THE L-WIGNER DISTRIBUTIONS AS A TOOL FOR ROBUST TIME-FREQUENCY SIGNAL ANALYSIS

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Abstract: *A robust time-frequency analysis, producing a rough parameters' estimation of a signal is presented. As a tool for this analysis, recently defined L-Wigner distribution is used. Its properties are also presented. The L-Wigner distribution is realized using an iterative formula starting from the Short time Fourier transform, via the Wigner distribution. The frequency domain window is used to eliminate the cross terms. The signal's parameters that may be easily estimated (even in the case when the signal is heavily corrupted by noise) are: the time instant and the instantaneous frequency corresponding to the maximal signal power. The theory is illustrated on the numerical example.*

1. INTRODUCTION

Higher order spectral analysis has been intensively studied during last few years. Higher order statistics, known as cumulants, and its Fourier transforms known as higher order spectra are often considered, but we refer here only to the review paper [1] and the references therein. Recently, higher order time-varying spectra have been defined and analyzed [2]. The basic representation in the time-varying higher order spectral analysis is the Wigner higher order spectra (the third order Wigner distribution was introduced in [3]) in analogy with the Cohen's general class [4], where the Wigner distribution plays that role. Recently, L-Wigner distribution (LWD), as a special and optimal case of the Multi-time Wigner higher order spectra has been introduced, [5,6,7,8]. Its pseudo form is given by:

$$PLWD_L(t,\omega)=\int_\tau x^{*L}(t-\frac{\tau}{2L})x^L(t+\frac{\tau}{2L})w_L(\tau)e^{-j\omega\tau}d\tau \qquad (1)$$

where $w_L(t)$ is the window function, usually real and even. For $L=1$ it is reduced to the pseudo Wigner distribution.

2. PROPERTIES OF THE LWD

Some of the basic properties of the LWD, corresponding to the ones of the Wigner distribution, are the following:

-The LWD is always real.

- If the signal is time shifted, $x(t-t_0)$, its LWD is time-shifted as well, $LWD_L(w,t-t_0)$.

-The modulated signal $x(t)exp(jw_0t)$ corresponds to the LWD shifted in frequency: $LWD_L(t,w-w_0)$.

-The LWD is time-limited to the interval $|t|<T$ if the signal is time limited, $x(t)=0$ for $|t|\geq T$.

-The LWD is band-limited to the interval $|w|<w_m$ if the signal is band-limited, $X(w)=0$ for $|w|\geq w_m$.

- Integral of the LWD over frequency is equal to the generalized signal power $|x(t)|$ 2L:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}LWD_L(t,\omega)d\omega=|x(t)|^{2L} \qquad (2)$$

- Integral over time and frequency is equal to

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the $2L$-th power of the 2L norm of signal, $\|x(t)\|_{2L}^{2L}$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} LWD_L(t,\omega)dtd\omega = \int_{-\infty}^{\infty}|x(t)|^{2L}dt = \|x(t)\|_{2L}^{2L} \quad (3)$$

3. ITERATIVE REALIZATION OF THE LWD

The realization of LWD may be efficiently done using recursive formula which follows from (1):

$$LWD_{2L}(\omega,t) = \frac{1}{\pi}\int_{\theta} LWD_L(\omega+\theta,t)LWD_L(\omega-\theta,t)d\theta \quad (4)$$

If $LWD_L(w,t)$ is cross-term free, then $LWD_{2L}(w,t)$ will be cross-term free as well, if the integration in (4) is performed through the frequency domain window $P(q)$. The modified form of (4) is:

$$MLWD_{2L}(\omega,t) = \frac{1}{\pi}\int_{\theta} P(\theta)LWD_L(\omega+\theta,t)LWD_L(\omega-\theta,t)d\theta \quad (5)$$

The width W_p of $P(q)$, $P(q)=0$ for $|q|>W_p$, may be determined from:

$$W_{w_L} < W_p < \min_{i,j}|\phi_i'(t)-\phi_j'(t)| - W_{w_L} \quad (6)$$

where W_{w_L} is the width of $W_L(w)$, while $f_i'(t)$ and $f_j'(t)$ are the instantaneous frequencies of the i-th and j-th component, respectively, in a multicomponent signal.

The starting iteration ($2L=1$) is:

$$MWD(\omega,t) = \frac{1}{\pi}\int_{\theta} P(\theta)STFT(\omega+\theta,t)STFT^*(\omega-\theta,t)d\theta \quad (7)$$

where $STFT(w,t)$ is the Short time Fourier transform defined as

$STFT(w,t)=FT_t\{x(t+\tau)w(t)\}$.

This way, the resulting LWD is cross-terms free if $STFT(w,t)$ is cross-term free (what is the case if the signal components do not overlap in the time-frequency plane) and if, at the same time, (6) is satisfied in each iteration. Note, if (6) can not be satisfied for some i,j, and t, then the cross-terms will appear at that instant t, between i-th and j-th signal component.

The discrete forms of (5) and (7), with a rectangular window P(q), are:

$$MWD(n,k) = |STFT(n,k)|^2 + 2\sum_{i=1}^{P}\mathrm{Re}\{STFT(n,k+i)STFT^*(n,k-i)\}$$

$$MLWD_{2L}(n,k) = LWD_L^2(n,k) + 2\sum_{i=1}^{P}\{LWD_L(n,k+i)LWD_L(n,k-i)\} \quad (8)$$

where P is the width of the discrete form of P(q). In [9], it is shown that the realization of $MWD(n,k)$ may be computationally very efficient. Here, we will only indicate that the oversampling in the modified Wigner distribution (8) is not necessary, because the aliasing components are removed in the same way as the cross-terms are, [9]. The same conclusions are valid for the LWD. This iterative approach may be extended to the wavelet and affine distributions, as well [8].

4.ROBUST SIGNAL ANALYSIS

Robust control widely exploits the fact that an infinity norm of a signal x(t) results in its essential supremum:

$$\|x(t)\|_\infty = \lim_{p\to\infty}\left(\int_{t_1}^{t_2}|x(t)|^p dt\right)^{1/p} = \operatorname*{ess\,sup}_{t\in[t_1,t_2]}|x(t)| \quad (9)$$

The same conclusion is valid if we consider a large number $L\to\infty$ in the LWD of a monocomponent signal. We may neglect all values of $LWD_L(w,t)$ except the maximal one in the time-frequency plane:

$$\lim_{L\to\infty}\frac{LWD_L(\omega,t)}{LWD_L(\omega_m,t_m)} = \begin{cases}0 & if \quad \omega\neq\omega_m \quad or \quad t\neq t_m\\ 1 & if \quad \omega=\omega_m \quad and \quad t=t_m\end{cases}$$

where (w_m,t_m) is a point where $LWD_L(w,t)$ reaches its essential supremum.

In this way one may determine the time

instant t and instantaneous frequency w where the power of signal x(t) reaches supremum.

The above analysis holds in the case of multicomponent signals if the cross terms are not present, what is explained in Sec.3.

5.NUMERICAL EXAMPLE

To illustrate this idea consider a signal $x_0(t)$:

$$x_o(t) = e^{-32(t+0.5)^2} e^{-j16\pi(t+0.5)^2}$$

with two echoes:

$$x(t) = x_o(t) + 0.707 x_o(t-0.5) + 0.5 x_o(t-1) + n(t)$$

Note that the maximal power is at t=-0.5 and w=0. These theoretical values are easily available from the plots in of a very inconvenient case, Fig.1., where the signal is corrupted by the Gaussian white noise with SNR=0 dB.

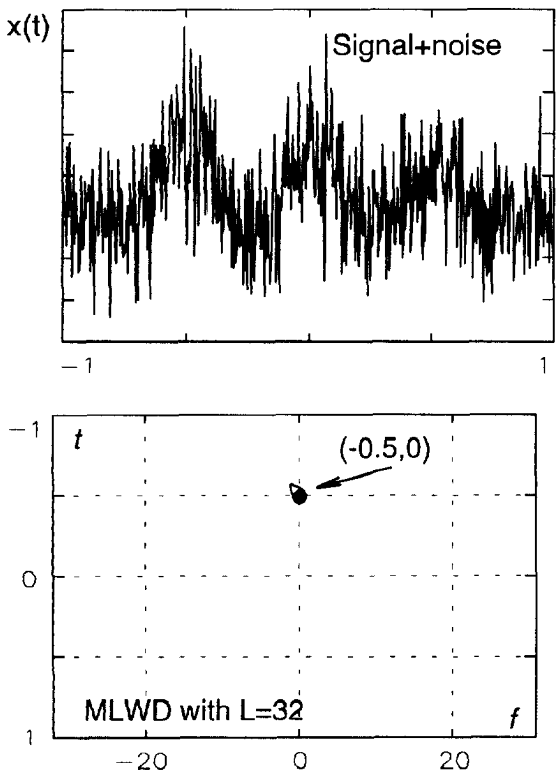

Fig.1. *Time-frequency representation of the signal x(t)*

6.CONCLUSION

The LWD, with the method for cross terms removal, is used to locate the point in the time-frequency plane where the signal assumes the maximal power. The efficiency of the distribution is illustrate on the numerical example with the noisy signal, with a very small signal to noise ratio.

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