, \underline{u})$ can be expressed simply as

$$P(\underline{r}|\hat{\tau}, \nu, \phi, \underline{u}) = \prod_{m=1}^{N} P(r_m|\hat{\tau}, \nu, \phi, u_{m-\tau}) \quad (2)$$

where

$$P(r_m|\hat{\tau}, \nu, \phi, u_{m-\tau}) = K_1 \exp\left(-K_2\left|r_m - u_{m-\tau}e^{j(2\pi\nu m+\phi)}\right|^2\right) \quad (3)$$

and $K_1$ and $K_2$ are determined by the noise power. Substituting (3) into (2) and rearranging terms results in the expression

$$P(\underline{r}|\hat{\tau}, \nu, \phi, \underline{u}) = A_1 \times A_2 \times \exp\left(2K_2\mathcal{R}e\left\{e^{-j\phi}\sum_{m=1}^{N} r_m\bar{u}_{m-\tau}e^{-j2\pi\nu m}\right\}\right) (4)$$

where $A_1 = K_1\exp\left(-K_2\sum_{m=1}^{N}|r_m|^2\right)$ and $A_2(\hat{\tau}) = \exp\left(-K_2\sum_{m=1}^{N}|u_{m-\hat{\tau}}|^2\right)$. The phase term is eliminated by averaging over the statistics of $\phi$ (generally assumed to be uniform on $[-\pi, \pi]$) resulting in the expression

$$P(\underline{r}|\hat{\tau}, \nu, \underline{u}) = \frac{A_1A_2}{2\pi}\times \int_\phi \exp\left(2K_2\mathcal{R}e\left\{e^{-j\phi}\sum_{m=1}^{N} r_m\bar{u}_{m-\hat{\tau}}e^{-j2\pi\nu m}\right\}\right) d\phi \quad (5)$$
$$= A_1A_2\mathcal{I}_0\left(2K_2\left|\sum_{m=1}^{N} r_m\bar{u}_{m-\hat{\tau}}e^{-j2\pi\nu m}\right|\right)$$

where $\mathcal{I}_0$ represents the modified zero order Bessel function of the first kind. The received signal energy, represented by the argument of the exponential $A_1$, is independent of the timing error $\tau$. Also, assuming that the number of samples $N$ in a received signal block is greater than or equal to twice the number of reference samples $M$, and restricting $u(t)$ to the class of constant envelope signals—both reasonable assumptions from a practical implementation point of view [3]—removes the dependence of the exponential $A_2$ on the timing error $\tau$. Since these, as well as the constant terms, are not relevant to the likelihood calculation, the multiplier $\frac{A_1 A_2}{2\pi}$ can be dropped in subsequent expressions.

If the Doppler shift $\nu$ were known, the resultant likelihood expression could be maximized over $\hat{\tau}$ to obtain the most likely estimate of the timing error $\tau$, given the particular Doppler value. Our assumption, however, is that $\nu$ is not known exactly hence, one possibility is to average out the Doppler term to obtain the expression

$$\Lambda(\hat{\tau}) = \int_{\nu} \mathcal{I}_0 \left( 2K_2 \left| \sum_{m=1}^{N} r_m \bar{u}_{m-\hat{\tau}} e^{-j2\pi\nu m} \right| \right) d\nu. \quad (6)$$

A closed form solution to this integral is not readily obtained although $\Lambda(\hat{\tau})$ can be evaluated precisely using numerical techniques. Unfortunately, the resultant metric, which would need to be calculated for all possible timing values in an estimation implementation, is extreamly difficult to evaluate in practice.

Alternately, if we assume that the Doppler shift $\nu$ does not significantly effect the argument of $\mathcal{I}_o$, a simplified approximation of the likelihood function $\hat{\Lambda}(\hat{\tau})$ can be obtained by avoiding the integration in (6). Moreover, the monotonic property of the Bessel function allows us to replace $\mathcal{I}_o(X)$ with $X$ or $X^2$. The resulting likelihood expression

$$\hat{\Lambda}_1(\hat{\tau}) = \left| \sum_{m=1}^{N} r_m \bar{u}_{m-\hat{\tau}} e^{-j2\pi\nu m} \right|^2 \quad (7)$$

represents the squared magnitude output of a matched filter or correlation receiver for $\hat{\tau} - \tau = \nu = 0$.

Because the value of $\nu$ will affect the Bessel function argument, the relationship between $\hat{\Lambda}_1(\hat{\tau})$ and $\nu$ must be examined to determine the conditions for which an estimator based on (7) will be valid. Hence we replace the term $\hat{\Lambda}_1(\tau)$ with $h(\hat{\tau}, \nu)$, (to underscore the dependence in (7) on the value of $\nu$), and note that since

$$h(\hat{\tau}, \nu) = \left| \sum_{m=1}^{N} (u_{m-\hat{\tau}} + n_m) \bar{u}_{m-\hat{\tau}} e^{-j2\pi\nu m} \right|^2$$

$$= \left| \sum_{m=1}^{N} (u_{m-\tau} \bar{u}_{m-\hat{\tau}} e^{-j2\pi\nu m}) + (n_m \bar{u}_{m-\hat{\tau}} e^{-j2\pi\nu m}) \right|^2,$$

it follows that

$$h(\hat{\tau}, \nu) = \left| e^{-j2\pi\tau} \sum_{n} u_n \bar{u}_{n-(\hat{\tau}-\tau)} e^{-j2\pi\nu n} + \eta \right|^2$$

$$= \left| e^{-j2\pi\tau} \chi_u(\tau - \hat{\tau}, \nu) + \eta \right|^2 \quad (8)$$

for $n = m - \tau$. The zero mean noise term involving $n_m$ has been replaced by the random variable $\eta$ which is also a zero mean Gaussian. The significance here is that the behavior of $\hat{\Lambda}_1$ can be completely separated into signal ($\chi_u(\tau - \hat{\tau}, \nu)$) and noise ($\eta$) contributions. More importantly though, the signal characteristics represented by the function $\chi_u(\tau - \hat{\tau}, \nu)$ singularly determines the effect of Doppler and timing errors on the likelihood function $\hat{\Lambda}_1$. Therefore, restrictions placed on $\chi_u(\tau - \hat{\tau}, \nu)$ will control, to some extent, the performance of the estimator implemented using equation 7.

## 3 Doppler Tolerant Signals

The peak sidelobe ratio (PSL) for reference signals corresponding to $\chi(\tau - \hat{\tau}, \nu)$ is defined as

$$\frac{\max_{\tau \neq \hat{\tau}, \nu \in [-\Omega, \Omega]} \left( |\chi_u(\tau - \hat{\tau}, \nu)|^2 \right)}{\min_{\tau = \hat{\tau}, \nu \in [-\Omega, \Omega]} \left( |\chi_u(\tau - \hat{\tau}, \nu)|^2 \right)} \quad (9)$$

for Doppler shifts in the range $[-\Omega, \Omega]$. This ratio determines, as a function of $\{u_m\}$ and $\Omega$, how well the estimator discriminates between correct and incorrect timing estimates when the received signal has a Doppler shift $\nu$ [3] Small PSL ratios ($\ll 1.0$) indicate that for all Doppler shifts in the specified range, the value of $\chi_u$ for a correct estimates is easily distinguished from those values corresponding to incorrect estimates. Because the denominator in (9) vanishes for all constant envelope signals when $\nu = \frac{1}{M}$, we must assume that at the input to the timing estimator, $\Omega < \frac{1}{M}$, (it is preferable that the Doppler range be even smaller to insure that the denominator, which varies only as a function of $\Omega$, deviates from its maximum value by no more than 15%). Sequences with PSL values that are minimum (for some choice of $\Omega$) over all possible constant envelope sequences in some class, (e.g. binary, quaternary, etc.), are referred to as *Doppler Tolerant (DT)*. The function $|\chi(\hat{\tau} - \tau, \nu)|^2$ for an ideally Doppler Tolerant reference sequence, (i.e. $PSL = 0$), is plotted in figure 1.

The effect of Doppler tolerance on an estimator which implements $\hat{\Lambda}_1(\hat{\tau})$ is seen by examining the statistics of the random variable $h(\hat{\tau}, \nu)$. By letting $X = \Re\{e^{-j2\pi\tau} \chi_u(\tau - \hat{\tau}, \nu) + \eta\}$ and $Y = \Im\{e^{-j2\pi\tau} \chi_u(\tau - \hat{\tau}, \nu) + \eta\}$ we find that

$$h(\hat{\tau}, \nu) = X^2 + Y^2 \quad (10)$$

where $X$ and $Y$ are both Gaussian random variables with the expected values $m_x$ and $m_y$ and common

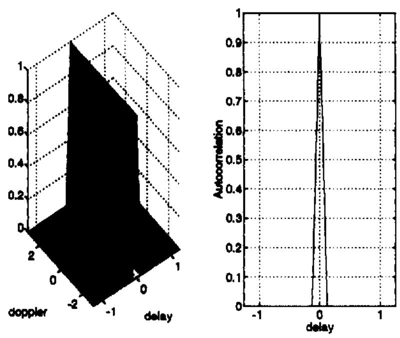


Figure 1: *Ideal* $|\chi(\tau-\hat{\tau},\nu)|^2$ *for* $\nu \in [-\Omega,\Omega]$. *Delay and Doppler axes normalized*

variance $\sigma_\eta$. The scalar, $h(\hat{\tau},\nu)$ has a non-central chi-square distribution with non-centrality parameter $s^2 = m_x^2 + m_y^2 = |\chi_u(\tau-\hat{\tau},\nu)|^2$, hence, for positive values of $h(\hat{\tau},\nu)$

$$P_h(h(\hat{\tau},\nu)) = \frac{1}{2\sigma_\eta^2} e^{-(s^2+y)/2\sigma_\eta^2} \mathcal{I}_o\left(\sqrt{h(\hat{\tau},\nu)}\frac{s}{\sigma_\eta}\right). \quad (11)$$

This distribution can be applied directly to the task of determining how different reference signals will affect the performance of the estimator. Particularly, for a given timing estimate $\hat{\tau}_i$ $i \in [1, M]$, the probability that this estimate is correct, $1 - \text{PE}$, is equal to the probability that the decision variable $h(\hat{\tau}_i,\nu)$ exceeds all others $h(\hat{\tau}_j,\nu)$, $i \neq j$. The joint statistics of all decision variables must be known to calculate this probability exactly, however if we view the estimator as a receiver making $M-1$ binary decisions between $h(\hat{\tau}_i,\nu)$ and the other estimator outputs, the behavior of the estimator can be gauged by considering the $M-1$ individual the error events,

$$h(\hat{\tau}_i,\nu) > h(\hat{\tau}_j,\nu) \quad \forall i \neq j \text{ when } \quad \tau = \hat{\tau}_j.$$

The probability of each error event ($\text{PE}(\tau)_{\text{event}}$) is represented by the sum of the area under the tails of the conditional distributions $P_h[h(\hat{\tau}_j,\nu) \mid \tau = \hat{\tau}_j]$ and $P_h[h(\hat{\tau}_i,\nu) \mid \tau = \hat{\tau}_j]$ for some $i \neq j$, i.e.,

$$\text{PE}(\tau_j\tau_i)_{\text{event}} = \int_{h<a} P_h[h(\hat{\tau}_j,\nu) \mid \hat{\tau}_j = \tau]\text{d}(h) + \int_{h>a} P_h[h(\hat{\tau}_j,\nu) \mid \hat{\tau}_j \neq \tau]\text{d}(h) \quad (12)$$

where $a$ is the value of $h(\hat{\tau}_i,\nu)$ for which

$$P_h[h(\hat{\tau}_j,\nu) \mid \hat{\tau}_j = \tau] \equiv P_h[h(\hat{\tau}_j,\nu) \mid \hat{\tau}_j \neq \tau].$$

Plots of typical distributions help to illustrate how both $\sigma_\eta$ and $s$ affect $\text{PE}(\tau_j\tau_i)_{\text{event}}$ for some value of $\tau$

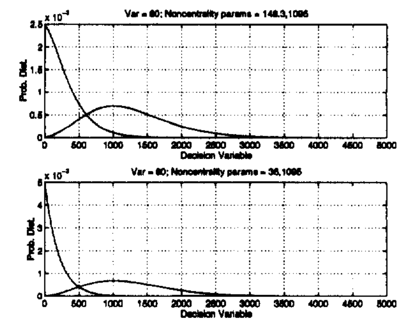

Figure 2: *Probability distributions for decision metrics corresponding to assumed correct and incorrect timing estimates.* $\sigma_\eta = 80$

(figure 2). Specifically, increasing $\sigma_\eta$ lowers the peak of the distribution functions which effectively increases the area under distribution tails and results in a higher likelihood that the event

$$h(\hat{\tau}_i,\nu) > h(\hat{\tau}_j,\nu) \quad i \neq j \text{ when } \quad \hat{\tau}_j = \tau$$

will occur. Similarly, denoting $s_1^2$ and $s_2^2$ as the non-centrality parameters for two probability functions, the value of $|s_1^2 - s_2^2|$ effects the separation between the distributions which, when decreased, results in a larger amount of area under the tails and thus a higher error event probability.

Clearly, in order for $\hat{\Lambda}_1(\hat{\tau})$ to be useful it is important to keep $\sigma_\eta$ as low as is possible and to select decision variables such that the separation between the distribution for $h(\tau,\nu)$ and the distributions for $h(\hat{\tau},\nu)$ is as large as possible $\forall \nu \in [-\Omega,\Omega]$ and $\hat{\tau} \neq \tau$ In practice we have little control over $\sigma_\eta$ however, the parameter affecting distribution separation, $s^2 = |\chi(\hat{\tau}-\tau,\nu)|^2$, depends entirely on the choice of reference signal. The performance is, therefore, limited by the worst case error event probability, ($\max_{\tau_i \neq \tau_j} \text{PE}(\tau_j\tau_i)_{\text{event}}$) which is minimized when

$$\left[\frac{\sin(\pi\Omega M)}{\sin(\pi\Omega)}\right]^2 - \left[\max_{\substack{\hat{\tau}\neq\tau \\ \nu\in[-\Omega,\Omega]}} |\chi(\hat{\tau}-\tau,\nu)|^2\right] \quad (13)$$

is maximized over the values of the reference samples $\{u_m\}$. The Doppler tolerance property characterized by a minimum PSL ratio equivalently minimizes the result in (13) and hence minimizes the worst case theoretical $\text{PE}(\tau_j\tau_i)_{\text{event}}$.

## 4 Performance Results

In [3], the topic of searching for Doppler tolerant (DT) signals is treated in considerable detail. While these details are beyond the scope of this discussion, some of the compiled sequences are of interest in an implementation of $\hat{\Lambda}_1(\hat{\tau})$. Particularly, the binary reference signals offer a realistic example of the type of signal that would be useful in a typical receiver implementing a matched filter synchronization mechanism [2]. The $M$ length signal samples found in [3] have been optimized for Doppler shifts restricted to the range $\nu \in [-\frac{1}{3M}, \frac{1}{3M}]$. This insures that the frequency dependent increase in PSL, discussed in the last section is limited to 2 dB.

In the implementation considered here, we assume without loss in generality, observation blocks of length $N = 80$ and reference signal samples of length $M = 40$, where, at the estimator input, timing errors in the range $\tau \in \{1, 2, \ldots\}$ are corrected when the worst case Doppler uncertainty is no more than $\nu = \frac{2}{3N} = \frac{1}{3M}$. Doppler errors larger than that can be easily reduced using methods such as the DFT techniques in [5].

The length 40 sequences that were used in our simulations are listed in table 1 along with the corresponding non–centrality values for the distributions $P_h[h(\hat{\tau}_j, \nu) \mid \hat{\tau}_j = \tau]$ and $P_h[h(\hat{\tau}_j, \nu) \mid \hat{\tau}_j \neq \tau]$, (i.e. $|\chi_u(0,\nu)|^2$ and $\max_{\tau \neq \hat{\tau},\ \nu \in [-\Omega,\Omega]} |\chi_u(\hat{\tau} - \tau, \nu)|^2$) two of which were used in the plots of figure 2. The worst case error event probabilities were determined for each sequence in the table based on the values of $s^2$ associated with the sequence. The results, calculated using the theoretical error expression (13) and confirmed via computer simulation are illustrated in figure 3.

## 5 Remarks and Conclusions

Results indicate that significant improvements in worst case probability, $\mathrm{PE}(\tau_j \tau_i)_{\text{event}}$ are achieved when the DT references (optimized over $\nu \in [-\Omega, \Omega]$) are used instead of arbitrary sequences or those optimized for zero Doppler. These results, illustrated in figure 3 were verified in a computer simulation of the implementation model for $\hat{\Lambda}_1(\hat{\tau})$. Interestingly, it was also noted that the values of $\mathrm{PE}(\tau_j \tau_i)_{\text{event}}$ for the sequences optimized for $\nu = 0$ were all less than those corresponding to the DT sequences except at a single value of $\hat{\tau} - \tau$, i.e. This demonstrates a clear justification for the use of the event probability as a system performance measure since the *worst case* $\mathrm{PE}(\tau_j \tau_i)_{\text{event}}$ would be obscured in the calculation of a typical error rate PE. Though not appropriate in communications where each bit error has an equally small impact on total system performance, *worst case* $\mathrm{PE}(\tau_j \tau_i)_{\text{event}}$ is a crucial measure in synchronization where timing errors can have a tremendous effect on the total system performance.

Table 1: *Length 40 Reference Sequences*

| **Finite Alphabet Sequence Type** | **Binary Trivial** $\{0, \pi\}$ | **Binary Min PSL** for $\nu = 0$ $\{0, \pi\}$ | **Binary DT Min PSL** $\nu \in [-\Omega, \Omega]$ $\{0, \pi\}$ |
|---|---|---|---|
| by Phase index | 0000000000<br>0000000000<br>0000000000<br>0000000000 | 0001000100<br>0110000001<br>1000000100<br>0110001000 | 0010001001<br>1000111101<br>0111010100<br>1100001110 |
| $\max_{\tau \neq 0, \nu}$ $\lvert\chi(\hat{\tau} - \tau, \nu)\rvert^2$ | 1060.9 | 143.0 | 36 |
| $\min \nu$ $\lvert\chi(0, \nu)\rvert^2$ | 1096.0 | 1096.0 | 1096.0 |

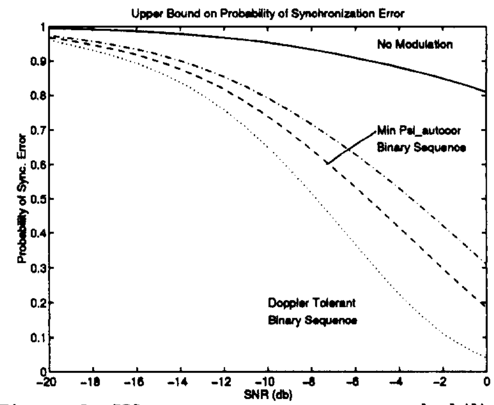

Figure 3: *Worst case error event probability*

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