Model Order Reduction of Multichannel Stable Systems*

Won Cheol Lee and S. Unnikrishna Pillai

Electrical Engineering Department
Polytechnic University
Five MetroTech Center
Brooklyn, New York 11201
e-mail: wlee@arma.poly.edu, pillai@arma.poly.edu

Abstract

This paper addresses the problem of approximating multichannel stable systems by lower order stable system transfer functions that interpolate the partial impulse response matrix sequence of the original system. This is achieved by making use of the left- and right-Schur recursion algorithms, and in this context, the classical Padé approximations that are also stable are shown to be a special case of this general formulation.

1. Introduction

This paper addresses the problem of approximating the class of all multichannel stable system transfer functions by lower order stable rational transfer functions that interpolate the given partial impulse response matrix sequence of the original system. Classical Padé approximations can generate left- and right-coprime rational representations of such systems in an optimal manner by matching the coefficients to a maximum extend. However, such systems need not be stable, and hence they may be unattractive from practical considerations. The theory of optimal Hankel-norm approximations[1] gives a satisfactory solution to this problem, and the present approach offers a viable alternative to this approximation problem. In this context, let $\mathbf{H}(z)$ represent the matrix transfer function of a discrete-time-invariant causal stable multichannel system with m inputs and m outputs. Then, the $m \times m$ matrix function[1]

$$\mathbf{H}(z) = \sum_{k=0}^{\infty} \mathbf{H}_k \, z^k \tag{1}$$

is analytic[2] in $|z| < 1$, and consequently it is free of poles in $|z| < 1$. The sequence $\mathbf{H}_k$, $k = 0 \to \infty$ represents the matrix impulse response of the system. If $\mathbf{H}(z)$ is rational, then every entry of $\mathbf{H}(z)$ is a rational function, and more conveniently, it can be written in the left-coprime form

$$\mathbf{H}(z) = \mathbf{P}^{-1}(z)\mathbf{Q}(z) \tag{2}$$

where $\mathbf{P}(z)$ and $\mathbf{Q}(z)$ are two matrix polynomials of orders p and q respectively. Here $\mathbf{P}(z)$ and $\mathbf{Q}(z)$ satisfy the left-coprime relation [2]

$$\mathbf{P}(z)\mathbf{X}(z) + \mathbf{Q}(z)\mathbf{Y}(z) = \mathbf{I}_m \, , \tag{3}$$

where $\mathbf{X}(z)$ and $\mathbf{Y}(z)$ are matrix polynomials and $\mathbf{P}(z)$ and $\mathbf{Q}(z)$ are unique upto multiplication by elementary polynomials[3] on their left. Similarly, in the rational case, $\mathbf{H}(z)$ also can be written in the right-coprime form

$$\mathbf{H}(z) = \mathbf{Q}_1(z)\mathbf{P}_1^{-1}(z) \, , \tag{4}$$

where $\mathbf{P}_1(z)$ and $\mathbf{Q}_1(z)$ are right-coprime matrix polynomials of orders p_1 and q_1 respectively.

If $\mathbf{H}(z)$ is also stable, then the determinants of $\mathbf{P}(z)$ and $\mathbf{P}_1(z)$ are nonzero in $|z| < 1$ and hence constant terms in $\mathbf{P}(z)$ and $\mathbf{P}_1(z)$ are nonsingular. In that case, equating (1) and (2), it is easy to show that

$$\mathbf{H}_k = -\sum_{i=1}^{p} \mathbf{P}_i \mathbf{H}_{k-i} \, , \quad k \geq p \tag{5}$$

where $\mathbf{P}_0 = \mathbf{I}$. Similarly, equating (1) and (4), we get

$$\mathbf{H}_k = -\sum_{i=1}^{p_1} \mathbf{H}_{k-i} \tilde{\mathbf{P}}_i \, , \quad k \geq p_1 \tag{6}$$

*This research work was supported by the Office of Naval Research under contract N00014-89-J-1512 P-5.

[1] Throughout this paper, upper and lower case bold letters are used to denote matrices, and the variable z is used to represent the unit delay operator. Also, $\mathbf{A}^*$ denotes the complex conjugate transpose of $\mathbf{A}$.

[2] A matrix $\mathbf{H}(z)$ is said to be analytic in some region $\mathbf{R}$, if every entry of $\mathbf{H}(z)$ is analytic in that region. A pole of $\mathbf{H}(z)$ is , by definition, the pole of at least one entry of $\mathbf{H}(z)$.

[3] A square matrix is said to be elementary (or unimodal) if its determinant is a nonzero constant.

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where $\tilde{\mathbf{P}}_0 = \mathbf{I}$. Thus, in the rational case, the impulse response matrix coefficients are linearly dependent beyond a certain stage, and letting $\mathcal{H}_k$ represent the block Hankel matrix generated from $\mathbf{H}_0, \mathbf{H}_1, \cdots \mathbf{H}_k$, it follows that

$$rank\, \mathcal{H}_{r-1} = rank\, \mathcal{H}_r\,, \quad r \geq p \text{ or } p_1\,. \tag{7}$$

Moreover, if $\mathbf{H}(z)$ possesses left- and right-coprime representations as in (2)–(4), then together with (7), it follows that $p = p_1$. Equating (1), (2) and (4), after some manipulations, it is easy to show that $q = q_1$. The above analysis shows that under some mild restrictions the left- and right-coprime representations of stable rational systems have the same order (p, q). However, as we show below it may be possible to obtain other lower order stable left and right coprime representations of rational systems that approximate the original rational system quite well.

Referring back to (1), given the partial impulse response $\mathbf{H}_k$, $k = 0 \to n$, the problem is to parametrize the class of all stable systems $\mathbf{H}(z)$ that interpolate this sequence. As we show below, this problem is closely related to the matrix bounded function extension problem, and in this context, it is useful to introduce the concept of multichannel bounded functions, and the related extension problem.

A matrix function $\mathbf{D}(z)$ is said to be bounded if

$$\begin{array}{ll} \text{(i)} & \mathbf{D}(z) \text{ is analytic in } |z| < 1 \\ \text{and} & \\ \text{(ii)} & \mathbf{I} - \mathbf{D}^*(z)\mathbf{D}(z) \geq 0\,, \quad \text{in } |z| < 1. \end{array} \tag{8}$$

The analyticity of $\mathbf{D}(z)$ in $|z| < 1$ allows the representation

$$\mathbf{D}(z) = \sum_{k=0}^{\infty} \mathbf{d}_k z^k\,, \tag{9}$$

that is valid in $|z| < 1$. If $\mathbf{D}(z)$ is rational then the above analyticity implies that $\mathbf{D}(z)$ is free of poles in $|z| \leq 1$ and hence $\mathbf{D}(z)$ represents a stable system.

From Schur's theorem $\mathbf{D}(z)$ in (9) represents a bounded function iff [3]

$$\mathbf{I} - \mathbf{D}_k^*\mathbf{D}_k \geq 0\,, \quad k = 0 \to \infty \tag{10}$$

where $\mathbf{D}_k$ represents the lower triangular block Toeplitz matrix generated from $\mathbf{d}_0 \to \mathbf{d}_k$.

Given a partial set of matrix coefficients $\mathbf{d}_0 \to \mathbf{d}_n$ that satisfy $\mathbf{I} - \mathbf{D}_n^*\mathbf{D}_n \geq 0$, the bounded function extension problem is to parametrize all functions of the form

$$\mathbf{D}(z) = \sum_{k=0}^{n} \mathbf{d}_k z^k + O(z^{n+1}) \tag{11}$$

that satisfy (8), or, equivalently (10).

An algorithm introduced by Schur to generate a sequence of new bounded functions is quite useful in this context, and using that approach it is possible to exhibit the class of all stable system transfer functions that interpolate the given partial impulse response.

2. Multichannel Schur Algorithm

Left-Inverse Form: Let $\mathbf{d}_k(z)$ represent a bounded matrix function. Then

$$\mathbf{d}_{k+1}(z) = \frac{1}{z}\mathbf{M}_k\left(\mathbf{d}_k(z) - \mathbf{S}_k\right)\left(\mathbf{I} - \mathbf{S}_k^*\mathbf{d}_k(z)\right)^{-1}\mathbf{N}_k^{-1} \tag{12}$$

represents a new bounded function [4], where $\mathbf{S}_k = \mathbf{d}_k(0)$, and $\mathbf{M}_k$ and $\mathbf{N}_k$ satisfy

$$\mathbf{M}_k^*\mathbf{M}_k = \left(\mathbf{I} - \mathbf{S}_k\mathbf{S}_k^*\right)^{-1}\,, \quad \mathbf{N}_k^*\mathbf{N}_k = \left(\mathbf{I} - \mathbf{S}_k^*\mathbf{S}_k\right)^{-1}\,. \tag{13}$$

It is easy to show that, in the rational case, the complexity of $\mathbf{d}_{k+1}(z)$ in terms of its left- or right-coprime order never exceeds that of $\mathbf{d}_k(z)$. Furthermore, model order reduction happens when the following condition is satisfied [4], i.e.,

$$\delta_0\left[\,\mathbf{d}_{k+1}(z)\,\right] = n - 1 \Longleftrightarrow \mathbf{I} - \mathbf{d}_{k*}(z)\mathbf{d}_k(z)|_{z=0} \equiv \mathbf{0}\,. \tag{14}$$

Thus, the Schur algorithm results in order reduction iff $\mathbf{d}_k(z)$ satisfies (14).

Rearranging (12), it is possible to express $\mathbf{d}_k(z)$ in terms of $\mathbf{d}_{k+1}(z)$. This gives, for $k \geq 0$,

$$\mathbf{d}_k(z) = \left[\mathbf{M}_k + z\mathbf{d}_{k+1}(z)\mathbf{N}_k\mathbf{S}_k^*\right]^{-1}\left[\mathbf{M}_k\mathbf{S}_k + z\mathbf{d}_{k+1}(z)\mathbf{N}_k\right], \tag{15}$$

and (15) represents the left-inverse form of the Schur algorithm. Using (15) iteratively n times, we obtain [4]

$$\mathbf{d}_0(z) = \left(\mathbf{A}_n(z) + z\mathbf{d}_{n+1}(z)\tilde{\mathbf{D}}_n(z)\right)^{-1}\left(\mathbf{B}_n(z) + z\mathbf{d}_{n+1}(z)\tilde{\mathbf{C}}_n(z)\right) \tag{16}$$

where $\mathbf{A}_n(z)$, $\mathbf{B}_n(z)$, $\mathbf{C}_n(z)$ and $\mathbf{D}_n(z)$ are matrix polynomials of order n that can be determined in a recursive manner as follows [4]:

$$\begin{aligned} \mathbf{A}_n(z) &= \mathbf{M}_n\left(\mathbf{A}_{n-1}(z) + z\mathbf{S}_n\tilde{\mathbf{D}}_{n-1}(z)\right) && (17) \\ \mathbf{B}_n(z) &= \mathbf{M}_n\left(\mathbf{B}_{n-1}(z) + z\mathbf{S}_n\tilde{\mathbf{C}}_{n-1}(z)\right) && (18) \\ \mathbf{C}_n(z) &= \left(\mathbf{C}_{n-1}(z) + z\tilde{\mathbf{B}}_{n-1}(z)\mathbf{S}_n\right)\mathbf{N}_n^* && (19) \\ \mathbf{D}_n(z) &= \left(\mathbf{D}_{n-1}(z) + z\tilde{\mathbf{A}}_{n-1}(z)\mathbf{S}_n\right)\mathbf{N}_n^*\,, && (20) \end{aligned}$$

with $\mathbf{A}_0(z) = \mathbf{M}_0 = \left(\mathbf{I} - \mathbf{S}_0\mathbf{S}_0^*\right)^{-1/2}$, $\mathbf{B}_0(z) = \mathbf{M}_0\mathbf{S}_0$, $\mathbf{C}_0(z) = \mathbf{N}_0^* = \left(\mathbf{I} - \mathbf{S}_0^*\mathbf{S}_0\right)^{-1/2}$ and $\mathbf{D}_0(z) = \mathbf{S}_0\mathbf{N}_0^*$.

Here, $\tilde{\mathbf{C}}_n(z) = z^n \mathbf{C}_n^*(1/z^*)$ and $\tilde{\mathbf{D}}_n(z) = z^n \mathbf{D}_n^*(1/z^*)$ for example represent matrix polynomials that are reciprocal to $\mathbf{C}_n(z)$ and $\mathbf{D}_n(z)$ respectively.

Right-Inverse Form: Once again, let $\mathbf{d}_k(z)$ represent a bounded function, and as before it is easy to show that [4]

$$\mathbf{d}_k(z) = \left[\mathbf{S}_k\mathbf{N}_k^* + z\mathbf{M}_k^*\mathbf{d}_{k+1}(z)\right]\left[\mathbf{N}_k^* + z\mathbf{S}_k^*\mathbf{M}_k^*\mathbf{d}_{k+1}(z)\right]^{-1}, \tag{21}$$

represents the right inverse form of Schur algorithm. Proceeding as in (15)–(16), equation (21) yields

$$\mathbf{d}_0(z) = \left(\mathbf{D}_n(z) + z\tilde{\mathbf{A}}_n(z)\mathbf{d}_{n+1}(z)\right)\left(\mathbf{C}_n(z) + z\tilde{\mathbf{B}}_n(z)\mathbf{d}_{n+1}(z)\right)^{-1} \tag{22}$$

where the matrix polynomials $\mathbf{A}_n(z)$, $\mathbf{B}_n(z)$, $\mathbf{C}_n(z)$ and $\mathbf{D}_n(z)$ are the same as in (16)–(19). Notice that the same set of matrix polynomials take part in both the left-inverse form and the right-inverse form of $\mathbf{d}_0(z)$.

In (16) and (22), $\mathbf{d}_{n+1}(z)$ represents an arbitray bounded matrix function, and for every such choice, $\mathbf{d}_0(z)$ represents a new bounded matrix function. It can be shown that the power series expansions of the matrix bounded functions $\mathbf{d}_0(z)$ in (16), (22) as well as $\mathbf{A}_n^{-1}(z)\mathbf{B}_n(z)$, $\mathbf{D}_n(z)\mathbf{C}_n^{-1}(z)$ all agree upto the first $n+1$ terms. Consequently, these terms must be independent of $\mathbf{d}_{n+1}(z)$, and they depend only on the polynomial matrices $\mathbf{A}_n(z)$, $\mathbf{B}_n(z)$, $\mathbf{C}_n(z)$ and $\mathbf{D}_n(z)$. Thus, for every arbitrary bounded matrix function $\mathbf{d}_{n+1}(z)$ in (16) and (22), we have the interpolation property

$$\mathbf{d}_0(z) = \sum_{k=0}^{n} \mathbf{d}_k z^k + O(z^{n+1}), \tag{23}$$

and the $\mathbf{d}_k$'s can be used to specify the matrix reflection $\mathbf{S}_k$, $k = 0 \to n$, and thereby the Schur polynomials $\mathbf{A}_n(z)$, $\mathbf{B}_n(z)$, $\mathbf{C}_n(z)$ and $\mathbf{D}_n(z)$. In particular,

$$\mathbf{S}_n = \left(\sum_{k=0}^{n-1} \mathbf{A}_k^{(n-1)}\mathbf{d}_{n-k}\right)\left(\mathbf{C}_0^{(n-1)^*} - \sum_{k=1}^{n} \mathbf{D}_{n-k}^{(n-1)^*}\mathbf{d}_{n-k}\right)^{-1}. \tag{24}$$

with $\mathbf{A}_k^(n-1)$ representing the coefficient of z^k in $\mathbf{A}_{n-1}(z)$. Equations (17)–(20) together with (13) and (24) completely specify the Schur recursions.

3. Parametrization of Rational Stable Systems

Returning back to (1), the problem is to obtain all stable systems that interpolate the partial impulse response matrix sequence $\mathbf{H}_0, \mathbf{H}_1, \cdots \mathbf{H}_n$. Notice that the impulse response sequence does not form part of a bounded function, and to make use of the above formulation, it is necessary to 'prepare' the given data. Towards this, consider the block lower triangular matrix $\mathcal{K}_n$ generated from $\mathbf{H}_0, \mathbf{H}_1, \cdots \mathbf{H}_n$ and let λ_n^2 represent the largest eigenvalue of $\mathcal{K}_n^*\mathcal{K}_n$. Then, clearly, the sequence

$$\mathbf{d}_k = \frac{1}{\kappa_n}\mathbf{H}_k, \quad \kappa_n > \lambda_n, \quad k = 0 \to n \tag{25}$$

satisfies (12) up to n, and hence these coefficient matrices qualify as the first $(n+1)$ coefficients of a bounded matrix function.

If $\mathbf{H}(z)$ or equivalently $\mathbf{d}_0(z)$ is rational to start with, then application of the Schur algorithm as in (12) or (21) will result in rational bounded matrix functions $\mathbf{d}_k(z)$, $k \geq 1$, whose complexity in terms of matrix orders does not exceed that of $\mathbf{H}(z)$. In fact, order reduction happens at some stage only if (14) is satisfied at that stage, and otherwise the order of $\mathbf{d}_k(z)$ remains the same as that at the previous stage. Thus, if $\mathbf{H}(z)$ represents a rational stable multichannel system with left-coprime representation of order (p,q) as in (2), with $p > q$, then as (14) shows the left-coprime order of $\mathbf{d}_{n+1}(z)$ is in general p, and moreover it has the representation

$$\mathbf{d}_{n+1}(z) = \mathbf{G}^{-1}(z)\mathbf{F}(z), \tag{26}$$

where $\mathbf{G}(z)$ and $\mathbf{F}(z)$ are matrix polynomials with orders p and $p-1$ respectively. Similarly the right coprime representation of $\mathbf{d}_{n+1}(z)$ can be expressed as

$$\mathbf{d}_{n+1}(z) = \mathbf{F}_1(z)\mathbf{G}_1^{-1}(z). \tag{27}$$

Substituting (26)–(27) into (16) and (22), we obtain

$$\begin{aligned}
\mathbf{H}(z) &= \kappa_n\left(\mathbf{G}(z)\mathbf{A}_n(z) + z\mathbf{F}(z)\tilde{\mathbf{D}}_n(z)\right)^{-1} \\
&\quad \times \left(\mathbf{G}(z)\mathbf{B}_n(z) + z\mathbf{F}(z)\tilde{\mathbf{C}}_n(z)\right) \\
&= \kappa_n\left(\mathbf{D}_n(z)\mathbf{G}_1(z) + z\tilde{\mathbf{A}}_n(z)\mathbf{F}_1(z)\right) \\
&\quad \times \left(\mathbf{C}_n(z)\mathbf{G}_1(z) + z\tilde{\mathbf{B}}_n(z)\mathbf{F}_1(z)\right)^{-1}
\end{aligned} \tag{28}$$

where the two forms in (28) represent the left- and right- coprime representations of $\mathbf{H}(z)$ respectively. Conversely, every rational system after n subsequent applications of the Schur algorithm has the above form, where $\mathbf{G}^{-1}(z)\mathbf{F}(z)$ and $\mathbf{F}_1(z)\mathbf{G}_1^{-1}(z)$ are two unique bounded matrix functions as in (26)–(27), and hence we can make use of the matrix order constraints of $\mathbf{H}(z)$ to obtain these unknown bounded functions in each case. Towards this, notice that the formal order of (28) is $n+p$, and to respect the MARMA(p,q) nature of $\mathbf{H}(z)$, we can equate the coefficients of the higher order terms to zero in (28).

Left-Coprime Representation: Referring to the left inverse form in (28), we have

$$\mathbf{H}(z) = \mathbf{P}^{-1}(z)\mathbf{Q}(z)\,, \tag{29}$$

where

$$\mathbf{P}(z) = \mathbf{G}(z)\mathbf{A}_n(z) + z\mathbf{F}(z)\tilde{\mathbf{D}}_n(z) \triangleq \sum_{k=0}^{n+p} \mathbf{P}_k z^k \tag{30}$$

and

$$\mathbf{Q}(z) = \kappa_n \left[\mathbf{G}(z)\mathbf{B}_n(z) + z\mathbf{F}(z)\tilde{\mathbf{C}}_n(z) \right] \triangleq \sum_{k=0}^{n+p} \mathbf{Q}_k z^k\,, \tag{31}$$

and to maintain $\mathbf{H}(z) \sim \text{MARMA}(p,q)$, we must have

$$\mathbf{P}_k = 0\,,\; k = p{+}1 \to p{+}n\,,\; \mathbf{Q}_k = 0\,,\; k = q{+}1 \to p{+}n\,. \tag{32}$$

It can be shown that [4],

$$\mathbf{P}_k = 0\,, k = p{+}1 \to p{+}n \Leftrightarrow \mathbf{Q}_k = 0\,, k = p{+}1 \to p{+}n\,. \tag{33}$$

As a result, we obtain n equations from the coefficients of z^{p+1}, z^{p+2}, $\cdots z^{p+n}$ and $p-q$ equations by equating the remaining $\mathbf{Q}_k = 0$, $k = q+1 \to p$ in (32). Thus we have $n+p-q$ linear block matrix equations and $2p$ unknowns $\mathbf{G}_k$, $k = 1 \to p$ and $\mathbf{F}_k$, $k = 0 \to p-1$. Clearly, for a unique solution $n \geq p+q$, and the minimum value of n is given by $p+q$. In that case, $n = p+q$ and the resulting $2p$ equations in $2p$ unknowns can be represented as

$$\mathbf{X}\mathbf{A} = \mathbf{b} \tag{34}$$

where $\mathbf{A}$ is given by

$$\left[\begin{array}{cccc|cccc}
\mathbf{A}_{p+q} & \mathbf{A}_{p+q-1} & \cdots & \mathbf{A}_1 & \mathbf{B}_0 & 0 & \cdots & 0 \\
0 & \mathbf{A}_{p+q} & \cdots & \mathbf{A}_2 & \mathbf{B}_1 & \mathbf{B}_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \mathbf{A}_{p-1} & \mathbf{B}_{p-2} & \mathbf{B}_{p-3} & \cdots & \mathbf{B}_{q-1} \\
0 & 0 & \cdots & \mathbf{A}_p & \mathbf{B}_{p-1} & \mathbf{B}_{p-2} & \cdots & \mathbf{B}_q \\
\hline
\mathbf{D}_0^* & \mathbf{D}_1^* & \cdots & \mathbf{D}_{p+q-1}^* & \mathbf{C}_{p+q}^* & 0 & \cdots & 0 \\
0 & \mathbf{D}_0^* & \cdots & \mathbf{D}_{p+q}^* & \mathbf{C}_{p+q-1}^* & \mathbf{C}_{p+q}^* & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \mathbf{D}_{q+1}^* & \mathbf{C}_{q+2}^* & \mathbf{C}_{q+3}^* & \cdots & \mathbf{C}_{p+1}^* \\
0 & 0 & \cdots & \mathbf{D}_q^* & \mathbf{C}_{q+1}^* & \mathbf{C}_{q+2}^* & \cdots & \mathbf{C}_p^*
\end{array}\right], \tag{35}$$

$$\mathbf{X} \triangleq [\mathbf{G}_p \;\cdots\; \mathbf{G}_2\; \mathbf{G}_1 \mid \mathbf{F}_{p-1} \;\cdots\; \mathbf{F}_1\; \mathbf{F}_0] \tag{36}$$

and

$$\mathbf{b} \triangleq [\mathbf{0} \;\cdots\; \mathbf{0}\; \mathbf{A}_{p+q} \;\cdots\; \mathbf{A}_{p+1} \mid \mathbf{B}_p \;\cdots\; \mathbf{B}_{q+1}]\,. \tag{37}$$

Here $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{C}_k$, $\mathbf{D}_k$, $k = 0 \to p+q$ represent the matrix coefficients of the Schur polynomials $\mathbf{A}_{p+q}(z)$, $\mathbf{B}_{p+q}(z)$, $\mathbf{C}_{p+q}(z)$ and $\mathbf{D}_{p+q}(z)$ respectively. Notice that at the correct stage, (34) is guaranteed to have a solution that results in a bounded function for $\mathbf{G}^{-1}(z)\mathbf{F}(z)$, and the parameters of the left-coprime representation of $\mathbf{H}(z)$ can be expressed as, for $k = 0 \to p$,

$$\mathbf{P}_k = \sum_{i=0}^{k} \mathbf{G}_i \mathbf{A}_{k-i} + \sum_{i=1}^{k} \mathbf{F}_{i-1} \mathbf{D}_{p+q-k+i}^* \tag{38}$$

and, for $k = 0 \to q$,

$$\mathbf{Q}_k = \kappa_n \left(\sum_{i=0}^{k} \mathbf{G}_i \mathbf{B}_{k-i} + \sum_{i=1}^{k} \mathbf{F}_{i-1} \mathbf{C}_{p+q-k+i}^* \right)\,, \tag{39}$$

with $\mathbf{G}_0 = \mathbf{I}$. The stability of $\mathbf{H}(z)$ and the required interpolation property follow from the bounded character of $\mathbf{G}^{-1}(z)\mathbf{F}(z)$. Since $\mathbf{P}(z)$ and $\mathbf{Q}(z)$ in (38)–(39) are computed without involving any spectral factorization, the nonminimum phase[4] characteristics of $\mathbf{H}(z)$ is preserved in this case.

The above computations assume the matrix orders p and q are known. Usually, these quantities are unknown, and they have to be evaluated from the given data. As we show below, the invariant characteristics of the bounded functions $\mathbf{d}_{p+q+1}(z)$ and $\mathbf{d}_{p+q+2}(z)$ in terms of their matrix orders, together with the Schur update rule can be used to determine p and q.

4. Model Order Selection:

Having determined $\mathbf{d}_{p+q+1}(z) = \mathbf{G}^{-1}(z)\mathbf{F}(z)$ at stage $n = p+q$, the Schur procedure can be repeated once again to obtain the next bounded matrix function

$$\mathbf{d}_{p+q+2}(z) = \mathbf{E}^{-1}(z)\mathbf{J}(z)\,. \tag{40}$$

Here, the bounded matrix functions $\mathbf{d}_{p+q+1}(z) = \mathbf{G}^{-1}(z)\mathbf{F}(z)$ and $\mathbf{d}_{p+q+2}(z) = \mathbf{E}^{-1}(z)\mathbf{J}(z)$ are related through (21) and, after a series of algebraic manipulations, we obtain the following criteria for model order determination [4]

$$\epsilon_0 \triangleq \mathbf{E}_p \mathbf{M}_{p+q+1} \mathbf{S}_{p+q+1} + \mathbf{J}_{p-1} \mathbf{N}_{p+q+1} \equiv \mathbf{0}\,. \tag{41}$$

From the above analysis, to obtain a 'reduced order' stable representation of a given system, it is sufficient to determine a stage (p_0, q_0) where the bounded function $\mathbf{d}_{p_0+q_0+1}(z)$ exists. Naturally this procedure can be also used to obtain stable rational approximation of nonrational systems as well [4].

Figure 1 shows the reduced order representation of a rational system with McMillan degree equals to 18. In Fig. 1, Solid lines represent the magnitude and

[4] A matrix function $\mathbf{H}(z)$ is said to be minimum phase if its determinant is free of poles and zeros in $|z| < 1$. A nonminimum phase system has no restrictions on the locations of its zeros.

phase plots of the original multichannel system. The dotted lines represent the left-coprime representation of the reduced stable system of degree 16 obtained using the present technique. The original rational system is given by

$$\mathbf{H}_o(z) = \begin{bmatrix} H_{11}(z) & H_{12}(z) \\ H_{21}(z) & H_{22}(z) \end{bmatrix},$$

where

$$H_{11} = \frac{-0.188+0.208z-0.231z^2}{1.0-1.342z+1.292z^2-0.685z^3+0.260z^4}$$

$$H_{12} = \frac{0.033-0.033z+0.069z^2+0.036z^3+0.041z^4}{1.0+0.261z+0.255z^2+0.073z^3-0.121z^4-0.017z^5+0.128z^6}$$

$$H_{21} = \frac{0.066-0.251z+0.536z^2-0.231z^3}{1.0-2.208z+2.681z^2-1.873z^3+0.708z^4}$$

$$H_{22} = \frac{-0.109-0.308z-0.435z^2+0.385z^3-0.170z^4}{1.0+0.167z+0.177z^2+0.085z^3-0.170z^4}.$$

The left-coprime representation using the proposed technique is given by

$$\mathbf{H}(z) = \mathbf{P}^{-1}(z)\mathbf{Q}(z)$$

where

$$\mathbf{P}(z) = \mathbf{I} + \begin{bmatrix} -0.027 & -0.093 \\ 0.000 & -2.041 \end{bmatrix} z + \begin{bmatrix} 0.403 & 0.103 \\ 0.000 & 2.489 \end{bmatrix} z^2 - \begin{bmatrix} 0.047 & 0.074 \\ -0.00 & 1.731 \end{bmatrix} z^3 + \begin{bmatrix} 0.309 & -0.008 \\ 0.000 & 0.512 \end{bmatrix} z^4 + \begin{bmatrix} -0.072 & 0.034 \\ 0.000 & 0.389 \end{bmatrix} z^5 + \begin{bmatrix} 0.112 & -0.031 \\ 0.000 & -0.489 \end{bmatrix} z^6 + \begin{bmatrix} -0.007 & -0.014 \\ 0.000 & -0.12 \end{bmatrix} z^7 - \begin{bmatrix} 0.007 & 0.037 \\ -0.00 & 0.120 \end{bmatrix} z^8$$

and

$$\mathbf{Q}(z) = \begin{bmatrix} -0.188 & 0.033 \\ 0.066 & -0.109 \end{bmatrix} - \begin{bmatrix} 0.045 & 0.032 \\ 0.239 & 0.067 \end{bmatrix} z + \begin{bmatrix} -0.106 & 0.101 \\ 0.505 & -0.047 \end{bmatrix} z^2 + \begin{bmatrix} -0.171 & 0.018 \\ -0.180 & 0.723 \end{bmatrix} z^3 + \begin{bmatrix} -0.195 & 0.000 \\ 0.024 & -1.687 \end{bmatrix} z^4 + \begin{bmatrix} -0.031 & 0.064 \\ 0.047 & 2.004 \end{bmatrix} z^5 - \begin{bmatrix} -0.008 & 1.359 \\ 0.111 & 1.485 \end{bmatrix} z^6 + \begin{bmatrix} 0.005 & 0.006 \\ 0.040 & 0.591 \end{bmatrix} z^7 + \begin{bmatrix} 0.000 & 2.316 \\ 0.000 & -0.120 \end{bmatrix} z^8.$$

5. Conclusions

A new approach to the identification and lower order approximation of multichannel stable systems from their impulse response data is described in this paper. This is achieved by making use of the left- and right-Schur algorithms, and in this context every stable system is represented in terms of four Schur polynomials, and an arbitrary bounded function. The central quantity behind every stable system is shown to be a bounded function, and its determination simultaneously determines the "denominator" as well as the "numerator" coefficients of the multichannel system.

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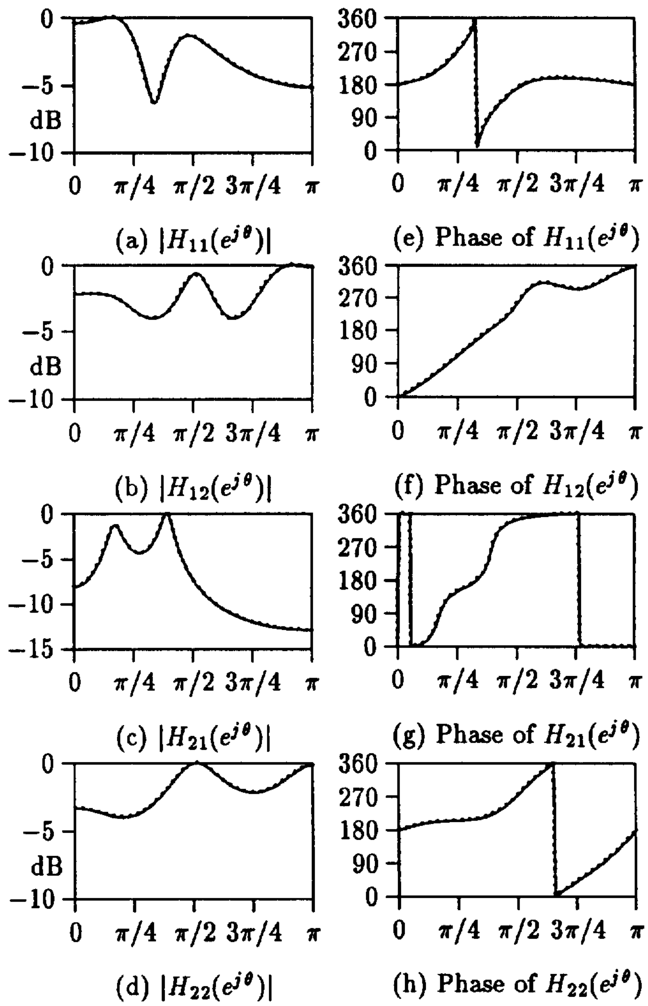

Figure 1: