Asymmetric FIR Digital Filters with Specified Group Delays

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Abstract

In this paper we present a new algorithm for designing asymmetric FIR digital filters. The proposed algorithm is based on extending the algorithms that we presented in [1].

1. Introduction

A new optimization algorithm is presented for designing asymmetric FIR digital filters. The new algorithm can design optimal filters that simultaneously meet specifications on the group delay and the magnitude of the frequency response.

Asymmetric FIR digital filters are needed for many practical applications. In particular, they are needed to synchronize radar echos with different time delays. In general, asymmetric FIR digital filters are used to shift discrete-time signals by specified fractions of a sampling period. These types of filters are designed to have specified group delays in their passbands. Asymmetric FIR digital filters with specified group delays are called "digital phase shifters" in [6]-[7]. Digital phase shifters can be viewed as polyphase filters [8] in the context of multirate applications.

Numerous researchers have proposed optimization methods for digital phase shifters and polyphase filters [6]-[8]. Most of these methods are based on the least-squares or minimax optimality criteria. In this paper we propose to use the peak-constrained least weighted squared error (PCLWSE) criterion. It is more general than the least-squares and minimax optimality criteria. In fact, the least-squares and minimax optimality criteria are special cases of the PCLWSE criterion.

This work was supported in part by the National Science Foundation under Grant MIP-9200581.

2. Formulation of the Design Problem

We formulate the desired frequency response as follows:

$$H_d(e^{j\omega}) = |H_d| e^{-j\omega\,(GD)} \qquad (1)$$

GD is the user specified group delay. If the group delay is specified to be anything other than the natural group delay of $(L-1)/2$, where L is the impulse response length, then the resulting FIR filter is asymmetric. The weighted squared error is defined by E as follows:

$$E = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j\omega}) \left| H\left(e^{j\omega}\right) - H_d\left(e^{j\omega}\right) \right|^2 d\omega \qquad (2)$$

The squared-error term can be expanded as follows:

$$\begin{aligned} \left| H(e^{j\omega}) - H_d(e^{j\omega}) \right|^2 &= (H - H_d)(H - H_d)^* \\ &= HH^* - HH_d^* - H_d H^* + H_d H_d^* \qquad (3) \\ &= |H|^2 - 2\,\mathrm{Re}\left[H H_d^* \right] + |H_d|^2 \end{aligned}$$

The squared magnitude of the frequency response can be formulated as

$$|H|^2 = HH^* = \boldsymbol{h}^T \boldsymbol{w} \boldsymbol{h}^T \boldsymbol{w}^* = \boldsymbol{h}^T \boldsymbol{w} \boldsymbol{w}^{T^*} \boldsymbol{h} = \boldsymbol{h}^T \boldsymbol{S} \boldsymbol{h}$$

where

$$\boldsymbol{w} = \left[1 \;\; e^{-j\omega} \;\; e^{-j2\omega} \;\; \cdots \;\; e^{-j(L-1)\omega} \right] \qquad (5)$$

and S is an LxL matrix with elements defined by

$$s(m, n) = e^{-j\omega\,(m-n)},\; m,n = 0, 1, \ldots, L-1 \qquad (6)$$

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Expanding the complex exponential with Euler's identities

$$s(m,n) = cos\omega(m-n) - jsin\omega(m-n) = q - jz \quad (7)$$

and substituting into 4, we get

$$|H|^2 = h^T(Q - jZ)h = h^TQh - jh^TZh \quad (8)$$

Since $|H|^2$ is real, the imaginary term in (8) must be zero. Thus (8) becomes

$$|H|^2 = \boldsymbol{h}^T\boldsymbol{Q}\boldsymbol{h} \quad (9)$$

where $\boldsymbol{Q}$ is an $L \times L$ matrix with real valued elements

$$q(m,n) = \cos\omega(m-n) \quad (10)$$

The following properties hold for $\boldsymbol{Q}$ because cosine functions are even.

$$q(m,n) = q(n,m), m,n = 0,\ldots,L-1 \quad (11)$$

$$q(m-n) = q(n-m) = q(|m-n|), m,n = 0, \ldots, L-1 \quad (12)$$

Equations (11) and (12) show that $\boldsymbol{Q}$ is a real, symmetric, Toeplitz matrix. The second term in (3) can be determined based on the desired frequency response, as follows:

$$\begin{aligned}\mathrm{Re}\left[HH_d^*\right] &= \mathrm{Re}\left[\boldsymbol{h}^T w|H_d|e^{j\omega(GD)}\right] \\ &= |H_d|\boldsymbol{h}^T\left[\mathrm{Re}(we^{j\omega(GD)})\right] \\ &= |H_d|\boldsymbol{h}^T\boldsymbol{g}\end{aligned} \quad (13)$$

where $\boldsymbol{g}$ is a vector with elements

$$g(m) = \cos\omega\ (GD - m) \quad m = 0,\ldots,L-1 \quad (14)$$

We now substitute (9) and (13) into (3) to get the final form of the squared error magnitude:

$$\left|H\left(e^{j\omega}\right) - H_d\left(e^{j\omega}\right)\right|^2 = \boldsymbol{h}^T\boldsymbol{Q}\boldsymbol{h} - 2|H_d|\boldsymbol{h}^T\boldsymbol{g} + |H_d|^2 \quad (15$$

By substituting (15) into (2) and taking advantage of the conjugate symmetry exhibited by $H(e^{j\omega})$ and $H_d(e^{j\omega})$, we obtain

$$E = \frac{1}{\pi}\int_0^{\pi} W(e^{j\omega})\left\{\boldsymbol{h}^T\boldsymbol{Q}\boldsymbol{h} - 2|H_d|\boldsymbol{h}^T\boldsymbol{g} + |H_d|^2\right\}d\omega \quad (16)$$

E can equivalently be written as a quadratic form

$$E = \frac{1}{\pi}\left[\boldsymbol{h}^T\boldsymbol{R}\boldsymbol{h} - 2\boldsymbol{h}^T\boldsymbol{P} + a_b\right] \quad (17)$$

in which

$$a_b = \frac{1}{\pi}\int_0^{\pi} W(e^{j\omega})|H_d|^2 d\omega \quad (18)$$

is independent of tthe impulse response coefficients and has no role in the minimization of E. $\boldsymbol{R}$ is an $(L \times L)$ real symmetric Toeplitz matrix with elements given by

$$r(m,n) = \int_0^{\pi} W(e^{j\omega})\cos\omega(m-n)d\omega \quad (19)$$

and $\boldsymbol{P}$ is a vector with elements given by

$$p(m) = \int_0^{\pi} W(e^{j\omega})\left|H_d\left(e^{j\omega}\right)\right|\cos\omega[GD - m] \quad (20)$$

$$m = 0,1,\cdots,L-1.$$

Since $W(e^{jw}) = 0$ in the transition region, and $H_d(e^{jw}) = 0$ in the stopband, we can rewrite $r(m,n)$ and $p(m)$ taking into consideration the contributions from the passband and stopband :

$$r_p(m,n) = \int_0^{\omega_p} W(e^{j\omega})\cos\omega(m-n)d\omega \quad (21)$$

$$r_s(m,n) = \int_{\omega_s}^{\pi} W(e^{j\omega})\cos\omega(m-n)d\omega \quad (22)$$

$$p(m) = \int_0^{\omega_p} W(e^{j\omega})\left|H_d\left(e^{j\omega}\right)\right|\cos\omega[GD - m] \quad (23)$$

Having developed the matrix representation of E, we now proceed to determine the impulse response $\boldsymbol{h}$ which minimizes E in (17). Differentiating E with respect to $\boldsymbol{h}$ and setting the result to zero we obtain

$$\frac{dE}{d\boldsymbol{h}} = \frac{1}{\pi}(2\boldsymbol{Rh} - 2\boldsymbol{P}) = 0 \tag{25}$$

$$2\boldsymbol{Rh} = 2\boldsymbol{P} \tag{26}$$

$$\boldsymbol{h} = \boldsymbol{R}^{-1}\boldsymbol{P} \tag{27}$$

Since calculating an inverse matrix is inefficient and R is toeplitz, we can use Levinson's algorithm to efficiently solve the Toeplitz system $\boldsymbol{R}\,\boldsymbol{h} = \boldsymbol{P}$ for $\boldsymbol{h}$. The resulting frequency response $H(e^{j\omega})$ at iteration K has peak ripples with magnitudes denoted by $H_p(k, i)$ and $H_s(k, i)$ in the passband and stopband respectively. The variable i refers to the ripple index. The errors caused by such peaks are defined as $|H_p(k, i) - G_p|$ in the passband and $|H_s(k, i)|$ in the stopband. The errors are fed to the algorithm which computes the weighting function as discussed in the next section.

3. Computation of the Weighting Function

The Ripple Weighted Least Squares (RWLS) algorithm presented in [1] is used to determine the weighting function for the asymmetric filters in this paper. The function is initiated by a weighting ratio C_p / C_s at iteration 0 where C_p and C_s are two uniform weights in the passband and stopband respectively. For subsequent iterations, the function becomes a series of jointed segments. A segment is born when a peak error in the frequency response as defined in the previous section exceeds the specification. For example

$|H_p(k, i) - G_p| > \delta_p$ produces $\gamma_p(k+1, i)$

$|H_s(k, i)| > \delta_s$ produces $\gamma_s(k+1, i)$

A segment, once generated at any iteration, is modified at later iterations as follows:

Stopband Weighting Function Updates:

$$\gamma_s(k+1,i) = B_s\left\{S_s\left(|H_s(k,i)|\right)\gamma_s(k,i)\right\} \tag{28}$$

where $S_s(x)$ is a saturating function defined by

$$S_s\left(|H_s(k,i)|\right) = S\left\{1 + \beta_s \frac{H_s(k,i) - \delta_s}{\delta_s}\right\} \tag{29}$$

and $B_s(x)$ is the clipping function

$$B_s(x) = \begin{cases} x, & x > C_s \\ C_s, & x \le C_s \end{cases} \tag{30}$$

Passband Weighting Function Updates:

$$\gamma_p(k+1,i) = B_p\left\{S_p\left(|H_p(k,i)|\right)\gamma_p(k,i)\right\} \tag{31}$$

S_p is a saturating function defined by

$$S_p\left(|H_p(k,i)|\right) = S\left\{1 + \beta_p \frac{H_p(k,i) - \delta_p}{\delta_p}\right\} \tag{32}$$

$$B_p(x) = \begin{cases} x, & x > C_p \\ C_p, & x \le C_p \end{cases} \tag{33}$$

β is a stepsize factor which controls the convergence rate. It can speed up or slow down convergence by taking on higher or lower values. If β is too large it can cause instability which occurs when the ripple errors begin to diverge. In order to avoid instability and achieve an overall faster convergence rate, we can allow β to have a large value for the first few iterations and gradually decay to a constant as convergence occurs.

As a great advantage over continuous weighting functions, the piecewise constant weighting function enables the integrals representing $r(m,n)$ and $p(m)$ to be computed analytically, instead of numerically. For example, with $\gamma_s(k,i)$,being the ith interval of the weighting function bounded by the frequencies $\omega_l(k)$ and $\omega_u(k)$, we have the following contribution to the $\boldsymbol{R}$ matrix which is easy to compute analytically.

$$r_s(m,n) = 2\sum_{i=1}^{I_s} \int_{\omega_l(k)}^{\omega_u(k)} \gamma_s(k,i)\cos\omega(m-n)d\omega$$

$$= 2\sum_{i=1}^{I_s} \gamma_s(k,i) \int_{\omega_l(k)}^{\omega_u(k)} \cos\omega(m-n)d\omega$$

4. Design Example

An asymmetric FIR filter with the following specifications was designed by using the algorithm in Section III: $L = 95$, $F_p = .0625$ cycles/sample, $F_s = .0804$ cycles /sample, $GD = 47.25$ samples, $DB_p = 1.0$ dB, and $DB_s = -45.0$ dB. The corresponding group delay, phase delay, and frequency response are shown in figures 1, 2, and 3, respectively.

5. Conclusion

A new algorithm for designing asymmetric FIR digital filters was presented in this paper. The new algorithm is based on extending the RWLS algorithm that we presented in [1] for symmetric FIR filter applications.

6. References

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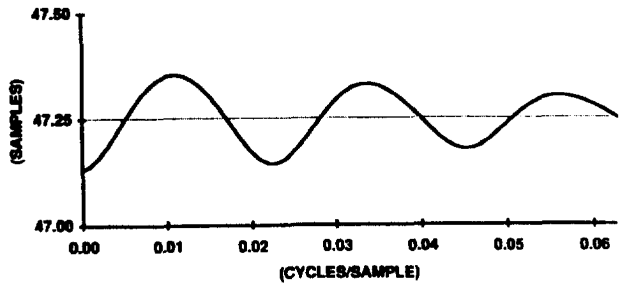

Fig. 1. Group delay in the passband.

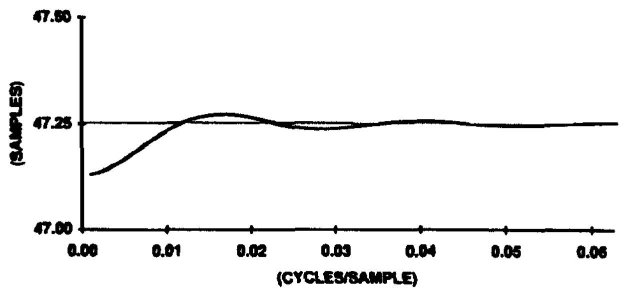

Fig. 2. Phase delay in the passband.

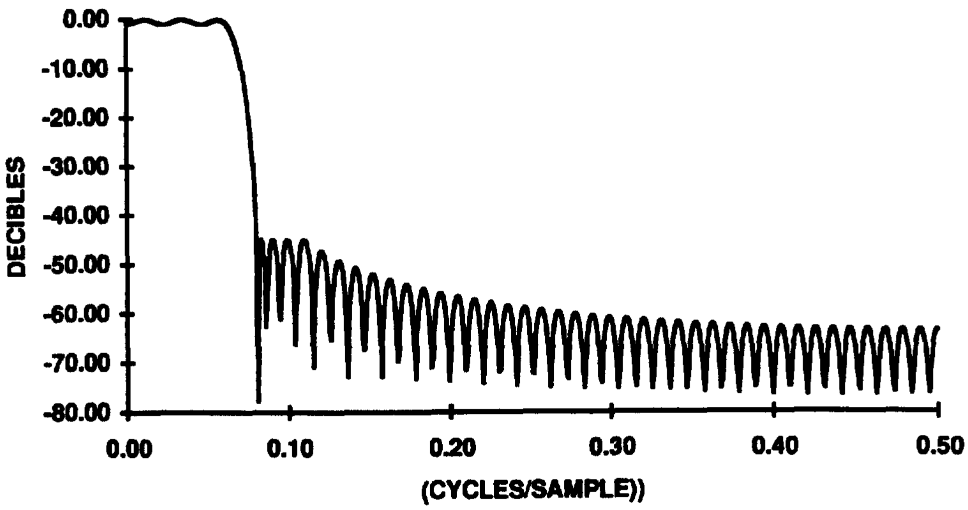

Fig. 3a. Frequency response for the design example.

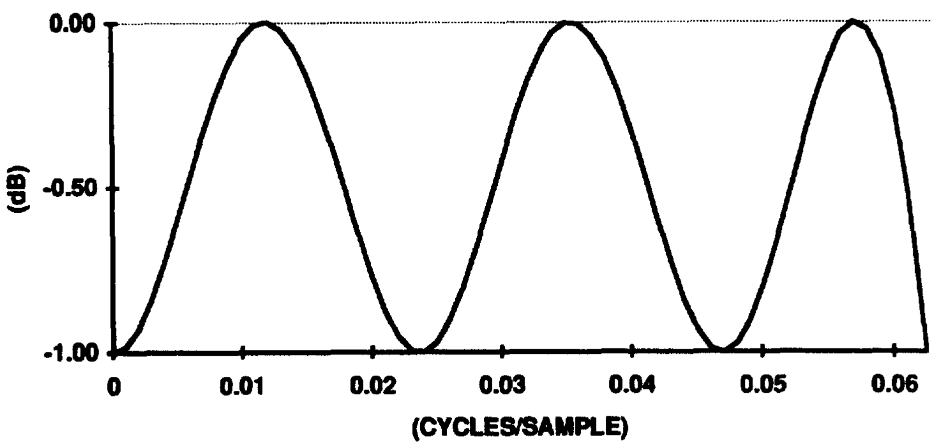

Fig. 3b. Frequency response in the passband.