A REAL-TIME IMPLEMENTABLE SUBCLASS OF QUADRATIC TIME-FREQUENCY REPRESENTATIONS

Dongsheng Wu and Joel M. Morris
Electrical Engineering Department, University of Maryland Baltimore County
5401 Wilkens Avenue, Catonsville, MD 21228-5398

Abstract

One of the major obstacles for quadratic time-frequency representations to be used in real(finite)-time applications is the computational complexity. In this paper, a real(finite)-time implementable subclass of Cohen's Class of Distributions (CCD) is defined using windowing techniques. The kernel function for each member in this subclass can be characterized by a necessary and sufficient real(finite)-time implementable condition (RTIC) imposed on "global" kernels and "local" kernels in the ambiguity function domain. By choosing special "local" kernels, one can obtain the spectrogram (SPG) and the pseudo-Wigner-Ville distribution (PWVD) as special cases of this subclass. Our results show that time-frequency representations suitable for real-time implementation can be obtained that provide better resolution tradeoff between the time and frequency domains than occurs with the SPG and better frequency interference than occurs with the PWVD.

1. Introduction

Time-frequency distributions (TFDs) are two-dimensional functions that are used to depict the signal's behavior at each frequency as a function of time. They have been utilized to study a wide range of signals, including speech, music, and other acoustical signals, biological signals, radar and sonar signals, and geophysical signals. Most TFDs of current interest are members of Cohen's bilinear class[1].

In the continuous-time case, each member of this class is given by [1]

$$\rho_x(t,f;\phi) = \int_\tau \int_\upsilon \phi(\upsilon,\tau) AF_x(\upsilon,\tau) e^{-j2\pi f\tau} e^{j2\pi\upsilon t} d\upsilon d\tau \qquad (1)$$

(unless otherwise indicated, the limits of integration extend from $-\infty$ to ∞) where ϕ is the kernel function that defines the particular distribution. $AF_x(\upsilon,\tau)$ is the Sussman ambiguity function (AF) of the given signal x(t) and is given as

$$AF_x(\upsilon,\tau) = \int_\mu x(\mu+\tfrac{\tau}{2}) x^*(\mu-\tfrac{\tau}{2}) e^{-j2\pi\mu\upsilon} d\mu . \qquad (2)$$

Because of the inherent bilinearity of the distributions in the Cohen's class, each member exhibits some degree of cross interference behavior when dealing with multicomponent signals [7]. The various kernels in Cohen's class, each leading to different distributions with distinct properties, are used to suppress, to some extent, these crossterms while retaining as much as possible the concentrated autoterms. Examples are the cone-shape kernel (CK) [3], the exponential kernel [2], RID kernel, generalized exponential kernel, and butterworth kernel [4], and the recently proposed radial-Butterworth kernel (RBK) [5].

Most members in Cohen's class have to take into account all the data along the time axis. This makes the real(finite)-time implementation impractical for these members if the signal under analysis is rather long. However, some members, such as the PWVD and the SPG, use only the neighboring data at every time point via a sliding window in the temporal coordinate. This approach can be implemented in real(finite)-time and, thus, achieve efficient computation.

In this paper, a subclass of Cohen's class is defined by imposing conditions on their corresponding kernels such that every member in this subclass can be implemented in real-time for analyzing long signals. Computational complexity for all members in this subclass is discussed..

2. The Real(Finite)-Time Implementable Subclass

To clarify the term *real(finite)-time implementable distributions*, we loosely define a quadratic time-frequency distribution as real(finite)-time implementable if the computation of the distribution for each time-

1058-6393/95 $4.00 © 1995 IEEE

frequency point can be done in a finite-length of time (*much shorter than the signal length*) regardless of the length of analyzed signal. In other words, the computation for each time-frequency point involves only a finite number of signal samples.

Furthermore, when we use the term *sliding window*, we assume that this window is a bounded function with finite support. We may also consider the Gaussian function as one of the sliding windows as generally done in applications.

Most members in Cohen's class have to take into account all the data along the time axis. This makes the real(finite)-time implementation impractical for these members if the signal under analysis is rather long. It can be shown that the WVD can achieve the best frequency concentration (thus, the best resolution if the interference is ignored) only if the analyzed signal is monocomponent and highly correlated over a long time interval. In this case, the more neighboring data used, the higher the resolution obtained. Unfortunately, these kinds of signals are generally man-made, e.g., the ideal chirp or sinusoids. Hence, in practical applications, it is reasonable to apply a sliding window to the data before the WVD is utilized and, thus, obtain the PWVD. Using the same idea, we define a subclass of Cohen's class such that every member in this subclass can be implemented in real(finite)-time.

We shall derive this subclass in the continuous-time case. Assume x(t) is a signal to be analyzed, and w(t) is a sliding window function used at time t_0 to obtain the windowed signal $\hat{x}_{t_0}(t) = x(t)w(t-t_0)$. We want to compute $\rho_x(t_0, f; \phi_l)$ only using data from the neighborhood of t_0, defined by the window w, where ϕ_l stands for a "local" kernel used at this time point. If we insert $\hat{x}_{t_0}(t)$ into Eq.(1), we obtain

$$\rho_{\hat{x}}(t_0, f; \phi_l) = \int_\upsilon \int_\tau \phi_l(\upsilon,\tau)\left[\int_\mu \hat{x}_{t_0}(\mu+\frac{\tau}{2})\hat{x}^*_{t_0}(\mu-\frac{\tau}{2})e^{-j2\pi\mu\upsilon}d\mu\right]e^{-j2\pi f\tau}e^{j2\pi\upsilon t_0}d\tau d\upsilon \tag{3}$$

And if we define a generalized ambiguity function $AF_{\hat{x}}(\xi,\zeta)$ by

$$AF_{\hat{x}}(\xi,\zeta) = \int_f \int_{t_0} \rho_{\hat{x}}(t_0, f; \phi_l)e^{-j2\pi\xi t_0}e^{j2\pi\zeta f}dt_0 df \tag{4}$$

we can also express it in the general form

$$AF_{\hat{x}}(\xi,\zeta) = \phi_g(\xi,\zeta)AF_x(\xi,\zeta) \tag{5}$$

where $\phi_g(\xi,\zeta)$ is a "global" kernel that embodies the windowing function w. After some derivation, we can obtain the following important relation:

$$\begin{aligned}\phi_g(\upsilon,\tau) &= \int \phi_l(\vartheta,\tau)AF_w(\vartheta-\upsilon,\tau)d\vartheta \\ &= \int_\vartheta \phi_l(\vartheta,\tau)AF^*_w(\upsilon-\vartheta,-\tau)d\vartheta \\ &= \phi_l(\upsilon,\tau) \underset{(\upsilon)}{\otimes} AF^*_w(\upsilon,-\tau)\end{aligned} \tag{6}$$

where $\underset{(\upsilon)}{\otimes}$ denotes the convolution on υ and * denotes the conjugate.

Equation (6) presents a necessary and sufficient condition for using a local kernel ϕ_l and window w to implement the distribution of Eq.(5) that is based on the global kernel ϕ_g. Note that, in the above derivation, we did not impose a finite-length condition on the window w. The idea of windowing signals, however, implies that one uses short windows to process long signals.

By choosing $\phi_l(\upsilon,\tau) = 1$, we have

$$\phi_g(\upsilon,\tau) = w(\tfrac{\tau}{2})w^*(-\tfrac{\tau}{2})$$

which is the PWVD's kernel; if $\phi_l(\upsilon,\tau) = \delta(\upsilon)$, we obtain the SPG's kernel

$$\phi_g(\upsilon,\tau) = \int_\mu w(\mu+\tfrac{\tau}{2})w^*(\mu-\tfrac{\tau}{2})e^{j2\pi\mu\upsilon}d\mu.$$

We define Eq.(6) the *real(finite)-time implementable condition* (RTIC). If a kernel satisfies the RTIC, we denote its induced distribution *real(finite)-time implementable* (RTI) since this distribution is computable for very long signals if the sliding window has reasonable finite time-duration.

3. Further Discussion

In the following discussion, we assumed that the sliding window is of finite length. The RTIC is important in that it provides a criterion to check whether a kernel and, thus, the corresponding distribution, is real-time implementable. We point out that the Choi-Williams (CW) distribution (CWD) is not real(finite)-time implementable while the ZAM with cone-shaped kernel in the time-lag domain does. Furthermore, given a real-time implementable global kernel, one can design a sliding window and a local kernel to implement the corresponding distribution; inversely, given a sliding window and a local kernel, one can obtain the global kernel to analyze the properties that the corresponding distribution may possess.

Given a kernel, e.g., the exponential kernel (CW kernel), one approach to implement the CWD for long signals is to apply a sliding window to the signals, and then apply the result to the CW formulation. However, there are several ways of windowing signals. For instance, one may shift the window by some fixed number of samples, compute the distribution for a short period of time, and then repeat the process. Using this approach, the resulting distribution, however, may no longer belong to Cohen's class because of the non-linearity of the shift operation. Another approach is to shift the window on a per sample basis, and then apply the result to the bilinear formulation as done in this paper. The resulting distribution, however, may no longer reflect the given kernel or possess some of the desirable properties. Distribution properties after windowing can be analyzed by using the global kernel $\phi_g(\upsilon,\tau)$.

Since the sliding window w has finite support, the resulting ambiguity function $AF_w(\upsilon,\tau)$ and, thus, the global kernel $\phi_g(\upsilon,\tau)$, are also finite-supported in τ as illustrated in figure 1:

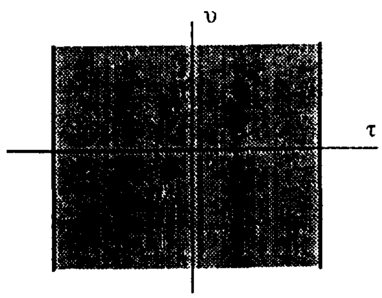

Fig. 1: Global kernel support in the AF domain

According to the relationship between the distribution properties and kernel constraints, we find several distribution properties such as the frequency-marginal property and group-delay property would be lost no matter what local kernels are chosen.

By inspecting the SPG's and PWVD's local kernels, one can design other kernels to obtain distributions superior to the SPG, which suffers the resolution tradeoff between time and frequency domains, and to the PWVD, which retains unwanted frequency interference.

Two approaches can be chosen for digital implementation of the distributions in this subclass. Depending on the properties of the sliding window and local kernel, one may chose one of the following approaches for the implementation of the distributions: (1) compute directly as in the derivation; and (2) compute the SPG using the sliding window, and then convolve with the smooth window derived from the local kernel in the time-frequency domain. The computational complexity for a member of this subclass can range from $O(NM\log M)$ to $O(NM^2\log M)$, where N is the length of the analyzed signal and M is the length of the sliding window. The SPG achieves the lower bound.

4. Numerical Example

Figures 2(a)-(f) illustrate the SPG, PWVD, and a real(finite)-time implementable distribution (using radial-Butterworth kernel [5] as the local kernel) of a signal comprised of three frequency hops and one chirp, respectively. In these plots, the analyzed signal is real and only the positive portion of the distributions is considered. In this example, a truncated Gaussian is used as the sliding window. Figures 2(a)-(b) show the SPG's global kernel in the AF domain and the corresponding distribution in the time-frequency domain. Figures 2(c)-(d) are the PWVD's global kernel and corresponding distribution, respectively. The PWVD has better resolution in the time domain, while exhibiting somewhat severe interference in the frequency domain. A compromise between these two distributions is obtained by using the radial-Butterworth function as the local kernel in the AF domain as given by

$$\phi_{RB}(\upsilon,\tau)=\frac{1}{1+\left(\dfrac{\upsilon^2+\tau^2}{r_0^2}\right)^M} \qquad (7)$$

Figures 2(e)-(f) show the resulting global kernel and corresponding distribution. The frequency interference is smoothed while good resolution is retained in the time domain.

5. Summary and Conclusions

In this paper, a real(finite)-time implementable subclass of Cohen's class is defined. By using a sliding window, we derived a necessary and sufficient Real (finite)-Time Implementable Condition (RTIC) imposed on the "global" and "local" kernels in the Ambiguity function domain. For each member of this subclass, only a finite number (much less than the signal length) of signal samples are used to compute the corresponding distribution for each time instant and, thus, is suitable for processing long signals. The properties of the resulting distributions are discussed in terms of the global kernels, and an example is provided to demonstrate how one can design a distribution that provides a compromise in performance between the SPG and PWVD.

References

[1] L. Cohen, "Generalized Phase-Space Distribution Functions," *J. Math. Phys.*, vol. 7, 1966, pp. 781-806.

[2] H. Choi and W. J. Williams, "Improved Time-Frequency Representation of Multicomponent Signals Using Exponential Kernels," *IEEE Trans. ASSP*, vol. 37, No. 6, June 1989, pp. 862-871.

[3] Y. Zhao, L. E. Atlas, and R. J. Marks, "The Use of Cone-Shaped Kernels for Generalized Time-Frequency Representations of Nonstationary Signals," *IEEE Trans. ASSP*, vol. ASSP-38, July 1990, pp. 1084-1091.

[4] A. Papandreou and G. F. Boudreaux-Bartels, "Generalization of the Choi-Williams Distribution and the Butterworth Distribution for Time-Frequency Analysis," *IEEE Trans. SP*, vol. 41, No. 1, Jan. 1993, pp. 463-472.

[5] D. Wu and J. M. Morris, "Time-Frequency Representations Using a Radial Butterworth Kernel," submitted to *IEEE-SP Int.l Symp. on Time-Freq. and Time-Scale Anal.*, Phila., PA, 1994.

[6] D. T. Barry, "Fast Calculation of the Choi-Williams Time-Frequency Distribution," *IEEE Trans. SP*, vol. 40, No. 2, Feb., 1992, pp. 450-455.

[7] K-B Yu and S. Cheng, "Signal Synthesis from Pseudo-Wigner Distribution and Applications," IEEE *Trans. ASSP*, vol. ASSP-35, No. 9, Sept., 1987, pp. 1289-1302.

ONR Grant #N00014-89-J1210

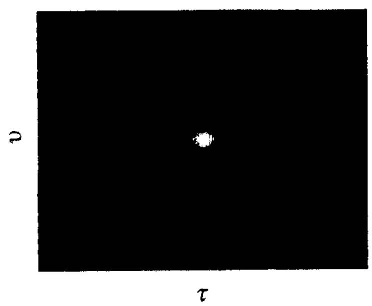

Fig. 2(a): Kernel of SPG using G_window(128,40)

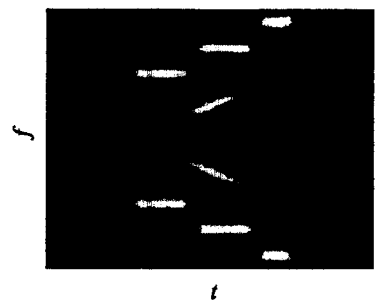

Fig. 2(b): SPG using kernel of (a)

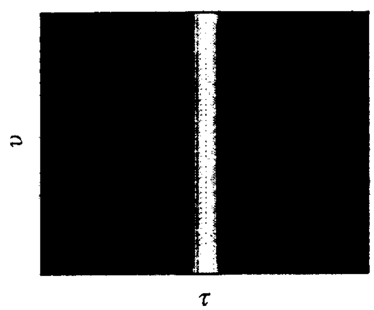

Fig. 2(c): Kernel of PWVD using G_window(128,40)

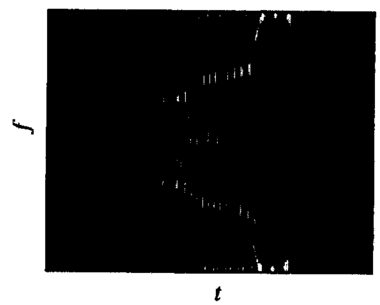

Fig. 2(d): PWVD using kernel of (c)

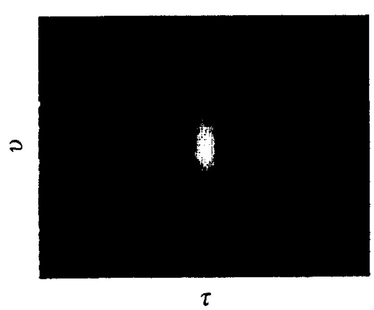

Fig. 2(e): Global kernel using RB local kernel and G_window(128,40)

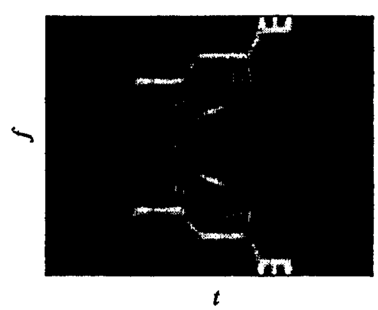

Fig. 2(f): Distribution for kernel of (e)