Approximation of Block Matrix Filters Via Heuristic Averaging Methods

Celestino A. Corral
Electronic Systems Division
Scientific-Atlanta, Inc.
3845 Pleasantdale Road, MS ATL-10M
Atlanta, GA 30340-4269

Claude S. Lindquist
Dept. of Electrical & Computer Engineering
University of Miami
P.O. Box 248294
Coral Gables, FL 33124

Abstract

This paper presents a heuristic averaging method for approximating a single-input single-output FIR digital block matrix filter (BMF) h by a structurally constrained BMF $\tilde{h}$ in order to exploit computational efficiency of the matrix implementation of convolution. The unique properties of the method are derived and applied to the circulant and Toeplitz matrix structures. Advantages and limitations of the method are considered. Simulations are submitted and discussed.

1 Introduction

Block matrix filters, first proposed by Burrus [2] [3] and later considered by others [1] [5] [7] [13], implement finite convolution [8] via the inner product of the matrix h with the input vector $\mathbf{x}$ to produce the corresponding output vector as

$$y = h\mathbf{x} \tag{1.1}$$

This results in improved sytem throughput for time-domain processing [4].

In addition to controlling filter responses via the desired structures, additional improvement in throughput can be obtained by constraining block matrix filters to have certain structures to exploit computational efficiency. For example, $N \times N$ circulant matrices require the storage of only N elements and output calculation viz Eq. (1.1) requires just shifting operations of those matrix elements. For non-causal Toeplitz matrices $2N-1$ elements are stored and a modified shifting algorithm can be used to implement Eq. (1.1).

In this paper we propose a method for obtaining matrix filters $\tilde{h}$ of the desired structures from general matrix filters h. It is shown that this method minimizes the average of the square of the difference between functions of the elements of the two matrices along the paths of symmetry of the matrix $\tilde{h}$. The *averaging method* is extended to the "arithmetic average" and "geometric average" methods.

2 Statement of the Averaging Method

Let $h, \tilde{h} \in \Re^{N\times N}$ be block matrix filters. Let the elements of $\tilde{h}$ be related through the following

$$\tilde{h}_{k,l} = \tilde{h}_{p(i),q(j)} \quad i,j,k,l \in \mathcal{Z} \tag{2.1}$$

where $p(i), q(j) : \mathcal{Z} \mapsto \mathcal{Z}$ are functions of i and j. For a circulant matrix $p(i) = mod_N(i+n_0)$ and $q(j) = mod_N(j+n_0)$, $i,j,n_0 \in \mathcal{Z}$ yielding the structure

$$\tilde{h} = \begin{pmatrix} \tilde{h}_{0,0} & \tilde{h}_{0,1} & \cdots & \tilde{h}_{0,N-1} \\ \tilde{h}_{0,N-1} & \tilde{h}_{0,0} & \cdots & \tilde{h}_{0,N-2} \\ \vdots & \cdots & \ddots & \vdots \\ \tilde{h}_{0,1} & \tilde{h}_{0,2} & \cdots & \tilde{h}_{0,0} \end{pmatrix} \tag{2.2}$$

For a Toeplitz matrix $p(i) = i + n$ and $q(j) = j + n$, $i,j,n \in \mathcal{Z}$ giving the resulting structure

$$\tilde{h} = \begin{pmatrix} \tilde{h}_{0,0} & \tilde{h}_{0,1} & \cdots & \tilde{h}_{0,N-1} \\ \tilde{h}_{1,0} & \tilde{h}_{0,0} & \cdots & \tilde{h}_{0,N-2} \\ \vdots & \cdots & \ddots & \vdots \\ \tilde{h}_{N-1,0} & \tilde{h}_{N-2,0} & \cdots & \tilde{h}_{0,0} \end{pmatrix} \tag{2.3}$$

We now introduce the heuristic:

Let $f(\tilde{h}_{k,l}), f(h_{i,j}) : \Re \mapsto \Re$ be well-defined functions of the elements of $\tilde{h}$ and h, respectively. Form a matrix M such that $m_{i,j} = (f(\tilde{h}_{k,l}) - f(h_{i,j}))^2$. We sum the elements of M along the paths of symmetry of $\tilde{h}$, where $\tilde{h}_{k,l} = \tilde{h}_{p(i),q(j)}$. Take the derivative as

$$\frac{\partial \sum_{k,l}(f(\tilde{h}_{k,l}) - f(h_{i,j}))^2}{\partial \tilde{h}_{k,l}} \quad \text{for all } k,l$$

$$= 2\sum_{k,l}(f(\tilde{h}_{k,l}) - f(h_{i,j}))\frac{\partial f(\tilde{h}_{k,l})}{\partial \tilde{h}_{k,l}} \tag{2.4}$$

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If we set Eq. (2.4) equal to zero we can solve for the values of $\tilde{h}_{k,l}$ as functions of $h_{i,j}$.

It is easy to show that this method minimizes the average of the square of the difference between functions of elements of the matrices h and $\tilde{h}$ along the paths of symmetry of $\tilde{h}$ [6]. We will now apply this heuristic to specific functions f for circulant and non-causal Toeplitz structures.

3 Arithmetic Averaged Block Matrix Filters

Let $f(h_{i,j}) = h_{i,j}$ and $f(\tilde{h}_{k,l}) = \tilde{h}_{k,l}$. We form the matrix M and take the derivative of the sum of the elements as in Eq. (2.4) and equating to zero we solve for the matrix $\tilde{h}$ as

$$\tilde{h}_{0,l} = \frac{1}{N}\sum_{i=0}^{N-1} h_{i,mod_N(i+l)} \quad l = 0, 1, \ldots N-1 \quad (3.1)$$

for circulant structures. For Toeplitz structures

$$\tilde{h}_{0,l} = \frac{1}{N-l}\sum_{i=0}^{N-1} h_{i,i+l} \quad l = 0, 1, \ldots N-1 \quad (3.2a)$$

$$\tilde{h}_{l,0} = \frac{1}{N-l}\sum_{i=0}^{N-1} h_{i+l,i} \quad l = 0, 1, \ldots N-1 \quad (3.2b)$$

where $i + l \leq N - 1$.

Therefore, the arithemtic averaged block matrix filters are simply the average of the elements of h along the path of symmetry of the elements of $\tilde{h}$. Only N^2 summations are required for circulant structures while $(N^2 + N - 2)/2$ summations are required for Toeplitz structures. This arithmetic average heuristic was first used by Lindquist [10] in order to get a quick approximation of a general matrix h with the desired structure of $\tilde{h}$.

Sensitivity analysis estimates the percentage drift of a dependent variable y which results from some percentage drift of two or more independent variables x_i. The sensitivity is given by the relation [9]

$$S^{y}_{x_i} = \frac{\partial y/y}{\partial x_i/x_i} = \frac{x_i}{y}\frac{\partial y}{\partial x_i} \quad (3.3)$$

It has been shown [6] that the sensitivities are

$$S^{\tilde{h}_{0,l}}_{h_{i,j}} = \frac{1}{N}\left(\frac{h_{i,j}}{\tilde{h}_{0,l}}\right) \quad (3.4)$$

for circulant structures and

$$S^{\tilde{h}_{0,l}}_{h_{i,j}} = \frac{1}{(N-l)}\left(\frac{h_{i,j}}{\tilde{h}_{0,l}}\right) \quad (3.5a)$$

$$S^{\tilde{h}_{l,0}}_{h_{i,j}} = \frac{1}{(N-l)}\left(\frac{h_{i,j}}{\tilde{h}_{l,0}}\right) \quad (3.5b)$$

for Toeplitz structures. The sensitivity is smallest along the main diagonal and increases to a maximum value of unity at $l = N - 1$.

4 Geometric Averaged Block Matrix Filters

Let $f(h_{i,j}) = \log(h_{i,j})$ and $f(\tilde{h}_{k,l}) = \log(\tilde{h}_{k,l})$. This requires that $h_{i,j} \geq 0$ so that $\tilde{h}$ is not complex in general. (If there are negative elements in h, they must occur an even number of times along the paths of symmetry of $\tilde{h}$ so that the $\tilde{h}_{k,l}$ are not complex. This is not guaranteed.) We now form the matrix M, take the derivative of the sum of the elements as in Eq. (2.4) and equate it to zero solving for the matrix $\tilde{h}$ as

$$\tilde{h}_{0,l} = \left(\prod_{i=0}^{N-1} h_{i,mod_N(i+l)}\right)^{1/N} \quad l = 0, 1, \ldots, N-1 \quad (4.1)$$

for circulant structures. For Toeplitz structures

$$\tilde{h}_{0,l} = \left(\prod_{i=0}^{N-1} h_{i,i+l}\right)^{1/(N-l)} \quad l = 0, 1, \ldots, N-1 \quad (4.2a)$$

$$\tilde{h}_{l,0} = \left(\prod_{i=0}^{N-1} h_{i+l,i}\right)^{1/(N-l)} \quad l = 0, 1, \ldots, N-1 \quad (4.2b)$$

where $i+l \leq N-1$. The geometric averaged structures require N^2 multiplies to generate a circulant $\tilde{h}$ and $(N^2 + N - 2)/2$ for a Toeplitz $\tilde{h}$.

For this method the sensitivities become [6]

$$S^{\tilde{h}_{0,l}}_{h_{i,j}} = \frac{1}{N} \quad (4.3)$$

for circulant structures and

$$S^{\tilde{h}_{0,l}}_{h_{i,j}} = \frac{1}{(N-l)} \quad (4.4a)$$

$$S^{\tilde{h}_{l,0}}_{h_{i,j}} = \frac{1}{(N-l)} \quad (4.4b)$$

for Toeplitz structures. The sensitivity is smallest along the main diagonal and increases to a maximum value of unity at $l = N - 1$.

5 Simulation Results

The input consists of a 32-bit sample vector described by the relation

$$\boldsymbol{x} = \boldsymbol{s} + \boldsymbol{n} \quad (5.1)$$

where s is the signal and n is additive white noise with signal-to-noise ratio (SNR) of 10dB. The matrix h to be approximated is calculated as shown in [11]. Figure 1 shows the matrix h, the estimation output and the detection output for $s = \sin \omega t$. Figure 2 shows the matrix h, the estimation and detection outputs for $s = t$, the ramp function.

Figure 3 shows the circulant arithmetic averaged $\tilde{h}$ and its corresponding estimation and detection output for the sine waveform. Note that the estimation output is essentially a phase-shifted sine wave. Detection output is good, however.

Figure 4 shows the circulant arithmetic averaged $\tilde{h}$ and its corresponding estimation and detection output for the ramp waveform. Again, the estimation output is poor, but detection performance is good.

Figure 5 shows the Toeplitz arithmetic averaged $\tilde{h}$ including its estimation and detection output for the sine waveform. Estimation performance is poor and shows the sensitivity of the Toeplitz averaged matrix filters. Detection performance is good.

Figure 6 shows the Toeplitz arithmetic averaged $\tilde{h}$ and its estimation and detection output for the ramp waveform. Estimation performance is very poor but detection output is good.

6 Conclusion

Block matrix filters exploit the computational efficiency of the matrix implementation of convolution. In addition, the nature of filter responses can be controlled. A structurally constrained BMF can further improve system throughput due to less memory requirement and more efficient variable handling. It is therefore important to consider a technique for deriving the structurally constrained BMF from a general BMF.

We have presented a general averaging method that provides for a direct and intuitively simple approach for the design of a structurally constrained BMF $\tilde{h}$ based on a general BMF h. Sensitivity analysis showed that the averaging filters are quite robust. Simulation results show that the arithmetic averaging filters are poor estimation performers but are good for detection applications.

In light of the inherent computational simplicity of the arithmetic averaging method, these approximating matrices may prove useful in improving system throughput for detection applications. The eventual success (or failure) of this method will depend on its continued use for different filter types (such as homomorphic filters [12]) and signal waveforms. It may even be possible to produce a taxonomy of averaging filters well-suited for certain applications.

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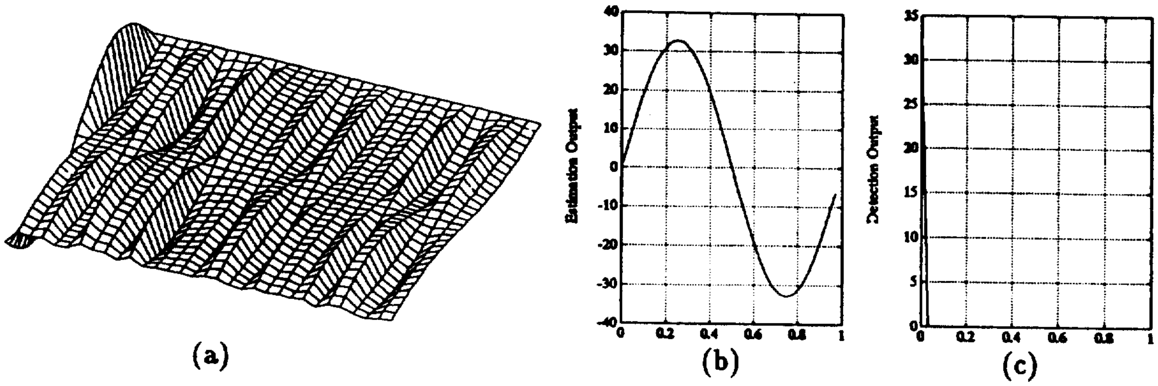

Figure 1. (a) Wiener matrix h, (b) estimation output, and (c) detection output for $= \sin \omega t$.

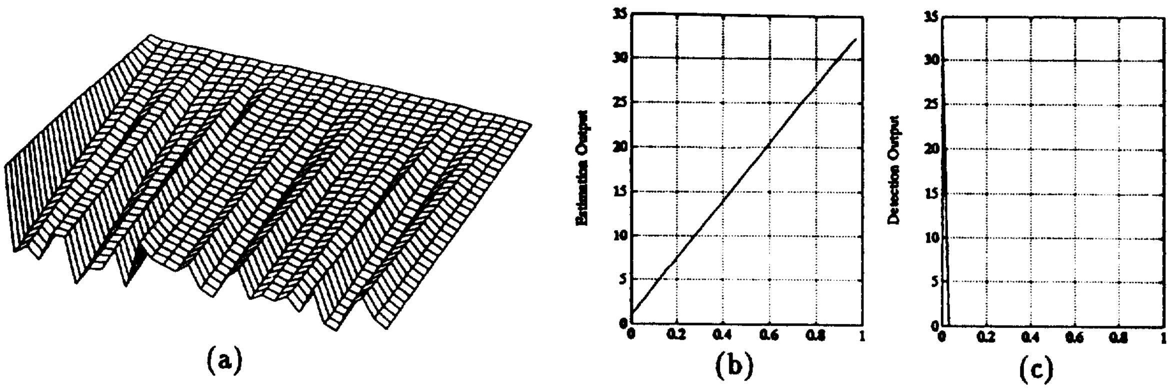

Figure 2. (a) Wiener matrix h, (b) estimation output, and (c) detection output for $= t$.

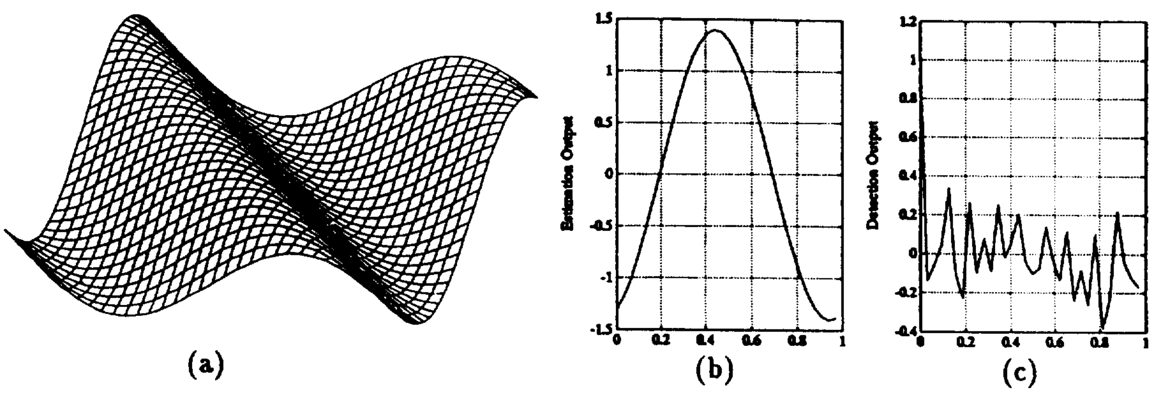

Figure 3. (a) Circulant matrix h, (b) estimation output, and (c) detection output for $= \sin \omega t$.

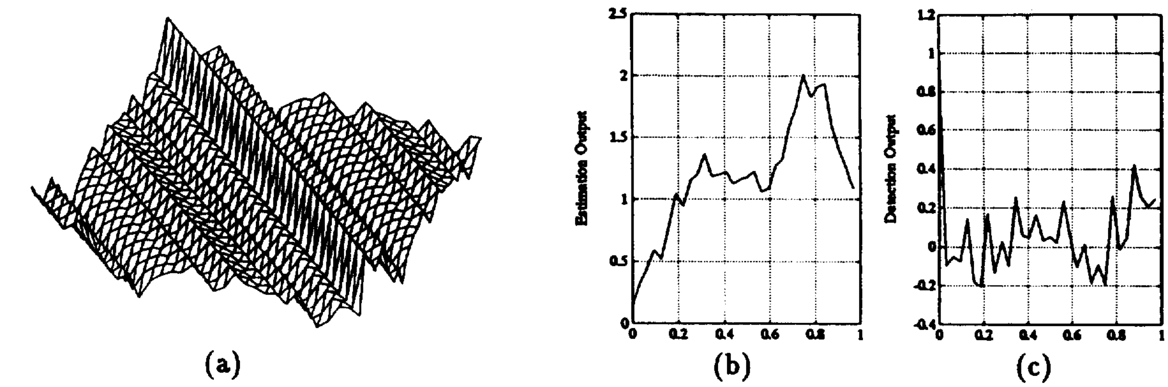

Figure 4. (a) Circulant matrix h, (b) estimation output, and (c) detection output for $= t$.

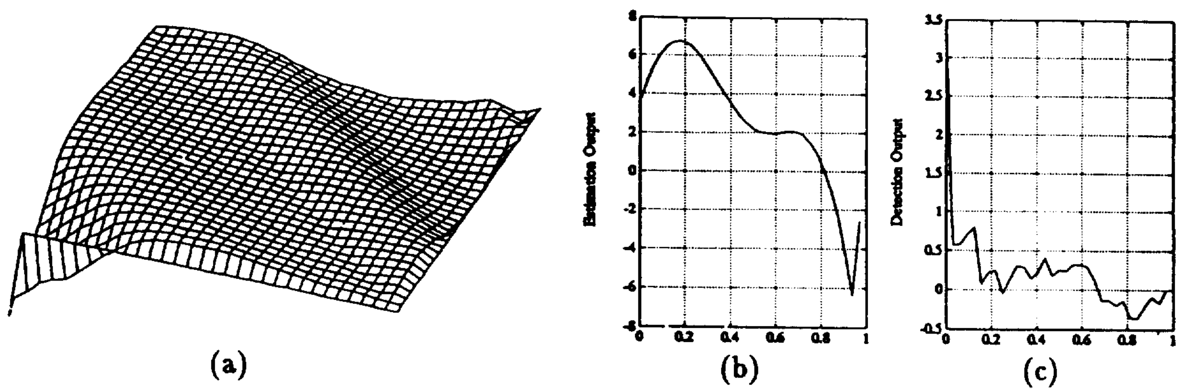

Figure 5. (a) Toeplitz matrix h, (b) estimation output, and (c) detection output for $= \sin\omega t$.

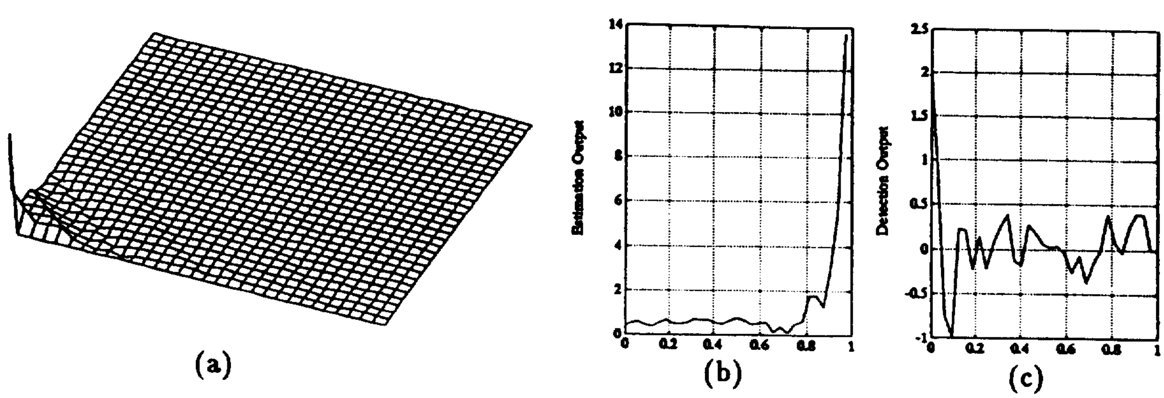

Figure 6. (a) Toeplitz matrix h, (b) estimation output, and (c) detection output for $= t$.