New results in System Identification of FIR systems

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Abstract

This paper deals with the problem of estimating the parameters of a MA model from the third order cumulant statistics of the noisy observations of the system output. The system is driven by an independent and identically distributed skewed sequence that is not observed. The proposed method does not rely on second order statistics and so it is not affected by symmetrically distributed additive coloured noise. Both batch least squares and recursive methods are developed which utilise all the available third order statistics.

1 Introduction

This paper introduces a new parametric approach to the identification of nonminimum phase linear time-invariant (NMP-LTI) systems. In particular a new method for nonminimum phase FIR system identification is considered . The main assumption is that the output sequence $\{x(k)\}$ is generated by ,

$$x(n) = \sum_{k=0}^{q} h(n-k)w(k). \tag{1}$$

Where $h(n-k)$ is the impulse response of the system and $w(k)$ is non-Gaussian, white, IID, with zero-mean $E\{w(k)\} = 0$, variance $\sigma^2 = E\{w(k)^2\}$, and skewness $\gamma_3 = E\{w(k)^3\}$ ($\gamma_3 \neq 0$). The observations of the signal $x(n)$ are in general noisy:

$$y(n) = x(n) + v(n). \tag{2}$$

The measurement noise sequence is assumed to be a zero mean, coloured sequence symmetrically distributed. Several methods utilising cumulant statistics for the identification of NMP LTI FIR systems have been proposed (for a review look in [1]). One of the best known approaches is that of Giannakis and Mendel [2]. The performance of this method was studied by Porat and Friendlander [3]. Various modifications were also proposed by Tugnait [4, 5]. Finally a method utilising all the available 2nd and 3rd order cumulant statistics was introduced in [6] by Alshebeili *et al.*.All of these methods are based on relations which associate 2nd order and 3rd order cumulants with system parameters. In this paper new methods are developed that allow us to recover the impulse response of the system $h(k)$, from only the third order cumulant statistics of a finite realisation of the observation sequence. The methods presented here require the solution of a system of linear equations which can be achieved using Least Squares methods. A new recursive solution is also proposed to the system of linear equations which is of independent interest but can also be used to claim the consistency (uniqueness) of the Least Squares solution.

An important problem in system identification is the determination of the system order. New methods are developed here for system order selection. The system order selection methods are closely related to the new parameter estimation methods presented.

2 Fundamental Relations

In this section a relationship between diagonal and non-diagonal third order cumulant slices of FIR systems is derived. This relationship is of fundamental importance because it is the basis of the system identification method. From the symmetry relations [1] in the lags of third order cumulants of stationary processes we know that

$$c_3(\tau-k, \tau-k) = c_3(k-\tau, 0).$$

The third order cumulants function for the process $\{x(n)\}$ at lags $k-\tau$ and 0 is defined by the the

1058-6393/95 $4.00

Brillinger and Rosenblatt [7] equation as follows:

$$c_{3,x}(k-\tau,0) = \gamma_{3,w} \sum_{i=0}^{q} h^2(i)h(i+k-\tau). \quad (3)$$

If we multiply both sides of (3) with $h(k)h(k+m)$ and then sum over k for $k=0$ to $k=q$, we have

$$\sum_{k=0}^{q} h(k)h(k+m)c_{3,x}(k-\tau,0) =$$
$$\gamma_{3,w} \sum_{k=0}^{q} h(k)h(k+m) \sum_{i=0}^{q} h^2(i)h(i+k-\tau). \quad (4)$$

Now if we change the order of the two sums in (3) we have,

$$\gamma_{3,w} \sum_{k=0}^{q} h(k)h(k+m) \sum_{i=0}^{q} h^2(i)h(i+k-\tau) =$$
$$\sum_{i=0}^{q} h^2(i)\gamma_{3,w} \sum_{k=0}^{q} h(k)h(k+m)h(k+i-\tau) =$$
$$\sum_{i=0}^{q} h^2(i)c_{3,x}(i-\tau,m).$$

So finally from the previous equation and (4) we have,

$$\sum_{k=0}^{q} h(k)h(k+m)c_{3,x}(k-\tau,0) =$$
$$\sum_{k=0}^{q} h^2(k)c_{3,x}(k-\tau,m) \quad (5)$$

Equation (5) relates the system response coefficients to diagonal and off-diagonal cumulant slices. We must note here that equation (5) is a special case of equation (19) in [4] for $n=0$. In the next section a new approach will be developed that exploits all the relevant statistics in the solution of (5).

3 MA Parameter Estimation

We select equation (5) for $1 \le m \le q$. Given m, equation (5) is non-trivial only if τ satisfies the condition $-m \le \tau \le 2q-m$. In the following it will be assumed that $h(0)=1$. The objective is to solve (5) with repsect to the unknown system parameters $h(1),\ldots,h(q)$.

3.1 Least Squares Method

In this section a Least Squares approach is presented to the solution of (5). Equation (5) is treated as a system of linear equations with respect to the unkowns $h(1),\ldots,h(q),h^2(1),\ldots,h^2(q)$, and $h(i)h(i+l)$ for $1 \le i \le q-1$ and $1 \le l \le q-i$. Then the system of linear equations

$$\sum_{k=0}^{q} h(k)h(k+m)c_{3,x}(k-\tau,0) -$$
$$\sum_{k=1}^{q} h^2(k)c_{3,x}(k-\tau,m) = c_{3,x}(-\tau,m), \quad (6)$$

for $1 \le m \le q$ and $-m \le \tau \le 2q-m$, can be expressed in a matrix form as follows:

$$\boxed{\mathbf{Bg} = \mathbf{d}.} \quad (7)$$

Where $\mathbf{g} = [h(1),\ldots,h(q),h^2(1),\ldots,h^2(q),h(1)h(2),\ldots,h(1)h(q),\ldots\ldots,h(q-1)h(q)]'$ is a $\frac{q^2+3q}{2}$ element vector, $\mathbf{d}$ is a $2q^2+q$ element vector, and $\mathbf{B}$ is a $(2q^2+q)\times\frac{q^2+3q}{2}$ matrix. The contents of $\mathbf{d}$ and $\mathbf{B}$ are determined according to (6). The least squares solution of this overdetermined system of linear equations is

$$\boxed{\mathbf{g} = (\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top\mathbf{d}.} \quad (8)$$

In practice the cumulants which the entries of $\mathbf{d}$ and $\mathbf{B}$ consist of, are not available and are replaced by their sample estimates $\hat{c}_{3,x}(\cdot,\cdot)$. Because of errors in the estimation of the cumulants the elements of the solution vector obtained from (8) will not comply with the theoretical structure of $\mathbf{g}$. In order to exploit all the available information hidden in $\mathbf{g}$, we can form the following matrix $\mathbf{R}$, in a similar fashion to [6]:

$$\mathbf{R} = \begin{bmatrix} 1 & h(1) & \cdots & h(q) \\ h(1) & h^2(1) & \cdots & h(1)h(q) \\ h(2) & h(2)h(1) & \cdots & h(2)h(q) \\ \vdots & \vdots & \ddots & \vdots \\ h(q-1) & h(q-1)h(1) & \cdots & h(q-1)h(q) \\ h(q) & h(q)h(1) & \cdots & h^2(q) \end{bmatrix}.$$

It is clear from the structure of $\mathbf{R}$ that its rank is one. $\mathbf{R}$ can be written in the following form:

$$\mathbf{R} = \mathbf{h}_q\mathbf{h}^\top = \begin{bmatrix} 1 \\ h(1) \\ \vdots \\ h(q) \end{bmatrix} \begin{bmatrix} 1 & h(1) & \cdots & h(q) \end{bmatrix} \quad (9)$$

In practice however, again due to estimation errors, its rank will be greater than 1. Now if the SVD of $\mathbf{R}$ is

$$\mathbf{R} = \sum_{k=1}^{q} \sigma_k \mathbf{u}_k \mathbf{v}'_k,$$

where $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_q \geq 0$. The system parameters can be found as

$$\boxed{h(n) = \sigma_1 u_{1,1} v_{n,1}.} \quad (10)$$

In many practical situations it is very useful to have a measure of "confidence" for the obtained least squares solution. From the previous discussion we can see that such a measure can be devised as follows:

$$\boxed{\lambda = \frac{\sigma_1}{\sum_{i=1}^{q} \sigma_i}} \quad (11)$$

where $0 < \lambda \leq 1$. The nearer λ is to 1 the more confident we can be for our solution.

3.2 A Recursive Algorithm

In this section it is shown that the unknown parameters $h(1), \ldots, h(q), h^2(1), \ldots, h^2(q)$, and $h(i)h(i+l)$ for $1 \leq i \leq q-1$ and $1 \leq l \leq q-i$ can be determined from (5), using a recursive algorithm. Taking (5) for $m = q$ and $\tau = -q$ we have

$$h(q) = \frac{c_{3,x}(q,0)}{c_{3,x}(q,q)}. \quad (12)$$

Again for $m = q$, if we take $\tau = -q+j$ for $j = 1, \ldots, q$ we have the following recursive equation:

$$h^2(j) = \frac{h(q)c_{3,x}(q,0) - \sum_{k=0}^{j-1} h^2(k) c_{3,x}(k+q-j,q)}{c_3(q,q)}.$$

For $m < q$ and $\tau = 2q - m$ we also have,

$$h(q-m)h(q) = \frac{h^2(q) c_{3,x}(q-m,q)}{c_{3,x}(q,q)} \quad (13)$$

Finally for $m < q$ and $\tau = 2q-m-j$, $j = 1, \ldots, q-m$,

$$h(q-j-m)h(q-j) = [\sum_{k=q-j}^{q} h^2(k) c_{3,x}(k-2q-m-j,m)$$

$$- \sum_{k=q-m-j+1}^{q-m} h(k)h(k+m)c_{3,x}(k-2q+m+j,0)]/c_{3,x}(q,q).$$

All the divisions are well conditioned since $c_{3,x}(q,q) \neq 0$. The above recursive algorithm shows that using only a subset of equations from (5) and assuming that $h(0) = 1$, we can recover uniquely the unknown parameters $h(1), \ldots, h(q), h^2(1), \ldots, h^2(q)$, and $h(i)h(i+l)$ for $1 \leq i \leq q-1$ and $1 \leq l \leq q-i$.

3.3 Uniqueness and Consistency of the Least Squares Method

The following result is a direct consequence of the uniqueness of the corresponding recursive closed form solution.

Theorem 1 *For the FIR system (1) and assuming that we are given the correct third order cumulant statistics, the matrix* $\mathbf{B}$ *in (8), is of full rank:*

$$\boxed{rank(\mathbf{B}) = \frac{q^2+3q}{2}.}$$

The previous theorem allows us to claim that the solution of the least squares problem in (8) is unique. The consistency of the proposed method depends on the convergence properies of sampled third-order cumulants which is studied in [8]. These results are well known and in summary we can say that, if the input random process $w(n)$ in (1) is stationary and its first six cumulants are absolutely summable, then the estimated $h(1), \ldots, h(q), h^2(1), \ldots, h^2(q)$, and $h(i)h(i+l)$ for $1 \leq i \leq q-1$ and $1 \leq l \leq q-i$, converge in probability to their true values.

4 Model Order Selection

In this section the problem of model order selection is considered. Many linear algebraic methods for system identification (including the one introduced here) assume that the model order is known and they fail completely if the model order is under- or over-estimated. Therefore it is crucial to be able to select the model order. The idea behind the proposed system order selection method is to select the order that results in the largest confidence measure λ defined by equation (11). The proposed method can be summarised in the following:

1. Assume that the system order is less than p. In practice it is usually possible to make such an assumption.
2. Assume that the model order is q, where $q = 1, \ldots, p$. For each value of q do the following:
   (a) From the given data calculate the sampled third order cumulants assuming that the system order is q.
   (b) Use the least squares method of section 3.1 to calculate the parameter vector $\mathbf{g}_q$.

(c) Form the matrix $\mathbf{R}_q$ as shown in section 3.1.

(d) Perform SVD on $\mathbf{R}_q$ and calculate the confidence measure λ_q from equation (11).

3. Select the order q that yields the maximum value of λ_q.

The described order selection algorithm does not depend on any second order statistical information and consequently it can be used in cases where there is coloured symmetrically distributed observation noise. It also offers the advantage of not requiring visual inspection or subjective thresholding as in the approaches in [9, 10].

5 Simulations

5.1 System Idenitification

This section presents simulation examples which test the performance of the algorithms introduced in the paper. Additive coloured noise is created as the output of the MA(4) model: $v(n) = 0.5u(n) - 0.25u(n-1) - 0.5u(n-2) + 0.25u(n-3) - 0.25u(n-4)$, where the sequence is an IID Gaussian sequence. The signals are generated using three different models. In all three cases the input signal $w(n)$ is zero-mean exponentially distributed i.i.d. sequence with $\gamma_{3,w} = 1$.

Model 1: $x(n) = w(n) - 1.40w(n-1) + 0.98w(n-2) + v(n)$. This a minimum phase system.

System Identification Model 1
SNR=2dB colored MA(4), N=1000

	Sec 3.1	[6]
Parameters	mean±std	mean±std
1	1±0.0	1±0.0
-1.4	-1.422±0.25	-0.53±0.482
0.98	0.9478±0.278	0.380±0.283
Normalised Error	0.147	0.544

Model 2: $x(n) = w(n) + 1.4w(n-1) - 2.2w(n-3) - 1.6w(n-4) + 1.9w(n-5) + v(n)$. This is a nonminimum phase system with zeros located at , $0.861 \pm j1.53$, -1.132 and -1.804. As we can see from the next table for low levels of noise both methods perform very well with the method in [6] having a slight advantage because it utilises more information.

System Identification Model 2
SNR=50dB colored MA(4), N=5000

	Sec 3.1	[6]
Parameters	mean±std	mean±std
1	1±0.0	1±0.0
1.4	1.344±0.094	1.40±0.1146
-2.2	-2.16±0.09	-2.20±0.11
-1.6	-1.57±0.119	-1.62± 0.126
1.9	1.85±0.121	1.92±0.132
Normalised Error	0.066	0.056

Model 3: $x(n) = w(n) + 0.1w(n-1) - 1.87w(n-2) + 3.02w(n-3) - 1.435w(n-4) + 0.49w(n-5)$. This is again a nonminimum phase system with zeros located at, $0.7 \pm j0.7$, $0.25 \pm j0.433$ and -2.0. In the next table we see results with virtually noise free data. It is important to note here that by combining the method of Sec 3.1 and [6] using information provided by the confidence factor λ results in the reduction of the average normalised error.

System Identification Model 3
SNR=100dB colored MA(4), N=4000

	Sec 3.1	[6]	$\lambda \geq 0.8$ Sec 3.1 else [6]
$h(\cdot)$	mean±std	mean±std	mean±std
1	1±0.0	1±0.0	1±0.0
0.1	0.089±0.175	-0.053±0.394	0.023±0.389
-1.87	-1.68±0.509	-1.663±0.513	-1.78±0.498
3.02	-2.87±0.777	2.922± 0.541	2.996±0.0.519
-1.435	1.297±0.383	-1.448±0.248	-1.46±0.258
0.49	0.34± 0.24	0.476±0.10	0.476±0.109
Error	0.1948	0.1945	0.168

Results with white noise (SNR=10dB's) are presented in the next table. In this case the combination of the two methods gives an error of 0.3121 which is worse than the error of the method of Sec. 3.1 alone. If we ignore the results of method Sec. 3.1 which have confidence factor $\lambda \geq 0.8$ then the error becomes 0.165. These result has been achieved with the exclusion of 26 results out of 100.

System Identification Model 3
SNR=10dB white MA(4), N=4000

	Sec 3.1	[6]
$h(\cdot)$	mean±std	mean±std
1	1±0.0	1±0.0
0.1	0.014±0.506	-0.225±0.1804
-1.87	-1.507±1.03	-1.085±0.162
3.02	2.34±1.183	2.131± 0.252
-1.435	-1.123±0.577	-1.033±0.202
0.49	0.321± 0.269	0.342±0.0702
Error	0.310	0.332

In the next two tables results are presented with additive coloured noise (SNR=10dB's).

System Identification Model 3 SNR=10dB colored MA(4), N=4000		
$h(\cdot)$	Sec 3.1	[6]
	mean±std	mean±std
1	1±0.0	1±0.0
0.1	0.046±0.433	-0.263±0.184
-1.87	-1.537±0.89	-1.02±0.157
3.02	2.37±1.067	2.0478± 0.229
-1.435	1.144±0.521	-0.983±0.207
0.49	0.328± 0.219	0.348±0.073
Error	0.308	0.362

System Identification Model 3 SNR=10dB colored MA(4), N=4000		
$h(\cdot)$	$\lambda \geq 0.8$ Sec 3.1 else [6]	$\lambda \geq 0.8$ else ignore
1	1±0.0	1±0.0
0.1	-0.207±0.194	0.084±0.227
-1.87	-1.137±0.304	-1.685±0.534
3.02	2.168±0.3678	2.721±0.604
-1.435	-1.04±0.276	-1.271±0.293
0.49	0.369±0.091	0.406±0.10
Error	0.327	0.197
results ignored with $\lambda < 0.8$		20 out of 100

5.2 Model Order Selection

In this section we illustrate the use of the system order selection algorithm introduced in this paper. To test the algorithm we generate signals using the model [10] :

$$x(n) = w(n) + 0.9w(n-1) + 0.385w(n-2) - 0.771w(n-3) + v(n) \quad (14)$$

where $w(n)$ is defined as before. We compare the method of Sec. 4 with the method based on equation (73) in [6].

System Order Selection SNR=10dB and 100 Monte Carlo Runs				
	Number of Succesful Selections			
Method	N=500	N=1000	N=2000	N=4000
Sec 3.2	88	95	97	99
[6]	84	93	94	99

6 Summary and Conclusions

In this paper a new parametric method for FIR system identification was introduced. The new method requires only third order statistics and consequently it is robust in the presence of additive coloured Gaussian noise. The performance of the algorithm is tested with computer simulations. The method presented here can be easily extended to fourth order cumulants and the performance can be analysed.

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