A Transformation Approach For Modeling and Detecting Non-Gaussian Signals

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Abstract

We present a new approach to the modeling of non-Gaussian complex random signals. The method transforms an underlying complex white gaussian sequence, whose magnitude is shaped by a Zero Memory Non-Linear (ZMNL) transformation. In this way, we match the magnitude pdf, and the power spectral density of the non-gaussian output. The ZMNL technique has the additional benefits of synthesizing complex, positive, or real valued signals by keeping only portions of the complex signal. This provides a quick simulation capability. The jpdf of a multivariate sample is easily computed from our model. We use this to form a likelihood detector for the presence of our non-gaussian versus a white gaussian signal. The dramatic improvement of the likelihood detector compared with a gaussian based quadratic detector is presented.

1 Introduction

There has been much discussion in the literature recently regarding the non-Gaussian behavior of certain physical processes. These include acoustic wave propagation in a random ocean and clutter returns in high resolution radars as discussed in Ewart [1] and Ward et al [2] respectively. This demonstrates a need for a model that incorporates non-Gaussian statistics and which can be used in signal processing. Further, these processes are often bandpass and modeled as complex, requiring a modeling approach that can handle complex seqeunces.

We present a new method for the modeling of non-Gaussian complex random signals. The method is based on an underlying complex gaussian sequence, whose magnitude is shaped by a Zero Memory Non-Linear (ZMNL) transformation. In this way, we match the magnitude pdf, and the power spectral density of the non-gaussian process. The ZMNL technique has the additional benefits of synthesizing complex, positive, or real valued signals by keeping only portions of the complex signal. This provides a quick simulation capability. The approach is an extension of Liu and Munson [4] from real to complex real signals.

The jpdf of a multivariate sample can be easily computed for our model. We use this to form a likelihood detector for the presence of our non-gaussian versus a white gaussian signal. The dramatic improvement of the likelihood detector compared with a gaussian based quadratic detector is presented.

2 Modeling

The specification of a covariance matrix, mean vector, and marginal distribution does not uniquely characterize a non-Gaussian sequence. Yet, in many situations this is the only information available. In practice, only these quantities can be accurately estimated since estimates of the higher order moments suffer from high sampling variances. Even if they are available, incorporating them into a modeling scheme is problematic. That is, we seek jpdfs not simply moments.

The tradeoffs among modeling candidates become issues of usability. The model presented in this section has the advantages of simplicity and generality, but, more importantly in this discussion, is the ability to compute the jpdf of a multivariate sample. This benefit, allows one to incorporate the model in signal processing architectures, such as the one developed in the next section.

The idea is to generate a non-Gaussian sequence as a distorted version of a correlated complex Gaussian sequence. The distortion, which is restricted to a Zero Memory Non-Linearity (ZMNL), acts only to shape the envelope of the complex Gaussian process. This provides the capability to incorporate into the model any prescribed marginal amplitude distribution.

A block diagram outline the model is seen in figure 1. We begin with a complex iid $N(0,1)$ $N \times 1$ sequence, $\tilde{\mathbf{x}}$. The complex Gaussian sequence, $\tilde{\mathbf{y}}$, is obtained from applying the filter H, $\tilde{\mathbf{y}} = H\tilde{\mathbf{x}}$, so that the resulting correlation of $\tilde{\mathbf{y}}$ is $C_y = H^{\dagger}H$. Here, $H^{\dagger}$ is the conjugate transpose of H. The mean of $\tilde{\mathbf{y}}$ is zero and has variance σ^2.

For a given covariance matrix, C_y, the filter H is obtained using an appropriate factorization algorithm. We use the Cholesky decomposition because it is computated efficiently and it provides a lower triangular matrix which results in a causal filter.

We write the complex Gaussian sequence, $\tilde{\mathbf{y}}$, in polar coordinates such that $\tilde{\mathbf{y}} = \mathbf{T}e^{j\boldsymbol{\theta}}$. The marginal distribution of $\mathbf{T}$ is Rayleigh and the marginal distribution of $\boldsymbol{\theta}$ is uniform. A sample of the resulting non-Gaussian sequence is written as

$$\mathbf{R}_n e^{j\boldsymbol{\theta}_n} = g_n(\mathbf{T}_n)e^{j\boldsymbol{\theta}_n}.$$

Here, the ZMNL, $g_n(\cdot)$, is applied to the amplitude and is obtained by solving

$$g_n(T) = F_{\mathbf{R}_n}^{-1}(F_{\mathbf{T}_n}(T)), \tag{1}$$

where $F_{\mathbf{R}_n}(\cdot)$ is the desired marginal distribution function for the n^{th} value in the sequence $\mathbf{R}$ and $F_{\mathbf{T}_n}(\cdot)$ is the Rayleigh distribution function. The derivation of this function is easily understood if we view it in two stages. First, it transforms the Rayleigh variates to uniform variates using the Rayleigh distribution function. Secondly, these uniform variates are transformed to variates of the desired distribution through its inverse distribution function.

The multivariate distribution for the non-Gaussian sequence, $\tilde{\mathbf{z}}$, is derived by applying the transformations to the multivariate Gaussian distribution of $\tilde{\mathbf{y}}$. The result of doing this yields

$$f_{\mathbf{z}}(z) = \frac{1}{(2\pi)^n|C_y|}\prod_{k=1}^{N}\frac{g_k^{-1}(R_k)f_R(R_k)}{R_k f_T\left(g_k^{-1}(R_k)\right)}\times \exp\left(-\frac{1}{2}\hat{z}^H C_y^{-1}\hat{Z}\right) \tag{2}$$

where
$\hat{z} = [g_1^{-1}(R_1)e^{j\theta_1} \quad g_2^{-1}(R_2)e^{j\theta_2} \quad \ldots \quad g_N^{-1}(R_N)e^{j\theta_N}]^T$, $R_k = |z_k|$, and $\theta_k = angle(z_k)$.

One crucial point that we must address is the covariance of the resulting non-Gaussian sequence. Clearly, because of the ZMNL transformation, the covariance of $\tilde{\mathbf{z}}$ will not be the same as the covariance of $\tilde{\mathbf{y}}$, C_y. To compensate for this, the effective change in

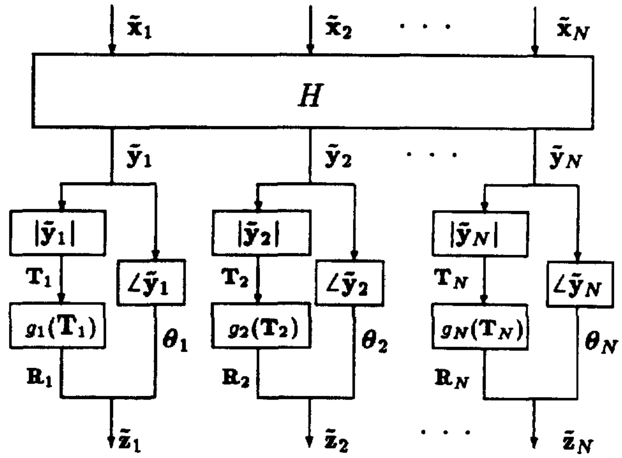

Figure 1: The ZMNL Envelope Mapping Model

correlation caused by the ZMNL must be computed and accounted for. This is accomplished by creating a mapping from the correlation of the complex Gaussian sequence, $\tilde{\mathbf{y}}$, to the correlation of the non-Gaussian sequence, $\tilde{\mathbf{z}}$. We define $\rho e^{j\phi}$ to be the correlation coefficient of two complex Gaussian points in the sequence $\tilde{\mathbf{y}}$. The covariance between two points in the the sequence $\tilde{\mathbf{z}}$ is computed from

$$<\tilde{\mathbf{z}}_n\tilde{\mathbf{z}}_m^*> = e^{j\phi}\int_0^{\infty}\int_0^{\infty} g_n(T_n)g_m(T_m)\times h(T_n,T_m)dT_n dT_m \tag{3}$$

where,

$$h(T_n,T_m) = \frac{T_nT_m}{\sigma^2(1-\rho^2)}\exp\left(-\frac{T_n^2+T_m^2}{2\sigma^2(1-\rho^2)}\right)\times I_1\left(\frac{\rho T_nT_m}{\sigma^4(1-\rho^2)}\right).$$

This equation is computed for a number of correlation coefficient values of the complex Gaussian sequence, $\tilde{\mathbf{y}}$, to obtain a map, $\rho e^{j\phi} \rightarrow <\tilde{\mathbf{z}}_n\tilde{\mathbf{z}}_m^*>$. This map is then inverted to find the proper correlation coefficient value needed in the Gaussian sequence to obtain the desired covariance value in the non-Gaussian sequence. It is important to note that the phase correlation is not affected by the transformation because the term $e^{j\phi}$ appears outside the integrand, independent of T, so we simply need to map the amplitude correlation, $\rho \rightarrow |<\tilde{\mathbf{z}}_n\tilde{\mathbf{z}}_m^*>|$.

Because equation (3) must be computed several times, simplifying the integral will greatly improve performance. Reed [3] decomposes the integral into a series of Laguerre polynomials. Replacing T_n with $\sigma\sqrt{2x}$, T_m with $\sigma\sqrt{2y}$, and ρ^2 with t the above equation is

$$< \tilde{\mathbf{z}}_n \; \tilde{\mathbf{z}}_m^* >= \frac{e^{j\phi}}{1-t}\int_0^\infty \int_0^\infty g_n(\sigma\sqrt{2x})g_m(\sigma\sqrt{2y})\times$$

$$\exp\left(-\frac{(x+y)/t}{1-t}\right) I_1\left(\frac{2\sqrt{xyt}}{1-t}\right) e^{-(x+y)}dxdy \quad (4)$$

With the use of the Hardy-Hill identity

$$\frac{(xyt)^{-1/2\alpha}}{1-t} I_\alpha\left(\frac{2\sqrt{xyt}}{1-t}\right)\exp\left(-\frac{t(x+y)}{1-t}\right) =$$

$$\sum_{k=0}^{\infty} \frac{L_k^{(\alpha)}(x)L_k^{(\alpha)}(y)t^k k!}{\Gamma(\alpha+k+1)}$$

where $L_k^{(\alpha)}(u)$ is the generalized Laguerre polynomial of order α defined by

$$L_k^{(\alpha)}(u) = \frac{e^u u^{-\alpha}}{k!}\frac{d^k}{du^k}(e^{-u}u^{k+\alpha})$$

equation (4) becomes

$$< \tilde{\mathbf{z}}_n \; \tilde{\mathbf{z}}_m^* >= \frac{e^{j\phi}}{1-t}\int_0^\infty \int_0^\infty g_n(\sigma\sqrt{2x})g_m(\sigma\sqrt{2y})\times$$

$$\sqrt{xyt}\sum_{k=0}^{\infty} \frac{L_k^{(1)}(x)L_k^{(1)}(y)t^k k!}{\Gamma(k+2)} e^{-(x+y)}dxdy.$$

Lastly, we substitute back in for ρ to obtain

$$< \tilde{\mathbf{z}}_n \tilde{\mathbf{z}}_m^* >= \frac{\rho e^{j\phi}}{1-\rho^2}\sum_{k=0}^{\infty}\frac{\rho^{2k}c_k^{(1)}c_k^{(2)}k!}{\Gamma(k+2)}$$

where $c_k^{(1)}$ and $c_k^{(2)}$ are the integral coefficients

$$c_k^{(1)} = \int_0^\infty e^{-x}\sqrt{x}g_1(\sigma\sqrt{2x})L_k^{(1)}(x)dx$$

$$c_k^{(2)} = \int_0^\infty e^{-y}\sqrt{y}g_2(\sigma\sqrt{2y})L_k^{(1)}(y)dy.$$

For the special case of a power law transformation, $g_1(T) = T^\alpha$ and $g_2(T) = T^\alpha$ Reed provides a further simplification yielding

$$< \tilde{\mathbf{z}}_n \tilde{\mathbf{z}}_m^* > \; = \rho e^{j\phi}(\sigma\sqrt{2})^{2\alpha}\Gamma\left(\frac{3+\alpha}{2}\right)^2 \times$$

$$_2F_1\left(\frac{1-\alpha}{2}, \frac{1-\alpha}{2}; 2; \rho^2\right)$$

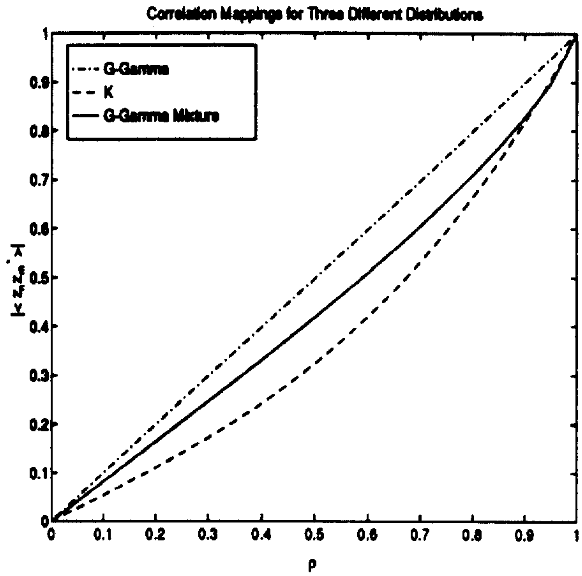

Figure 2: Covariance Maps For 3 Different Marginal Distributions

where $_2F_1(a,b;c;z)$ is the Gaussian hypergeometric function

$$_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)z^n}{\Gamma(c+n)\Gamma(n+1)}.$$

Computing the integral in (3) is also possible using strictly numerical integration. The integrand is generally a smooth function and is evaluated efficiently using the trapezoidal rule. Furthermore, if it is assumed that $g_1(T) = g_2(T)$ the integrand is symmetric about $T_1 = T_2$ reducing the computation in half. To get a rough idea of the computational complexity, on a SPARC 10 evaluating the integral 20 times took roughly 3/4 of a second. Examples of what these mappings look like are seen in figure 2.

A sequence was generated using this model. The real portion of this sequence is seen in figure 3 before and after the ZMNL transformation is applied (i.e. the $Re\{\tilde{\mathbf{y}}\}$ and the $Re\{\tilde{\mathbf{z}}\}$ respectively). The marginal amplitude distribution for this non-Gaussian sequence is Weibull, which has pdf

$$f_{\mathbf{R}}(R) = ba^{-b}R^{b-1}e^{-(R/a)^b},$$

and the correlation is exponential with a correlation length of 20. The distribution parameters are $a = 0.6351$, and $b = 0.9$. What is evident in this figure is that the ZMNL acts to accentuate the large values and suppress the smaller ones, producing a sequence with a spikey appearance, which is characteristic of heavy-tailed non-Gaussian sequences.

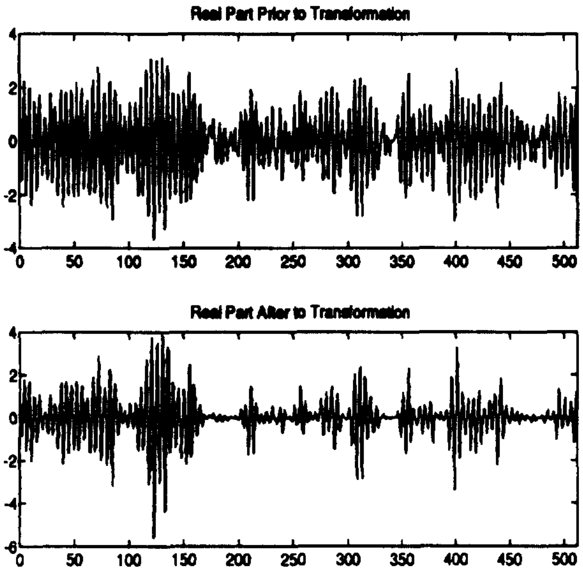

Figure 3: Gaussian (G) top and Non-Gaussian (NG) bottom

3 Detection

Our modeling scheme presented in the previous section contains a multivariate distribution (2) which we incorporate into a detection problem. We consider the scenario in which we have two alternative hypothesis, H_0 and H_1,

$$\begin{aligned} H_0 : \tilde{\mathbf{z}}_n &= \mathbf{w}_n \\ H_1 : \tilde{\mathbf{z}}_n &= \mathbf{s}_n \end{aligned}$$

where $\tilde{\mathbf{z}}_n$ is our received vector at time n, $\tilde{\mathbf{w}}_n$ is a complex uncorrelated Gaussian vector, and $\tilde{\mathbf{s}}_n$ is a complex correlated non-Gaussian vector with each vector having dimension N. Each hypothesis is equally likely providing the following likelihood ratio

$$\frac{f_{\mathbf{z}}(z_k|H_1)}{f_{\mathbf{z}}(z_k|H_0)} \gtrless 1.$$

We are basing our test on a single observation of length N. We tested two different detectors. The first uses our model as the non-Gaussian pdf for $f_{\mathbf{z}}(z_k|H_1)$, while the second uses the multivariate Gaussian pdf for $f_{\mathbf{z}}(r_k|H_1)$, which reduces to a quadratic detector.

This comparison provides insight as to how a Gaussian detector fails when the statistics are markedly non-Gaussian. In both detection schemes $f_{\mathbf{z}}(z_k|H_0)$ is the uncorrelated multivariate Gaussian pdf.

Under H_1 the received sequence has an exponential correlation and a generalized gamma marginal amplitude distribution which has pdf

$$f_{\mathbf{R}}(R) = \frac{|b|}{\Gamma(k)} a^{-bk} R^{bk-1} e^{-(R/a)^b}. \tag{5}$$

where $\mathbf{R}_n = |\tilde{\mathbf{s}}_n|$. The parameters selected were $a = 0.3094$, $b = 1.003$, $k = 2.283$, which yield a variance of 1. The SNR, defined as

$$SNR = 10log_{10}\left(\frac{< s^2 >}{< w^2 >}\right),$$

is fixed at 0 dB (i.e. the variance of the white noise is fixed at 1). To obtain error probabilities 50000 independent trials were run under each hypothesis for the two detector types. The non-Gaussian data was generated using the transformation model. Figure 4 shows the results. As is evident from the curve, as we increase the dimensionality of our received vector the non-Gaussian detector improves dramatically while the performance of the Gaussian detector remains poor. As one would expect, there is an increase in the distance between the non-Gaussian distribution and the white Gaussian distribution as the dimensionality is increased.

Further simulations were carried out where the tail behavior of the marginal distribution was adjusted. From (5) it is clear that the exponential term is going to dominate the behavior of the distribution for large values of $\tilde{\mathbf{R}}$, thus, we control the tail behavior by varying the parameter b. k is fixed at 1, and a is selected so that the variance remains constant at 1. The white Gaussian sequence, $\tilde{\mathbf{w}}_k$, is also selected to have a variance of 1 yielding a $SNR = 0$ dB. Some special cases are $b = 2$ which is the Rayleigh case (Gaussian statistics) and $b = 1$ which is the exponential case. Figure 5 shows the performance of the two detectors as we vary b. As expected, the two have similar performance at $b \approx 2$ because we have not deviated much from Gaussian statistics. However, for smaller values of b the performance of the non-Gaussian detector is clearly superior.

4 Conclusion

We have presented a model of a complex non-Gaussian sequence which allows us to incorporate both a prescribed marginal amplitude distribution and covariance. The method models the non-Gaussian sequence as a Zero Memory Non-Linear (ZMNL) transformation of the envelope of a complex Gaussian distribution. With the appropriate selection of the

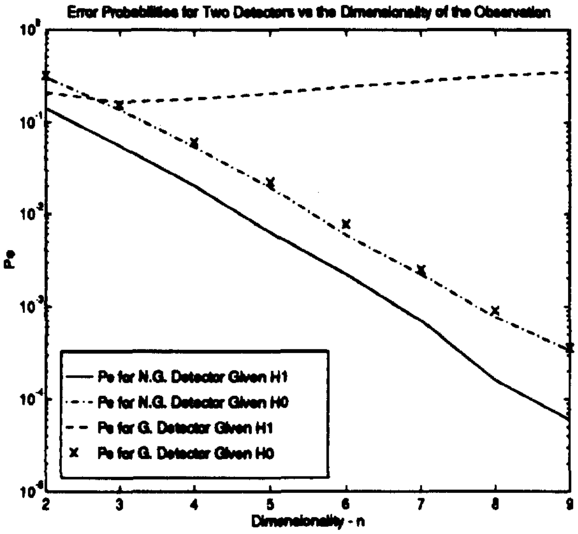

Figure 4: Performance Comparison

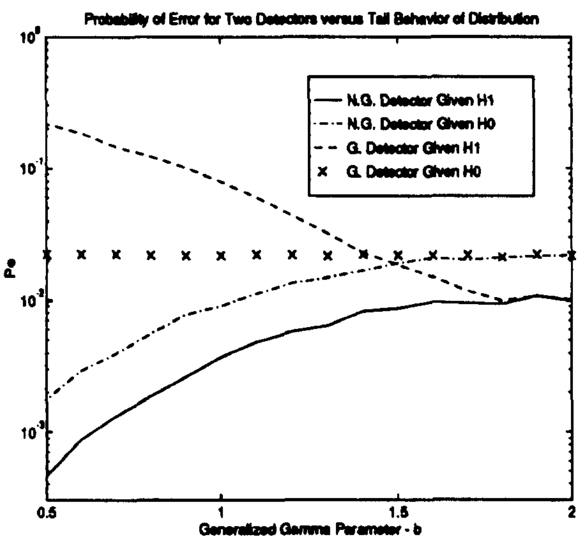

Figure 5: Performance versus tail behavior of G-Gamma Distribution

ZMNL the model is able to match any amplitude distribution. A particular covariance is matched by correlating the complex Gaussian sequence and accounting for any perturbation caused by the ZMNL.

This model provides an efficient method to generate complex, positive, and real valued non-Gaussian seqeunces. Further, the jpdf of a multivariate sample can be computed from the model. Using this, we have shown that incorporating non-Gaussian statistics in a detection problem greatly improves the performance when the statistics are non-Gaussian. As expected, our results show that including non-Gaussian statistics becomes increasingly important the more the marginal distribution deviates from Rayleigh (Gaussian) distribution.

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