Analytic Interpolation Problems and Lattice Filter Design

T. Boros J. Kamali T. Kailath

Information Systems Laboratory
Stanford University
Stanford, CA 94305

Abstract

In this paper the so-called *analytic* interpolation problem is addressed and solved. The objective is to find the family of rational interpolants which are analytic in a certain region of the complex plane. It turns out that the usual linear fractional map cannot be used to describe the solution set conveniently. Instead, an *affine* parametrization formula is proposed as the natural framework to impose analyticity constraint on the interpolants. All solutions of the interpolation problem are characterized in terms of a *generating system*, which can be obtained efficiently via a fast recursive algorithm. The recursive procedure can be used to update the solutions whenever a new interpolation constraint is added to the input data set. It is shown that the analytic interpolation problem is solvable if and only if the corresponding unconstrained problem is solvable, *i.e.*, if and only if the interpolation data-set is consistent. The above results have many applications in different areas such as stable lattice filter design, channel identification, and Q-parametrization of stabilizing controllers.

1 Introduction

Rational interpolation problems have many applications in various fields, such as digital filter design, circuit theory, system identification, and control theory. Interpolation problems can be classified according to the additional constraints which are imposed on the rational interpolants. Hence, in *Schur-type* problems (such as the Nevanlinna-Pick, Carathéodory-Fejér, and Hermite-Fejér problems) the rational interpolants have to be Schur functions (with H^∞-norm smaller than one) [11]. In *minimal* interpolation problems the McMillan degrees of the rational interpolants have to be as small as possible [1],[2],[6]. In *unconstrained* interpolation problems the interpolants do not have to satisfy any additional requirement [4],[3],[5].

In this paper we consider the *analytic* interpolation problem in which the rational interpolants have to be analytic in a certain subset of the complex plane. In engineering terminology we wish to parametrize the family of all *stable* linear filters whose transfer functions satisfy the given interpolation conditions.

2 Problem Statement

In a general tangential framework the analytic interpolation problem can be stated as follows:

Problem 1 (Analytic Interpolation Problem) *Let $\alpha_0, \ldots, \alpha_{n-1}$ complex points, and let σ be an arbitrary subset of the complex plane (σ may or may not contain α_i). With each point α_i associate a positive integer r_i and two row vectors u_i and v_i partitioned as*

$$u_i = \begin{bmatrix} u_1^{(i)} & u_2^{(i)} & \ldots & u_{r_i}^{(i)} \end{bmatrix},$$
$$v_i = \begin{bmatrix} v_1^{(i)} & v_2^{(i)} & \ldots & v_{r_i}^{(i)} \end{bmatrix},$$
$$u_j^{(i)} \in \mathbb{C}^{1\times p}, \qquad v_j^{(i)} \in \mathbb{C}^{1\times q}.$$

Given the nodes α_i and the associated vectors u_i and v_i, parametrize all rational interpolants $Y(z) \in \mathbb{C}^{p\times q}(z)$ which are analytic in $\sigma \cup \{\alpha_0, \ldots, \alpha_{n-1}\}$, and satisfy the tangential interpolation conditions

$$u_i \mathcal{T}_Y^{r_i}(\alpha_i) = v_i, \qquad i \in \{0, 1, \ldots, m-1\}, \tag{1}$$

where the block-Toeplitz operator $\mathcal{T}_Y^k(z)$ is defined as

$$\mathcal{T}_Y^k(z) = \begin{bmatrix} Y(z) & \frac{1}{1!}Y^{(1)}(z) & \frac{1}{2!}Y^{(2)}(z) & \cdots & \frac{1}{(r_i-1)!}Y^{(k-1)}(z) \\ & Y(z) & \frac{1}{1!}Y^{(1)}(z) & \cdots & \frac{1}{(r_i-2)!}Y^{(k-2)}(z) \\ & & Y(z) & \cdots & \frac{1}{(r_i-3)!}Y^{(k-3)}(z) \\ & & & \ddots & \vdots \\ & & & & Y(z) \end{bmatrix}.$$

■

In the scalar case Problem 1 will have following simple form:

Problem 2 (Scalar Case) *Given the distinct points $\alpha_0, \ldots, \alpha_{n-1}$, the associated multiplicities $r_0, \ldots, r_{n-1}$, and the complex numbers β_{ij}, $i \in \{0, \ldots, m-1\}$, $j \in \{0, \ldots, r_i - 1\}$, find all rational functions $y(z)$ which are analytic in $\sigma \cup \{\alpha_0, \ldots, \alpha_{n-1}\}$ such that*

$$y^{(j)}(\alpha_i) = \beta_{ij}, \qquad \begin{array}{l} i \in \{0, \ldots, m-1\}, \\ j \in \{0, \ldots, r_i - 1\}. \end{array}$$

■

In particular cases, the set σ can be the open unit disc, or the open right half plane, etc.

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3 Solvability Condition

The analytic interpolation problem is solvable whenever the interpolation data is consistent. In order to give a necessary and sufficient condition for solvability, we construct two arrays F and G from the interpolation data as follows:

$$F = \begin{bmatrix} F_0 & & & \\ & F_1 & & \\ & & \ddots & \\ & & & F_{m-1} \end{bmatrix}, \quad (2)$$

$$G = \begin{bmatrix} U_0 & V_0 \\ U_1 & V_1 \\ \vdots & \vdots \\ U_{m-1} & V_{m-1} \end{bmatrix} \equiv [\, U \quad V \,], \qquad J = I_p \oplus -I_q,$$

where $F_i \in \mathbb{C}^{r_i \times r_i}$, $U_i \in \mathbb{C}^{r_i \times p}$, and $V_i \in \mathbb{C}^{r_i \times q}$ are given by

$$F_i = \begin{bmatrix} \alpha_i & & & \\ 1 & \alpha_i & & \\ & \ddots & \ddots & \\ & & 1 & \alpha_i \end{bmatrix}, \quad U_i = \begin{bmatrix} u_1^{(i)} \\ u_2^{(i)} \\ \vdots \\ u_{r_i}^{(i)} \end{bmatrix}, \quad V_i = \begin{bmatrix} v_1^{(i)} \\ v_2^{(i)} \\ \vdots \\ v_{r_i}^{(i)} \end{bmatrix}.$$

Now the following theorem applies.

Theorem 1 (Solvability Condition) *Problem 1 is solvable if and only if the columns of V lie in the controllable subspace of the pair $\{F, U\}$*, i.e.,

$$Im\, V \in Im [\, U \quad F\, U \quad F^2\, U \quad \ldots \quad F^{r_{\max}-1}\, U \,],$$

where $r_{\max} = \max\{r_0, r_1, \ldots, r_{m-1}\}$. ■

Algorithm 2 in the next section provides a computationally oriented recursive procedure for checking the consistency of the input data.

It is interesting to note that the solvability condition of the analytic interpolation problem coincides with the solvability condition of the unconstrained interpolation problem (see [5]). In other words, if there exist an unstable system satisfying some given interpolation conditions then there must exist a family of stable systems which satisfy the same interpolation conditions.

4 Main Algorithm

Our approach to analytic interpolation is based on displacement structure concepts (see, *e.g.*, [10], [11], [5]). We give a recursive solution to the problem by factorizing a structured matrix R which satisfies the non-Hermitian displacement equation

$$R - F\, R\, A^* = G\, J\, B^*. \quad (3)$$

The advantage of the recursive method is that the rational interpolants can be simply updated whenever a new interpolation condition is added to the input data-set. The arrays F and G in (3) are constructed directly from the interpolation data as shown in (2), while the arrays A and B are free parameters which can be used to impose additional constraints, such as analyticity, on the rational interpolants. In order to obtain a recursive algorithm, we need to assume that A is lower triangular.

In what follows we slightly modify the main algorithm in [5] in order to incorporate the analyticity constraint on the interpolants. Throughout the discussion $f_0, \ldots, f_{n-1}$ and $a_0, \ldots, a_{n-1}$ denote the diagonal elements of F and A.

Algorithm 2 (Main Algorithm) *Start the recursion with $G_0 = G$, and $F_0 = F$. Repeat the following steps for $i = 0, 1, \ldots, n-1$:*

Step 1 *Let $g_i = [\, g_{ip} \quad g_{iq} \,]$, where g_{ip} denotes the first p elements and g_{iq} denotes the last q elements of g_i.*

- *If $g_{ip} = 0$, but $g_{iq} \neq 0$ then the interpolation data is contradictory.*
- *If both $g_{ip} = 0$ and $g_{iq} = 0$ then the interpolation data is redundant. Set $\Theta_i = I_{p+q}$, and go to Step 3.*

Step 2 *Choose a non-singular matrix Θ_i which transforms g_i into proper form with a single non-zero entry in one of the first p positions:*

$$g_i \Theta_i = [\, \bar{g}_{ip} \;\; 0 \,], \quad \text{where } \bar{g}_{ip} = [0 \ldots 0 * 0 \ldots 0].$$

Step 3 *Choose $a_i \in \mathbb{C}$ so as to satisfy*

$$1/a_i^* \notin \Omega \cup \{f_0, \ldots, f_{n-1}\}$$

Step 4 *Obtain the lattice section $\Theta_i(z)$ as*

$$\Theta_i(z) = \Theta_i \begin{bmatrix} I_{j-1} & & \\ & \frac{z - f_i}{1 - a_i^* z} & \\ & & I_{r-j} \end{bmatrix}.$$

Step 5 *Update the generator G_i as*

$$\begin{bmatrix} 0 \\ G_{i+1} \end{bmatrix} = \Phi_i\, G_i\, \Theta_i \begin{bmatrix} 0_{j-1} & & \\ & 1 & \\ & & 0_{r-j} \end{bmatrix} + G_i\, \Theta_i \begin{bmatrix} I_{j-1} & & \\ & 0 & \\ & & I_{r-j} \end{bmatrix},$$

where $\Phi_i = (F_i - f_i\, I)\,(I - a_i^*\, F_i)^{-1}$. The matrix F_i is obtained from F by deleting the first i rows and columns. ■

Remarks:

1. The above algorithm automatically checks whether the interpolation data is contradictory, or redundant.

2. The analyticity constraint on the interpolants is imposed in Step 3.

3. There is no need to choose the diagonal elements of A in advance. They can be chosen "dynamically" at the moment when they are needed in the algorithm.

4. Although we do not need this feature for solving the interpolation problem, Algorithm 2 obtains an LDU decomposition for the non-Hermitian matrix R (see [5]). Moreover, the presented scheme requires only $\mathcal{O}(rn^2)$ operations, whereas a general LDU decomposition method (such as Gauss elimination) would require $\mathcal{O}(n^3)$ operations.

5 Linear Fractional Parametrization

Algorithm 2 gives rise to the cascade system

$$\Theta(z)=\Theta_0(z)\Theta_1(z)\ldots\Theta_{n-1}(x)=\begin{bmatrix}\Theta_{11}(z) & \Theta_{12}(z)\\ \Theta_{21}(z) & \Theta_{22}(z)\end{bmatrix},$$

which allows us to parametrize the family of analytic interpolants (see [4] and [5]).

Theorem 3 (Linear Fractional Param.) *All solutions of Problem 1 are given by the linear fractional formula*

$$Y(z) = -(\Theta_{11}(z)P(z) + \Theta_{12}(z)Q(z)) \times(\Theta_{21}(z)P(z) + \Theta_{22}(z)Q(z))^{-1}, \quad (4)$$

where the rational parameters $P(z)$ and $Q(z)$ must be analytic in $\sigma\cup\{\alpha_0,\ldots,\alpha_{n-1}\}$, and they must be chosen so as to satisfy the additional condition

$$\det\{\Theta_{21}(z)P(z)+\Theta_{22}(z)Q(z)\}\neq 0 \quad \text{in} \quad z\in\sigma\cup\{\alpha_0,\ldots,\alpha_{n-1}\}. \quad (5)$$

■

Remark: Note that the special choices $P(z) = 0$ and $Q(z) = I$ satisfy condition (5), therefore $Y(z) = \Theta_{12}(z)\Theta_{22}^{-1}(z)$ is a particular solution of the analytic interpolation problem.

The linear fractional map (4) has an interpretation in terms of feed-back lattice filters (or, equivalently, in terms of discretized transmission lines). The interpolating function $Y(z)$ is obtained as the negative transfer function from the p top-left inputs to the q bottom-left outputs of the so-called *scattering cascade*

$$\Sigma(z)\overset{\text{def}}{=}\begin{bmatrix}\Theta_{11}-\Theta_{12}\Theta_{22}^{-1}\Theta_{21} & -\Theta_{12}\Theta_{22}^{-1}\\ \Theta_{22}^{-1}\Theta_{21} & \Theta_{22}^{-1}\end{bmatrix},$$

when the right-hand side is attached to the load $K(z) = -P_R(z)Q_R^{-1}(z)$ (see Fig. 1). Each step of the recursive algorithm gives rise to a first order section in the cascade system $\Sigma(z)$. The sections $\Sigma_i(z)$ are designed in such a way that the closed-loop transfer function $Y(z)$ satisfies the prescribed interpolation conditions independently of the load $K(z)$. The interpolating function $Y(z)$ can be updated by adjoining additional sections to the cascade system $\Sigma(z)$.

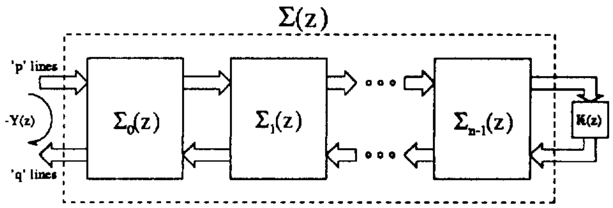

Figure 1: Lattice filter interpretation of the linear fractional parametrization formula.

6 Affine Parametrization

Unfortunately, the characterization in (4) cannot be readily used in practice, because it is not immediately clear how to check whether the parameters $P(z)$ and $Q(z)$ satisfy condition (5). In order to overcome this difficulty we propose another method in which the linear fractional formula (4) is replaced by an *affine* parametrization formula.

Lemma 4 *Assume that Θ_i at Step 2 of Algorithm 2 is constructed in upper triangular form, so that*

$$g_i\Theta_i=\begin{bmatrix}0 & \ldots & 0 & * & * & \ldots & *\end{bmatrix}\begin{bmatrix}1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 1 & * & \cdots & * \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1\end{bmatrix}=\begin{bmatrix}0 & \ldots & 0 & * & 0 & \ldots & 0\end{bmatrix},$$

where '$*$' denotes non-zero elements. Then the generating system $\Theta(z)$ must have the form

$$\Theta(z) = \begin{bmatrix}\Pi(z) & -H(z)\\ 0 & I_q\end{bmatrix},$$

where $\Pi(z)\in\mathbb{C}^{p\times p}(z)$ and $H(z)\in\mathbb{C}^{p\times q}(z)$ are rational matrices such that

$$\det\Pi(z)\sim\prod_{i=0}^{m-1}(z-\alpha_k)^{r_k}.$$

■

An upper triangular generating system, in turn, gives rise to the following affine parametrization formula:

Theorem 5 (Affine Parametrization) *All solutions of Problem 1 can be parametrized as*

$$Y(z) = H(z) + \Pi(z)K(z), \quad (6)$$

where the rational parameter $K(z)$ must be analytic in $\sigma\cup\{\alpha_0,\ldots,\alpha_{n-1}\}$. ■

Remark: It follows from (6) that $H(z)$ itself is a particular solution of the analytic interpolation problem. If a_i is chosen so that $a_i = 0$ for all i, then $H(z)$ must be a *polynomial* (rather than a rational) matrix function. If, in addition, $p = q = 1$ then $H(z)$ is identical to the *Hermite interpolating polynomial* (see, *e.g.*, [9]).

Figure 2 shows the lattice-filter interpretation of the affine parametrization formula in the scalar case when $p = q = 1$, and $a_0 = \ldots = a_{n-1} = 0$. Since $\Theta_i(z)$ is upper triangular for all i, the lattice filter degenerates into a *transversal filter*. The scattering cascade $\mathbf{\Sigma}(z)$

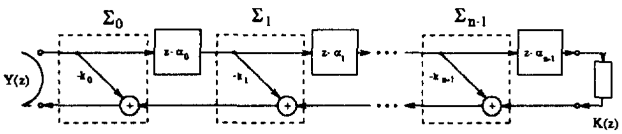

Figure 2: Lattice filter interpretation of the affine parametrization formula.

in the above example plays the role of a *universal prefilter*. It modifies the transfer function of an arbitrary stable system $K(z)$ so that the overall transfer function $Y(z)$ remains stable and, in addition, it satisfies the prescribed interpolation conditions. The prefilter $\mathbf{\Sigma}(z)$ depends only on the interpolation data, and it is independent of the stable system $K(z)$.

7 Summary and Conclusions

We have presented a fast recursive procedure which can be used to solve analytic rational interpolation problems. We have parametrized the family of analytic interpolants in terms of an affine parametrization formula. Such results have applications in various fields such as stable IIR filter design, system identification, deconvolution, and optimal control.

For example, in a system identification problem the partial input and output sequences $u_0, \ldots, u_{n-1}$, and $v_0, \ldots, v_{n-1}$ are measured (see Fig. 3), and the objective is to determine a stable rational transfer function $Y(z)$ which satisfies the following relation:

$$\begin{bmatrix} u_0 & \cdots & u_{n-1} \end{bmatrix} \begin{bmatrix} Y(0) & \cdots & \frac{1}{(n-1)!}Y^{(n-1)}(0) \\ & \ddots & \\ & & Y(0) \end{bmatrix} = \begin{bmatrix} v_0 & \cdots & v_{n-1} \end{bmatrix}.$$

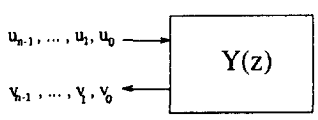

Figure 3: System identification.

The analytic interpolation problem has another interesting application in control theory in connection with the Q-parametrization of stabilizing controllers. Namely, all controllers $C(z)$ which stabilize a given plant $P(z)$ can be parametrized as $C(z) = Q(z)\,(I - P(z)\,Q(z)\,)^{-1}$, where $Q(z)$ is a suitable proper rational matrix function. If the plant $P(z)$ is stable, the only requirement on the parameter $Q(z)$ is stability, *i.e.*, analyticity in the exterior of the open unit disc (see [7]). If, however, the plant $P(z)$ is unstable then $Q(z)$ has to satisfy certain interpolation conditions at the unstable poles of $P(z)$.

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