Least p^{th} Power Design and Characterization of Affine Phase Complex 2-D Filters and the Min-Max Approximation

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Abstract

Affine phase 2-D filters are designed using the p^{th} power norm to measure its deviation from an idealized afiine phase filter response not necessarily symmetric. Necessary and sufficient conditions are derived in terms of its coefficients for the filter to be affine phase. The 2-D design is obtained in closed form for $p=2$. Next the design is generalized for $2<p$ where the solution is not obtainable in closed form but by the use of the complex Newton method iteration. Convergence is attained after a modest number of iterations by making p increase by a constant factor at each iteration. For large p solution approaches the Min-Max. Several examples provided.

1. Introduction

Complex coefficients 1-D FIR filters have recently been developed, [2], for radar/sonar clutter supression and other applications where nonsymmetrical filter response is desired. Complex coefficient FIR 1-D filter design has been treated by a few authors; more notably Preuss derived the theory and algorithm for the Min-Max criterion, [1] and created a Remez-type algorithm. Jaffer in [2], uses a least squares to obtain the complex coefficients of a FIR filter in closed form. A characterization of affine phase complex FIR filters is also given in [2]. Proper attention is given to the filter response in transition bands as well. Using the least p^{th} power filter design is not new. Fahmy and Lampropoulos, [5], proposed a modified p^{th} power criterion to design real 2-D filters however their Newton iterates were not computed in closed form nor any comments were made relative to the behaviour of the solution for large values of p. More recently the Iterative Reweighted Least Squares (IRLS) technique has been applied, [6], to compute the least p^{th} power filter design for real FIR filters. This approach is essentially a rearrangement of the Newton method and it is unusual in the sense that the power p is gradually increased in the successive iterations. The emphasis of this paper is complex 2-D FIR filter design and characteristic properties. We derive structure results as well as efficient algorithms for the filter formation. Our aim is the efficient computation of the least p^{th} power and and use it as an approximation to the minmax design for which there is no characterization in 2D. We treat the problem entirely from a complex variables point of view and do not decompose the filter coefficients into its real and imaginary components. This we believe adds elegance and compactness to our solution. The contribution of this paper are:

1. Necessary and sufficient conditions for complex FIR 2-D filter to posses an affine phase.
2. Closed form derivation of affine phase filter coefficients for $p = 2$.
3. Application of the complex Newton iteration to obtain the least p^{th} power solution for $p > 2$. Computation of complex gradient and Hessians in closed form.
4. Examples of 2-D filter design with $p = 2$, 15, and 60

2. Conditions for Complex 2-D FIR Filters to Possess Linear Phase

Linear phase response filters are desirable because sinusoids in their passband are processed without any phase distortion. Jaffer in [2] derived necessary and sufficient conditions for complex 1-D FIR filters to have an affine phase. Here we extend his results to the 2-D case. Let

$$H(f_1,f_2)=\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}h_{n,m}e^{-2\pi jnf_1}e^{-2\pi jmf_2}$$

be the frequency response of a 2-D complex FIR filter with coefficients $\{h_{n,m}\}_{n,m}$. We will require the phase of $H(f_1,f_2)$ be affine, i.e.,

$$H(f_1,f_2)=|H(f_1,f_2)|e^{-2\pi j\alpha_1 f_1}e^{-2\pi j\alpha_2 f_2}e^{j\beta}$$

What conditions on the filter coefficients does this entail? Note that $H(f_1,f_2)e^{2\pi j\alpha_1 f_1}e^{2\pi j\alpha_2 f_2}e^{-j\beta}$ is real for all (f_1,f_2). This implies that

$$H(f_1,f_2)e^{2\pi j\alpha_1 f_1}e^{2\pi j\alpha_2 f_2}e^{-j\beta}=\overline{H}(f_1,f_2)e^{-2\pi j\alpha_1 f_1}e^{-2\pi j\alpha_2 f_2}e^{j\beta}$$

for all (f_1,f_2).

Expressing this equation in terms of the filter coefficients we obtain

$$\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}h_{n,m}e^{-2\pi jnf_1}e^{-2\pi jmf_2}=\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\overline{h}_{n,m}e^{2\pi jnf_1}e^{2\pi jmf_2}e^{-4\pi j\alpha_1 f_1}e^{-4\pi j\alpha_2 f_2}e^{2j\beta}$$

We now write this equations in terms of summations involving only the variable n and f_1.

$$\sum_{n=0}^{N-1}\left(\sum_{m=0}^{M-1}h_{n,m}e^{-2\pi jmf_2}\right)e^{-2\pi jnf_1}=\sum_{n=0}^{N-1}\left(\sum_{m=0}^{M-1}\overline{h}_{n,m}e^{2\pi jmf_2}e^{-4\pi j\alpha_2 f_2}e^{2j\beta}\right)e^{2\pi jnf_1}e^{-4\pi j\alpha_1 f_1}$$

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Let $c_n(f_2)=\sum_{m=0}^{M-1} h_{n,m} e^{-2\pi jmf_2}$, then we get

$$\sum_{n=0}^{N-1} c_n(f_2) e^{-2\pi jnf_1} = \sum_{n=0}^{N-1}\left(\bar{c}_n(f_2) e^{-4\pi j\alpha_2 f_2} e^{2j\beta}\right) e^{2\pi jnf_1} e^{-4\pi j\alpha_1 f_1}$$

The set of complex exponentials $\{1, e^{-2\pi jf_1}, e^{-2\pi j2f_1}, \ldots, e^{-2\pi j(N-1)f_1}, e^{-4\pi j\alpha_1 f_1}, e^{2\pi jf_1} e^{-4\pi j\alpha_1 f_1}, \ldots, e^{2\pi j(N-1)f_1} e^{-4\pi j\alpha_1 f_1}\}$ is linearly independent unless there are coincident pairs of frequencies. Either all frequencies can be paired or at least the lowest frequency $(N-1)f_1$ remains unpaired. In the later case,

$$c_{N-1}(f_2)=\sum_{m=0}^{M-1} h_{N-1,m} e^{-2\pi jmf_2} = 0$$

for all f_2. This now implies that:

$h_{N-1,m} = 0$, for $m = 0, \ldots, M-1$

which means that the order of the filter is less than hypothesized. This contradiction now forces all frequencies to be paired as:

$-(N-1)f_1 = -2\alpha_1 f_1; \ldots; \; -f_1 = (N-2)f_1 - 2\alpha_1 f_1; \; 0 = (N-1)f_1 - 2\alpha_1 f_1$

This is satisfied when

$$\alpha_1 = \frac{(N-1)}{2}$$

Also the coefficients of the paired frequencies must be equal. This yields

$c_n(f_2) = \bar{c}_{N-1-n}(f_2) e^{-4\pi j\alpha_2 f_2} e^{2j\beta}$

for n=0,1,...,N-1 and all f_2.

By expanding this last equation in terms of the filter coefficients we obtain

$$\sum_{m=0}^{M-1} h_{n,m} e^{-2\pi jmf_2} = \sum_{m=0}^{M-1} \bar{h}_{N-1-n,m} e^{2\pi jmf_2} e^{-4\pi j\alpha_2 f_2} e^{2j\beta} \quad \forall f_2$$

By exploiting the linear independence of the exponentials and repeating the frequency pairing argument from above we obtain

$$\alpha_2 = \frac{(M-1)}{2} \quad \text{and} \quad h_{n,m} = \bar{h}_{N-1-n,M-1-m} e^{2j\beta} \;\; \forall\, n,m$$

3. Linear Phase 2-D FIR Complex Filter Design using Weighted pth Power Approach

We define a filter design using the conjugate symmetric properties of the 2-D filter in terms of its coefficients.

Let $z_D(f_1,f_2)$ be the desired frequency response of the filter; $z_D(f_1,f_2)$ can be expressed as

$z_D(f_1,f_2) = a_D(f_1,f_2) e^{-2\pi j(\alpha_1 f_1 + \alpha_2 f_2)}$

where $a_D(f_1,f_2)$ and $e^{-2\pi j(\alpha_1 f_1+\alpha_2 f_2)}$ are the desired amplitude and frequency response of the filter. Then the filter is formed by minimizing the weighted pth power error:

$$G_p(\mathbf{H}) = \int_0^1\int_0^1 W(f_1,f_2)\left|z_D(f_1,f_2) - H(f_1,f_2)\right|^p df_1 df_2$$

$W(f_1,f_2)$ is a weighting function that balance tradeoffs between passbands and stopbands. Also usually $W(f_1,f_2)$ = 0, on the transition bands. p is a power larger or equal to 1. When p = 2 we our criterion is the least squares design; as $p \Rightarrow \infty$ we get the minmax design.

The filter frequency response can be expressed in terms of the filter coefficients as

$H(f_1,f_2) = \vec{e}_{N,M}(f_1,f_2)^* \vec{h}$

where

$$\vec{h} = \begin{bmatrix} h_{0,0} \\ h_{1,0} \\ \vdots \\ h_{N-1,0} \\ h_{0,1} \\ h_{1,1} \\ \vdots \\ h_{N-1,1} \\ \vdots \\ h_{0,M-1} \\ \vdots \\ h_{N-1,M-1} \end{bmatrix}; \; \vec{e}_{N,M}(f_1,f_2) = \begin{bmatrix} 1 \\ e^{2\pi jf_1} \\ \vdots \\ e^{2\pi j(N-1)f_1} \\ e^{2\pi jf_2} \\ e^{2\pi j(f_1+f_2)} \\ \vdots \\ e^{2\pi j((N-1)f_1+f_2)} \\ \vdots \\ e^{2\pi j((M-1)f_2)} \\ \vdots \\ e^{2\pi j((N-1)f_1+(M-1)f_2)} \end{bmatrix}$$

Because of the derived conjugate symmetric constraints on the coefficients we have that

$\vec{h} = J_{NM}\bar{\vec{h}}$

where the matrix J_{NM} is the exchange matrix of order NM; it has ones on the cross diagonal and zeros elsewhere. Multiplication by J_{NM} on the left flips the rows of the vector about the center.

It should be noted that the (N(k-1)+n)th entry is $e^{2\pi j((n-1)f_1+(k-1)f_2)}$

3.1 Case p=2 (Least Squares Design)

The advantage of least square filter design is the closed form of its solution which allows for fast programming of various types of filters. For the 1-D case least squares FIR filter designs have been treated by several authors. In [2] the complex case is analyzed. We expand the objective function $G_2(\mathbf{H})$ in terms of $\mathbf{H}$ and obtain

$$G_2(\mathbf{H}) = \int_0^1\int_0^1 W(f_1,f_2)\left|a_D(f_1,f_2)\right|^2 df_1 df_2 - \int_0^1\int_0^1 W(f_1,f_2)\bar{z}_D(f_1,f_2)H(f_1,f_2) df_1 df_2$$
$$-\int_0^1\int_0^1 W(f_1,f_2) z_D(f_1,f_2)\bar{H}(f_1,f_2) df_1 df_2 + \int_0^1\int_0^1 W(f_1,f_2) H(f_1,f_2)\bar{H}(f_1,f_2) df_1 df_2$$
$$= a - \vec{c}^*\vec{h} - \vec{h}^*\vec{c} + \vec{h}^* E \vec{h}$$

where

$$a = \int_0^1\int_0^1 W(f_1,f_2)\left|a_D(f_1,f_2)\right|^2 df_1 df_2$$

$$\vec{c} = \int_0^1\int_0^1 W(f_1,f_2) z_D(f_1,f_2)\,\vec{e}_{N,M}(f_1,f_2) df_1 df_2$$

$$E = \int_0^1 \int_0^1 W(f_1, f_2)\, \vec{e}_{N,M}(f_1, f_2)\vec{e}_{N,M}(f_1, f_2)^* df_1 df_2$$

It should be noted that **E** is a Hermitian matrix which has the following special block structure when NM is even.

$$\mathbf{E} = \begin{bmatrix} E_{11} & E_{12} \\ E_{12}^* & E_{11} \end{bmatrix}$$

where all submatrices are of dimension NxM/2.

This can be seen from the following

E can be expressed as an integrated Kronecker product of matrices as:

$$E = \int_0^1 \int_0^1 W(f_1, f_2)\left(\vec{e}_M(f_2)\vec{e}_M(f_2)^* \right) \otimes \left(\vec{e}_N(f_1)\vec{e}_N(f_1)^* \right) df_1 df_2$$

Since $\vec{e}_N(f_1)\vec{e}_N(f_1)^*$ and $\vec{e}_M(f_2)\vec{e}_M(f_2)^*$ are Toeplitz matrices, their Kronecker product will be Toeplitz-block Toeplitz matrix

$$E = \begin{bmatrix} F & \bar{g}_1 F & \bar{g}_2 F & \cdots & \bar{g}_{M-1} F \\ g_1 F & F & \bar{g}_1 F & \ddots & \vdots \\ g_2 F & g_1 F & F & \ddots & \bar{g}_2 F \\ \vdots & \ddots & \ddots & \ddots & \bar{g}_1 F \\ g_{M-1} F & \cdots & g_2 F & g_1 F & F \end{bmatrix}$$

From this it is obvious that the above 2x2 block decomposition of E is correct.

Also if the weight function **W** factors as

$$W(f_1, f_2) = W_1(f_1) W_2(f_2)$$

we can express F as a NxN Toeplitz Hermitian matrix whose n, n' entry is

$$F_{n,n'} = \int_0^1 W_1(f_1)\, e^{2\pi j(n-n')f_1} df_1 \quad ; \quad 1 \le n,\ n' \le N$$

$$g_m = \int_0^1 W_2(f_2)\, e^{2\pi j m f_2} df_2 \quad ; \quad 1 \le m \le M-1$$

Minimization of the objective function $G_2(\mathbf{H})$ is a quadratic minimization problem with respect the variable $\vec{h}$. We must account now for the constraint $\vec{h} = J_{NM}\bar{\vec{h}}$. It is fairly clear that such a vector has only half of its components independent and it can be represented as

$$\vec{h} = \begin{bmatrix} \vec{h}_L \\ \cdots \\ J_{NM/2}\bar{\vec{h}}_L \end{bmatrix}$$

where $\vec{h}_L$ is an arbitrary NxM/2 complex vector.

We shall use analogous representations for **E** , valid when M is even.

A similar decomposition of $\vec{c}$ allows to express as

$$\vec{c} = \begin{bmatrix} \vec{c}_U \\ \cdots \\ \vec{c}_L \end{bmatrix}$$

where both vectors $\vec{c}_U$ and $\vec{c}_L$ are of dimension NxM/2.

We plug this into $G_2(\mathbf{H})$ and obtain

$$G_2(\mathbf{H}_L) = a - \left[\vec{c}_U^* : \vec{c}_L^*\right]\begin{bmatrix} \vec{h}_L \\ \cdots \\ J_{NM/2}\bar{\vec{h}}_L \end{bmatrix} - \left[\vec{h}_L^* : \left(J_{NM/2}\bar{\vec{h}}_L\right)^*\right]\begin{bmatrix} \vec{c}_U \\ \cdots \\ \vec{c}_L \end{bmatrix}$$

$$+\left[\vec{h}_L^* : \left(J_{NM/2}\bar{\vec{h}}_L\right)^*\right]\begin{bmatrix} E_{11} & E_{12} \\ E_{12}^* & E_{11} \end{bmatrix}\begin{bmatrix} \vec{h}_L \\ \cdots \\ J_{NM/2}\bar{\vec{h}}_L \end{bmatrix}$$

i.e.,

$$G_2(\vec{h}_L) = a - \left(\vec{c}_U^* + \vec{c}_L^T J_{NM/2}\right)\vec{h}_L - \vec{h}_L^*\left(J_{NM/2}\bar{\vec{c}}_L + \vec{c}_U\right) +$$

$$2\vec{h}_L^* E_{11}\vec{h}_L + \vec{h}_L^* E_{12} J_{NM/2}\bar{\vec{h}}_L + \vec{h}_L^T J_{NM/2} E_{12}^* \vec{h}_L$$

By applying the rules of complex partials derivatives from [3], i.e. treating $\vec{h}_L$ and $\bar{\vec{h}}_L$ as independent variables, using the chain rule for derivatives and the commutativity of multiplication in the vector inner product we obtain:

$$\frac{\partial G_2(\vec{h}_L)}{\partial \vec{h}_L} = -\left(\vec{c}_U^* + \vec{c}_L^T J_{NM/2}\right) + 2\vec{h}_L^* E_{11} + \vec{h}_L^T J_{NM/2} E_{12}^* + \vec{h}_L^T \bar{E}_{12} J_{NM/2} = 0$$

$$0 = -\left(\bar{\vec{c}}_U + J_{NM/2}\vec{c}_L\right) + 2\bar{E}_{11}\bar{\vec{h}}_L + \left(\bar{E}_{12} J_{NM/2} + J_{NM/2} E_{12}^*\right)\vec{h}_L$$

The conjugate-transpose of this equation for $\vec{h}_L$ is

$$0 = -\left(\vec{c}_U + J_{NM/2}\bar{\vec{c}}_L\right) + 2E_{11}\vec{h}_L + \left(E_{12} J_{NM/2} + J_{NM/2} E_{12}^T\right)\bar{\vec{h}}_L$$

These equations can be written in block form as

$$\begin{bmatrix} 2E_{11} & \left(E_{12} J_{NM/2} + J_{NM/2} E_{12}^T\right) \\ \left(\bar{E}_{12} J_{NM/2} + J_{NM/2} E_{12}^*\right) & 2\bar{E}_{11} \end{bmatrix}\begin{bmatrix} \vec{h}_L \\ \bar{\vec{h}}_L \end{bmatrix} = \begin{bmatrix} \left(\vec{c}_U + J_{NM/2}\bar{\vec{c}}_L\right) \\ \left(\bar{\vec{c}}_U + J_{NM/2}\vec{c}_L\right) \end{bmatrix}$$

By multiplying these equations on the left by the block matrix: $\begin{bmatrix} I & 0 \\ 0 & J_{NM/2} \end{bmatrix}$

they simplified to

$$E\begin{bmatrix} \vec{h}_L \\ J_{NM/2}\bar{\vec{h}}_L \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \left(\vec{c}_U + J_{NM/2}\bar{\vec{c}}_L\right) \\ \left(J_{NM/2}\bar{\vec{c}}_U + \vec{c}_L\right) \end{bmatrix}$$

Since E is Toeplitz-block-Toeplitz we can apply fast algorithms to solve for the upper half of this Toeplitz-block-Toeplitz system of linear equations, [7].

Explicit solution of the equation gives

$$\vec{h}_L = \frac{1}{2}\left(E_{11} - E_{12}E_{11}^{-1}E_{12}^*\right)^{-1}\left(\vec{c}_U + J_{NM/2}\bar{\vec{c}}_L - E_{12}E_{11}^{-1}\left(J_{NM/2}\bar{\vec{c}}_U + \vec{c}_L\right)\right)$$

EXAMPLE:Bandpass Filter

Here we construct a filter whose passband and stopband are defined from:

$f_{11p} = 0.4; f_{12p} = 0.6; f_{11s} = 0.34; f_{12s} = 0.66$

$f_{21p} = 0.4; f_{22p} = 0.6; f_{21s} = 0.36; f_{22s} = 0.64;$;

We select: M=24, N=16

Also we assume that the weight function factors in terms of one dimensional weights $W(f_1, f_2) = W_1(f_1) W_2(f_2)$. W_1 and W_2 are equal to 1 in their respective passbands and equal to 3 at the stopbands.

They are set to 0 at the transition bands. Grey scale 3D filter response plots show good passband behaviour and sidelobes near 30db.

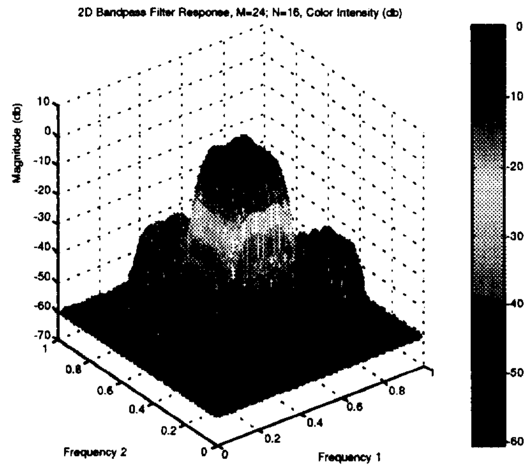

Fig 1. 3D plot of the 2D filter frequency response

3.2 Case 2. p^{th} power design (p>2)

Using the p^{th} power approach (p>2) gives a balanced solution between the least squares and minimax approach. It should be noted that as $p \Rightarrow \infty$, the p^{th} power solution approaches the minmax, [10].

Previously, several authors have treated p^{th} power minimization mainly in 1-D and for real FIR filters, [6] and [8]. The real 2-D case has been treated in part , [5], however the error function used was the sum of the p^{th} powers of the absolute values of real and imaginary components which is not equal to the magnitude of the error raised to the p^{th} power. Although the real Newton method was used here the Hessian was not computed in closed form nor any special properties of the structure of the Hessian matrix was adressed. Also the computational speed of the algorithm was not optimized.

Characterization of the minmax solution is by no means an easy matter. Even for the real 1D case its foundations lay on a deep theorem in approximation theory by Weierstrass. The complex 1D case was treated in [1] and it is already more complicated than the real one. Here due to the lack of a straighforward chracterization in 2D real or complex we propose the use of the least p^{th} power design to approximate minmax 2D filters.

A key advantage of the p^{th} power approach is that the objective function to be minimized is convex, twice differentiable. Thus a descent algorithm based on Newton's method can be made to converge to the true solution, by scaling back the update if necessary, no matter where it is started, [10]. The convergence rate is quadratic on a neighborhood of the solution.

The objective function to be minimized is

$$G_p(\vec{h}_L) = \int_0^1\int_0^1 W(f_1,f_2)\left| z_D(f_1,f_2) - \vec{e}_{N,M}(f_1,f_2)^* \begin{bmatrix} \vec{h}_L \\ \cdots \\ J_{NM/2}\bar{\vec{h}}_L \end{bmatrix} \right|^p df_1 df_2$$

$$= \int_0^1\int_0^1 W(f_1,f_2)\left[R(\vec{h}_L)\bar{R}(\vec{h}_L)\right]^{p/2} df_1 df_2$$

where

$$R(\vec{h}_L) = z_D(f_1,f_2) - \vec{e}_{N,M}(f_1,f_2)^* \begin{bmatrix} \vec{h}_L \\ \cdots \\ J_{NM/2}\bar{\vec{h}}_L \end{bmatrix}$$

is the discrepancty between the desired and produced filter response.

Under the assumption that M is even

$$\vec{e}_{N,M}(f_1,f_2) = \begin{bmatrix} \vec{e}_{N,M/2}(f_1,f_2) \\ \cdots\cdots\cdots\cdots \\ e^{\pi jMf_2}\vec{e}_{N,M/2}(f_1,f_2) \end{bmatrix}$$

Thus $R(\vec{h}_L)$ can be written as

$$R(\vec{h}_L) = z_D(f_1,f_2) - \vec{e}_{N,M/2}(f_1,f_2)^*\vec{h}_L - e^{-\pi jMf_2}\vec{e}_{N,M/2}(f_1,f_2)^* J_{NM/2}\bar{\vec{h}}_L$$

Computation of complex first and second partials derivatives of $G_p(\vec{h}_L)$ with respect to $\vec{h}_L$ are obtained next by using formulas from, [4],. These will be used in the implementation of the complex Newton method for finding the minimum of a function.

The Newton recursion for $\vec{h}_L$ say $\{\vec{h}_L^{(k)}\}_k$ is written as

$$\vec{h}_L^{(k+1)} = \vec{h}_L^{(k)} +$$

$$\left(\bar{F}_{\vec{h}_L^T\vec{h}_L} F_{\vec{h}_L^*\vec{h}_L}^{-1} F_{\vec{h}_L^T\vec{h}_L} - \bar{F}_{\vec{h}_L^*\vec{h}_L}\right)^{-1}\left(\overline{\nabla G}_{\vec{h}_L} - \bar{F}_{\vec{h}_L^T\vec{h}_L} F_{\vec{h}_L^*\vec{h}_L}^{-1}\nabla G_{\vec{h}_L}\right)$$

where the matrices and gradients of the second and first derivatives of $G_p(\vec{h}_L)$ are defined by

$$\nabla G_{\vec{h}_L} = \frac{\partial G_p(\vec{h}_L)^T}{\partial \vec{h}_L}\Bigg|_{\vec{h}_L=\vec{h}_L^{(k)}}$$

$$F_{\vec{h}_L^T\vec{h}_L} = \frac{1}{2}\left(\frac{\partial^2 G_p(\vec{h}_L)}{\partial \vec{h}_L^T \partial \vec{h}_L} + \frac{\partial^2 G_p(\vec{h}_L)^T}{\partial \vec{h}_L^T \partial \vec{h}_L}\right)\Bigg|_{\vec{h}_L=\vec{h}_L^{(k)}}$$

$$F_{\vec{h}_L^*\vec{h}_L} = \frac{\partial^2 G_p(\vec{h}_L)^T}{\partial \vec{h}_L^* \partial \vec{h}_L}\Bigg|_{\vec{h}_L=\vec{h}_L^{(k)}}$$

These gradients and Hessians can be computed in closed form, [9], in terms the error and weight functions and frequency steering matrices for the grids.

Its computation load is substantial but one can exploit the block Toeplitz/Hankel structure of the Hessian matrices in the iteration to reduce the computational load

Implementation of this algorithm is accomplished with increasing p and starting at p=2 in order to remain within the region of convergence of Newtons method. A typical sequence for p may be:

$p_0 = 2;$

For $k \geq 1$; $p_k = \min\{\alpha p_{k-1}, p\}$, where $1 < \alpha \leq 1.5$

Examples: In the examples below we have set the stopband of the filter to extend from [0,0.3] and [0.7,1.0] on each dimension. The passband is the same as example 1. In filters 2 and 4 the weighting function is the same as in 1. In filter 3 the weighting function is 1 on the passband, 0 o the transition band and uniformly equal to 9 on the stopband.

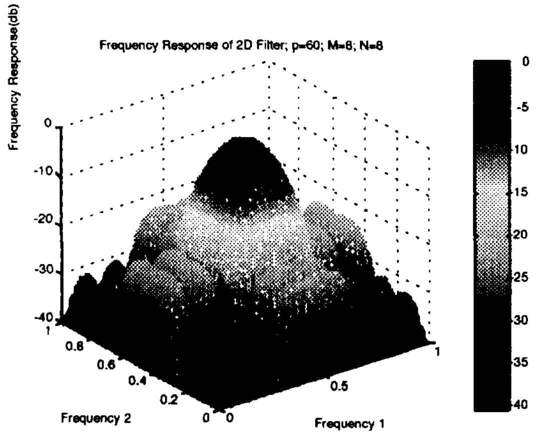

Fig. 2. Weight function is product of 1D weight functions. Convergence in 20 iterations. α=1.5

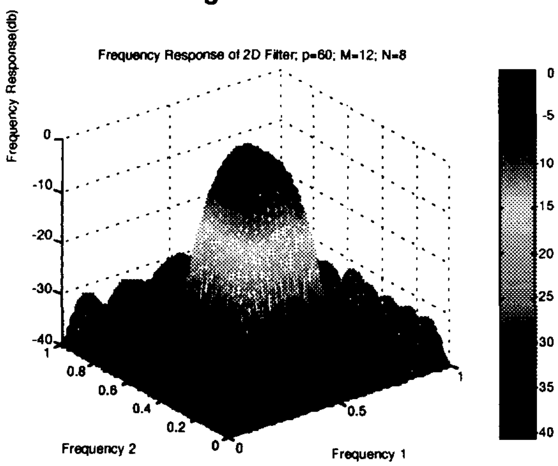

Fig. 3. Weight function is equal to 9 on stopband. Not the product 1D weight functions. Convergence in 35 iterations. α=1.3

CONCLUSIONS

We developed algorithms for 2D complex FIR filter design based on the least p^{th} power criterion. For p = 2 we derived the filter coefficients in closed form and necessary and sufficient conditions were stated to allow the filters to be affine phase. The complex Newton method was applied for 2 < p to obtain a recursive solution to the minimum p^{th} power filter design. The algorithm includes an increasing value of p for each iteration. For large p its solution approximates the minmax.. Several examples 2D filter designs were included.

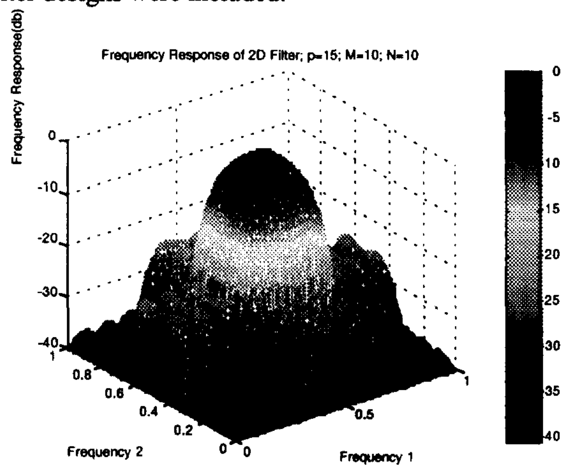

Fig 4. Weight function is the product of 1D weight functions. Convergence in 15 iterations. α=1.3

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