Adaptive Systems with Coefficients Indicating Frequency

Gerard Coutu
RPI@Hartford Graduate Center

Monique Fargues
Naval Postgraduate School

Samuel D. Stearns
Sandia National Laboratories

David J. Plaza
Granby, CT

Abstract

The theory and performance of adaptive filters and predictors with coefficients that indicate frequency content is examined. The first structure considered is a parallel bank of IIR bandpass filters and the second is a frequency sampling filter. The error surfaces for both systems are derived for the cases of filtering and prediction. The frequency interpretation of the results is straightforward for the case of filtering and more complicated, but still useful, for the case of prediction. The theory is supported by adaptive implementations.

1 Introduction

The general field of adaptive signal processing uses systems with both IIR and FIR transfer functions to perform tasks that include filtering, prediction, identification, and control. However, it is often important to have an understanding of the activity of the signal in the frequency domain as the adaptive processing is occuring. In the present paper, two general adaptive systems are examined.

The first structure considered is a parallel bank of IIR bandpasses with adaptive center frequencies and fixed gains. Each bandpass is derived from a first order Butterworth filter by a bilinear transformation.

The second topic investigated is the adaptive frequency sampling filter and predictor. The frequency sampling method of designing and implementing FIR filters is based directly on frequency domain criteria [1], as will be shown later. The coefficients of the frequency domain filter are equal in magnitude to those of the Discrete Fourier Transform (DFT) equally spaced around the unit circle. Recently, there have been two investigations of adaptive implementations of the frequency sampling structure. The first examined the frequency sampling filter [2]. The second employed an adaptive frequency sampling predictor to the problem of seismic signal compression [3].

The analysis of both the IIR and frequency sampling systems follow. Theoretical derivations of the error surface for both the filter and the predictor, as well as the LMS update equations, are included.

2 Theory

2.1 IIR Filterbank

It is well known that a second order bandpass filter based on the bilinear transformation of a first order Butterworth lowpass has the transfer function [1]

$$H_k(z) = \frac{a_{0k}(1-z^{-1})}{1-b_{1k}z^{-1}-b_{2k}z^{-2}} \qquad (1)$$

where

$$a_{0k} = \frac{1}{1+D_k}, b_{1k} = \frac{D_k E_k}{1+D_k}, b_{2k} = \frac{1-D_k}{1+D_k} \qquad (2)$$

$$D_k = \cot(\frac{\pi}{2}(\frac{f_{bk}}{f_0})), E_k = 2\cos(\pi(\frac{f_{ck}}{f_0})) \qquad (3)$$

The subscript k indicates the possibility of many bandpasses, for k= 1,N. The folding frequency, one half of the sampling frequency, is f_0. The center frequency of the bandpass is f_{ck}. The 3 dB bandwidth of the bandpass is f_{bk}. Let x(n) be the input. Then the associated difference equation for a single bandpass is $y_k(n) = a_{0k}(x(n)-x(n-2))+b_{1k}y(n-1)+b_{2k}y(n-2)$, and the overall output is the sum of each of the parallel IIR filters, $y(n) = \sum_{k=0}^{N} y_k(n)$, with a transfer function $H(z) = \sum_{k=0}^{N} H_k(z)$.

2.2 Error Surface for IIR Filter

The form of the error surface is of the form is[4]

$$\zeta = \phi_{dd}(0)+\frac{1}{2\pi j}\oint_c (H(z^{-1})\Phi_{xx}(z)-2\Phi_{dx}(z))H(z)\frac{dz}{z} \qquad (4)$$

for input signal x(n) and desired signal d(n). Consider the case of a single bandpass with a transfer function given by Eq(1). Let the input signal, x(n), be a sum of a single sinusoid plus unity variance white noise. Thus, $\phi_{xx}(n) = \frac{1}{2}\cos(\frac{2\pi n}{N}) + \delta(n)$. Let the desired signal be an uncorrupted version of the input so that $\phi_{dd}(n) = \frac{1}{2}\cos(\frac{2\pi n}{N})$. Since the signal and noise are uncorrelated, $\phi_{dd}(n) = \phi_{dx}(n)$. Taking the corresponding z-transforms and substituting into Eq(4) yields

$$\zeta = \frac{1}{2} + \frac{3}{2}\frac{a_{0k}^2(r_1^2-1)^2}{b_{2k}(r_1-r_2)(r_1-r_1')(r_1-r_2')r_1} \quad (5)$$
$$+\frac{3}{2}\frac{a_{0k}^2(r_2^2-1)^2}{b_{2k}(r_2-r_1)(r_2-r_1')(r_2-r_2')r_2}$$
$$+\frac{3}{2}\frac{a_{0k}^2}{b_{2k}r_1r_2r_1'r_2'} + \frac{a_{0k}}{r_1r_2}$$
$$+\frac{a_{0k}^2(r_1^2-1)^2(\cos(\frac{2\pi}{N})-r_1)}{b_{2k}(r_1-r_2)(r_1-r_1')(r_1-r_2')(1-2r_1\cos(\frac{2\pi}{N})+r_1^2)}$$
$$+\frac{a_{0k}^2(r_2^2-1)^2(\cos(\frac{2\pi}{N})-r_2)}{b_{2k}(r_2-r_1)(r_2-r_1')(r_2-r_2')(1-2r_2\cos(\frac{2\pi}{N})+r_2^2)}$$
$$-\frac{a_{0k}(r_1^2-1)(\cos(\frac{2\pi}{N})-r_1)}{(r_1-r_2)(1-2r_1\cos(\frac{2\pi}{N})+r_1^2)}$$
$$-\frac{a_{0k}(r_2^2-1)(\cos(\frac{2\pi}{N})-r_2)}{(r_2-r_1)(1-2r_2\cos(\frac{2\pi}{N})+r_2^2)}$$
$$-\frac{a_{0k}(r_1^2-1)}{(r_1-r_2)r_1} - \frac{a_{0k}(r_2^2-1)}{(r_2-r_1)r_2}$$

where

$$r_{1,2} = [\frac{b_{1k}}{2}] + / - [\frac{1}{2}(b_{1k}^2 + 4b_{2k})^{\frac{1}{2}}] \quad (6)$$

$$r_{1,2}' = [-\frac{b_{1k}}{2b_{2k}}] - / + [\frac{1}{2}(\frac{b_{1k}^2}{b_{2k}^2} + \frac{4}{b_{2k}})^{\frac{1}{2}}] \quad (7)$$

The details of the contour integrals can be found elsewhere [5]. In general, the error surface is not quadradic. The error surface is quadradic if only the gains, a_{0k}, are adapted.

For the present investigation, a_{0k} and b_{2k} are fixed, and b_{1k} is adapted. Specifically, the parameter to be adapted is center frequency. The normalized error surface as a function of center frequency is found by substitution of Eqs(2) and (3) into Eq(5). It is shown in Figure (1). Note that the minimum occurs at the desired center frequency. Thus, the frequency of the signal can be determined directly from the adapted center frequency after convergence. Even though the error surface is nonquadradic, it is unimodal and useful for search algorithms.

2.3 Error Surface for IIR Predictor

The case of the predictor differs in two ways. First the transfer function is multiplied by z^{-1} for one-step prediction. Secondly, the input signal also contains noise since it is a delayed version of the input. The resulting error surface is of the form

$$\zeta = \frac{3}{2} + \frac{3}{2}\frac{a_{0k}^2(r_1^2-1)^2}{b_{2k}(r_1-r_2)(r_1-r_1')(r_1-r_2')r_1} \quad (8)$$
$$+\frac{3}{2}\frac{a_{0k}^2(r_2^2-1)^2}{b_{2k}(r_2-r_1)(r_2-r_1')(r_2-r_2')r_2}$$
$$+\frac{3}{2}\frac{a_{0k}^2}{b_{2k}r_1r_2r_1'r_2'} + (\frac{a_{0k}}{r_1r_2}$$
$$-\frac{a_{0k}(r_1^2-1)}{(r_1-r_2)r_1} - \frac{a_{0k}(r_2^2-1)}{(r_2-r_1)r_2})\cos(\frac{2\pi}{N})$$
$$+\frac{a_{0k}^2(r_1^2-1)^2(\cos(\frac{2\pi}{N})-r_1)}{b_{2k}(r_1-r_2)(r_1-r_1')(r_1-r_2')(1-2r_1\cos(\frac{2\pi}{N})+r_1^2)}$$
$$+\frac{a_{0k}^2(r_2^2-1)^2(\cos(\frac{2\pi}{N})-r_2)}{b_{2k}(r_2-r_1)(r_2-r_1')(r_2-r_2')(1-2r_2\cos(\frac{2\pi}{N})+r_2^2)}$$
$$-\frac{a_{0k}(r_1^2-1)(\cos(\frac{2\pi}{N})-r_1)}{(r_1-r_2)(1-2r_1\cos(\frac{2\pi}{N})+r_1^2)}$$
$$-\frac{a_{0k}(r_2^2-1)(\cos(\frac{2\pi}{N})-r_2)}{(r_2-r_1)(1-2r_2\cos(\frac{2\pi}{N})+r_2^2)}$$

Just as with the case of the filter, the error surface is not quadradic (unless only a_{0k}, are adapted). The normalized error surface as a function of center frequency, with a_{0k} and b_{2k} fixed, is shown in Figure (2)and is bimodal. The global minimum occurs near (but not at) the frequency of interest, but there is an additional minimum at zero. A global maximum, that occurs just beneath the lower 3 dB cutoff of the bandpass, separates the two minima. Thus, if the initial center frequency of the bandpass is higher than the frequency of the signal, search techniques will work well. However, unlike the case of filtering, the minimum value of the error surface for this IIR predictor gives a center frequency close to, but not exactly at, the center frequency of the signal. This is the "penalty" of prediction for this configuration.

2.4 LMS IIR Implementation

The form of the LMS algorithm for a fixed gain and a variable center frequency is found by differentiating y_k(n)(defined in Sec(2.1)) with respect to f_{ck} to get

$$\frac{\partial y_k(n)}{\partial f_{ck}} = \beta_k(n) = -\frac{2\pi\sin(\pi\frac{f_{ck}}{f_0})}{f_0}\frac{D_k}{1+D_k}y(n-1) \quad (9)$$

$$+b_{1k}\beta_k(n-1)+b_{2k}\beta_k(n-2)$$

The update equation after n iterations becomes

$$f_{ck}(n+1)=f_{ck}(n)+2\mu e(n)\beta_k(n) \quad (10)$$

where e(n) = d(n) - y(n), with y(n) as the sum of the individual bandpass filters. Note that $\frac{\partial y(n)}{\partial f_{ck}}=\frac{\partial y_k(n)}{\partial f_{ck}}$. The updated center frequencies are then used to calculated new values of E_k and b_{1k} for use during the next iteration. This method ensures the stability of the IIR filter because of the implicit use of the bilinear transformation.

2.5 Frequency Sampling Filter Bank

The frequency sampling filter is a Finite Impulse Response (FIR) filter. In general, the FIR transfer function is H(z)=$\sum_{m=0}^{N-1} b(m)z^{-m}$ The key step in the formulation of the frequency sampling filter is to express the the coefficients, b(m), as the inverse DFT of B(k), ie, b(m) = $\frac{1}{N}\sum_{k=0}^{N-1}$ B(k) $e^{\frac{j2\pi mk}{N}}$. When substituted, the transfer function becomes

$$H(z)=\frac{1}{N}(1-z^{-N})\sum_{k=0}^{N-1}\frac{B(k)}{1-e^{\frac{j2\pi k}{N}}z^{-1}} \quad (11)$$

Let $\bar{H}(k)$ represent H(z) evaluated at z= $e^{\frac{2j\pi k}{N}}$, ie, $\bar{H}(k)=H(e^{\frac{2j\pi k}{N}})$. In order to create a smooth reconstruction between sample points, it has been shown [1] that the sample set of magnitudes, $\bar{H}(k)$, for the transfer function are chosen, due to phase considerations, such that $B(k)=(-1)^k\bar{H}(k)$. Finally, since B(k) are real and B(k) and B(N-k) are complex conjugates, it follows that B(N-k)=B(k). Furthermore, it is convenient to assume B($\frac{N}{2}$) is zero if N is even. Lastly, the cancellation of zeros and poles will not be exact and are placed on a circle of radius r slightly less than one. With these stipulations, the transfer function becomes

$$H(z)=\frac{2(1-r^N z^{-N})}{N}\times \quad (12)$$

$$\sum_{k=0}^{\frac{N}{2}-1}\frac{(-1)^k\bar{H}(k)[1-rz^{-1}\cos(\frac{2\pi k}{N})]}{1-2rz^{-1}\cos(\frac{2\pi k}{N})+r^2z^{-2}}$$

The implementation of the overall transfer function is the cascade of an all zero FIR filter with a parallel bank of IIR bandpass filters. Let

$$w(n)=\frac{2}{N}[x(n)-x(n-N)] \quad (13)$$

The output then becomes

$$y(n)=\sum_{k=0}^{\frac{N}{2}-1}(-1)^k\bar{H}(k)[w(n)-r\cos(\frac{2\pi k}{N})w(n-1)] \quad (14)$$

$$+\sum_{k=0}^{\frac{N}{2}-1}2r\cos(\frac{2\pi k}{N})y_k(n-1)-r^2\sum_{k=0}^{\frac{N}{2}-1}y_k(n-2)$$

with the overall output, y, as the sum of the outputs of the bandpass filters, y_k.

2.6 Error Surface for Frequency Sampling Filter

In forming the theory of the frequency sampling filter, it is important to bear in mind that the overall filter is FIR of the form given in Eq(1). Let d be the desired signal and x be the input signal in the adaptive formulation. From Eq(5), for the frequency sampling filter,

$$\zeta=\phi_{dd}(0)+\frac{2}{N^2}\sum_{k=0}^{\frac{N}{2}-1}\sum_{p=0}^{\frac{N}{2}-1}(-1)^{k+p}\bar{H}(k)\bar{H}(p)\times \quad (15)$$

$$\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}(-1)^{m+n}\times$$

$$\cos(\frac{2\pi(nk+mp)}{N}-\pi(n+m))\phi_{xx}(n-m)$$

$$-\frac{4}{N}\sum_{p=0}^{\frac{N}{2}-1}(-1)^{p+m}\bar{H}(p)\sum_{m=0}^{N-1}\cos(\frac{2\pi mp}{N}-\pi m)\phi_{dx}(-m)$$

An examination of the error surface indicates that it is indeed quadradic in $\bar{H}(k)$.

2.7 Error Surface for Frequency Sampling Predictor

When the filter is implemented as an adaptive line enhancer [4], the input to the filter is a delayed version of the input signal, whereas the desired reference signal is undelayed. It is clear that the transfer function for the case of the Δ -step predictor is $z^{-\Delta}$ times the transfer function of the filter in the presence of noise and, thus, the error surface is quadradic. However, if the coefficients are to be taken as an indication of frequency, it is most instructive to compare the coefficients of the Δ -step, frequency sampling predictor to the coefficients of the frequency sampling filter (Δ = 0), derived above.

When implemented as a Δ -step predictor, the coefficients of the predictor become $H_p(k) = \bar{H}(k)\exp(\frac{j2\pi k\Delta}{N})$ when compared with the coefficients of the filter, $\Delta = 0$. Furthermore, it is imposed that $H_p(k) = H_p(N-k)$. Substitution into the frequency selective structure and careful trigonometry results in

$$H_p(z) = \frac{2(1 - r^N z^{-N})}{N} \times$$
$$\sum_{k=0}^{\frac{N}{2}-1} [\frac{(-1)^k \bar{H}(k)\cos(\frac{2\pi k\Delta}{N})}{1 - 2rz^{-1}\cos(\frac{2\pi k}{N}) + r^2 z^{-2}} \times$$
$$[1 - rz^{-1}(\cos(\frac{2\pi k}{N}) + \tan(\frac{2\pi k\Delta}{N})\sin(\frac{2\pi k}{N})]] \quad (16)$$

for a practical implementation.

For each IIR filter, the gain is multiplied by the cosine of a term that has Δ in its argument. The cosine is zero when the argument is

$$\frac{2\pi k\Delta}{N} = \frac{\pi}{2} + m\pi; k = \frac{N(1+2m)}{4\Delta} \quad (17)$$

when m is an integer. These resulting predictive coefficients will be zero even if the corresponding frequencies exist in the data. The predictor will attempt to accomodate the presence of such a frequency in the data by adjusting the predictive coefficients on either side of the zero. For one-step prediction, there zeros at $k = \frac{N(1+2m)}{4}$. If N is even, then there is a zero at a frequency at one half of the maximum allowable limit allowed by the Nyquist criterion.

In general, the multiplication of the coefficients by the cosine term is the prediction penalty for the frequency sampling system subject the aforementioned assumptions. Sometimes a judicious choice of N can eliminate the problem by causing the zero to occur between at frequencies that correspond to the coefficients.

2.8 LMS Frequency Selective Implementation

To find the LMS update equations, take the partial derivative of the output, y(n), Eq(16), with respect to $\bar{H}(k)$ to get

$$\frac{\partial y(n)}{\partial \bar{H}(k)} = \frac{\partial y_k(n)}{\partial \bar{H}(k)} \quad (18)$$
$$= \alpha_k(n) = (-1)^k(w(n) - r\cos(\frac{2\pi k}{N})w(n-1))$$
$$+2r\cos(\frac{2\pi k}{N})\alpha_k(n-1) - r^2\alpha_k(n-2)$$

The update equations for the coefficients become

$$\bar{H}_{n+1}(k) = \bar{H}_n(k) + 2\mu e(n)\alpha_k(n) \quad (19)$$

3 Summary and Conclusions

The IIR filter is shown for the case fo a single bandpass with a bandwidth of 100 Hz and a sampling frequency of 10 kHz. The signal is a noisy sine wave with a frequency of 1 kHz with the same sampling rate. The desired signal was a noncorrupted sinusoid of the same frequency. The initial center frequency was set to a nominal value of 1005 Hz. The f_{ck}versus iteration is shown in Figure(3).It is clear that the bandpass filter tracks the center frequency of the sinusoid to 1kHz.

Two comparable cases are run for the IIR predictor. The signal to be tracked is the same as the filter. The desired signal is the same as the input signal except that there is a difference of a single step delay between the two. For the first case, initial center frequency is set to 1200 Hz. The theoretical minimum of the error surface for the IIR predictor will occur at 1030 Hz. Figure(4) shows that f_{ck} goes to 1028 Hz. Next, if the IIR predictor is started with a nominal value of 800 Hz, to the left of the maximum, the adaptation will be in the direction of dc, the local minima, as is shown in Figure(5).

For the frequency sampling filter, the input signal was white noise, variance one half, plus nine sinusoids, equally spaced, from 1-9 kHz, with a sampling frequency of 20 kHz. Four cases are considered.

The first case has $\Delta = 0$ and the desired signal to be the nine sines. Figure(6) shows that the peak frequencies are distinguished by the magnitude of the coefficients.The second case has $\Delta = 0$ but the desired signal is identical to the input (sines plus noise). Figure(7) shows the gains of the transfer function that best approximate an all-pass for the frequency sampling structure.The third case has $\Delta = 10$ step prediction. Eq(20) indicates that the ideal coefficients will be zero at k = 5, 15, 25,...,5(2m + 1). Note that there are no such frequencies in the signal. Figure(8) shows that the peak frequencies are picked out and nulls are introduced as per the theory. Finally, the last case is for $\Delta = 1$ step prediction. Eq(20) indicates that there will be a zero at k = 50. There is a frequency in the signal at this coefficient. Figure(9) shows that the values of the coefficients are skewed in an effort to have the surrounding bandpass filters matched the aforementioned frequency. In this case the frequency interpretation is problematic.

In conclusion, we have shown that a straightforward frequency interpretation is possible for both the IIR and frequency selective filters. As for the case of prediction, a frequency interpretation is feasible if care is taken.

References

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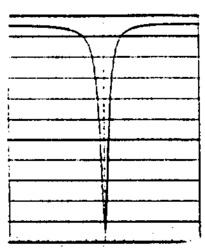

Figure 1: IIR Filter,ζ v Frq,min@1000Hz

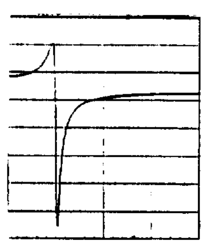

Figure 2: IIR Predictor,ζ v Frq,min@1030 Hz

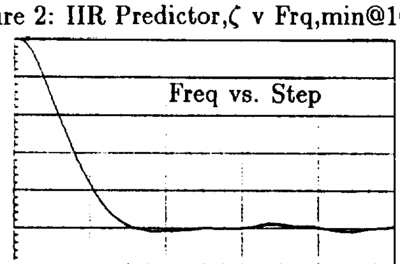

Figure 3: IIR Filter Start 1005Hz,End 1000Hz

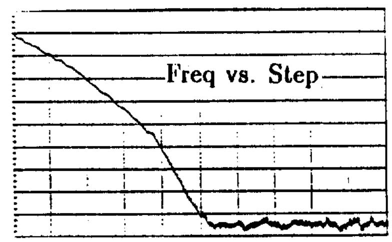

Figure 4: IIR Predictor Start 1200Hz,End 1028Hz

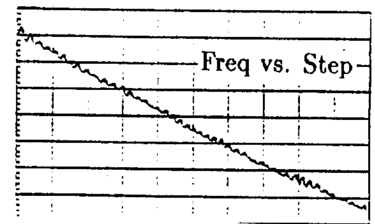

Figure 5: IIR Predictor Start 800Hz,End 735Hz

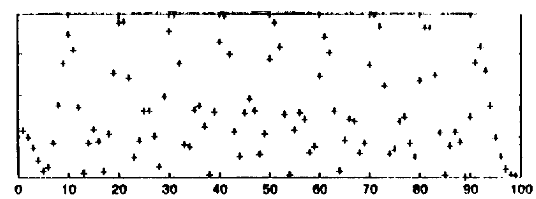

Figure 6: FIR Filter $\Delta = 0$

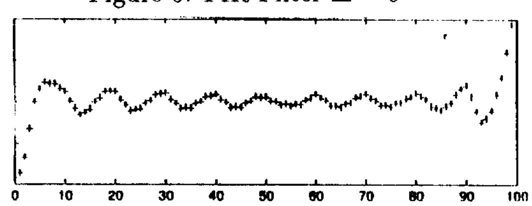

Figure 7: FIR $\Delta = 0$, Noisy Desired Signal

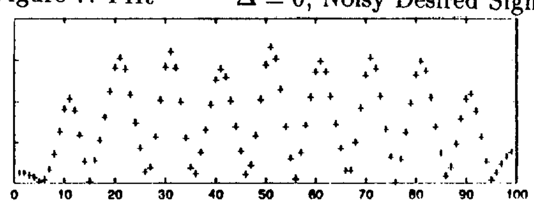

Figure 8: FIR Predictor,$\Delta = 10$

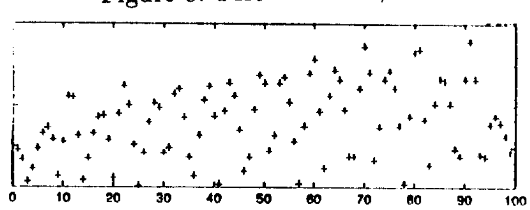

Figure 9: FIR Predictor,$\Delta = 1$