INSTANTANEOUS FREQUENCY ESTIMATION: CONFIDENCE BOUNDS USING THE BOOTSTRAP

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ABSTRACT

This paper uses a nonparametric method for estimating confidence intervals for the instantaneous frequency. The method, known as the bootstrap, is computer-intensive but requires very little in the way of modelling and assumption. The confidence intervals are based on the least-squares instantaneous frequency estimator. Simulation results demonstrate the potential of the method in situations where theoretical analysis cannot at all, or only with considerable effort, provide the results.

1. INTRODUCTION

The instantaneous frequency (IF) of a signal is a parameter which is of significant importance. In many applications such as seismic, radar, sonar, communications, and biomedical engineering, the IF is a good descriptor of some physical phenomenon, because it describes the location of the signal spectral peak as it varies with time. A comprehensive treatment of the instantaneous frequency and its estimation is given in [2, 3].

This paper presents the estimation of confidence intervals for the IF. Such intervals, based on a particular estimator give an indication of the possible nearness of an estimate to the true IF. Using the bootstrap, we present a method of setting the confidence intervals, based on the least-squares estimator of the IF.

The bootstrap is a computer based method, which substitutes considerable amounts of computation in place of theoretical analysis [6, 7, 11]. It is very attractive to the practitioner because it requires very little in the way of modelling, assumptions, or analysis, and can be applied in an automatic way [19].

The paper is organised as follows. The next section discusses the estimation of the IF based on the method of least-squares. In section 3, we give an overview of the bootstrap and propose a method of setting confidence intervals. Sections 4 presents simulation results.

2. LEAST-SQUARES BASED INSTANTANEOUS FREQUENCY ESTIMATION

The concept of the IF was introduced to describe the time-varying nature of the spectral content of a deterministic nonstationary signal $s(t)$. It is defined as

$$f(t) = \frac{1}{2\pi}\frac{d}{dt}\arg\{z(t)\},$$

where $z(t)$ is the *analytic* signal associated with $s(t)$. The analytic signal, can be represented in the exponential form as $z(t) = A(t)\,e^{j\phi(t)}$, where $A(t)$ is the envelope and $\phi(t)$ is the phase of the signal.

For a discrete-time signal $s[n]$, the IF can be considered as the output of a discrete-time differentiator, driven by the phase sequence $\phi[n] = \phi(t)\mid_{t=n\Delta}$, where Δ is the sampling period assumed to be unity throughout the paper.

2.1. Signal Model

Suppose we are given a finite number of observations of a discrete-time signal modelled by a complex exponential affected by additive noise,

$$z[n] = Ae^{j\phi[n]} + u[n], \tag{1}$$

where $u[n]$ is zero-mean complex independent noise with variance σ_u^2, and A is the real amplitude of the signal, assumed to be a constant. Let the phase $\phi[n]$ be modelled by a polynomial of order p

$$\phi[n] = \sum_{k=0}^{p} a_k n^k. \tag{2}$$

The amplitude A and the phase coefficients collected in a column vector $\mathbf{a} = (a_0, \ldots, a_p)^T$ are deterministic but unknown constants. The aim is to estimate the IF, given by

$$f[n] = \frac{1}{2\pi}\sum_{k=1}^{p} k\,a_k\,n^{k-1}, \tag{3}$$

and possibly A. Based on the obtained estimator, we wish to set a confidence interval for the IF.

We define the signal-to-noise power ratio (SNR) as A^2/σ_u^2, which is assumed to be large ($A^2/\sigma_u^2 \gg 1$), so that the signal model in (1) can be approximated by [18, 5]

$$z[n] \approx A\exp\{j(\phi[n] + \varepsilon[n])\} = A\,\exp\{j\tilde{\phi}[n]\}, \tag{4}$$

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with $\varepsilon[n]$ being a real zero-mean independent noise sequence with variance $\sigma_\varepsilon^2 = \sigma_u^2/(2A^2) \ll 1$.

In this contribution, we consider the least-squares IF estimator [3, 12, 18] based on observations $z[n]$, $n = 0, \ldots, N-1$. Without loss of generality, we set $A = 1$ in the signal model (1).

2.2. The Least-Squares Method

Tretter's original idea [18] was to use the least-squares (LS) fit to the unwrapped phase, in order to estimate a_0 and a_1 of a constant frequency tone in noise. Tretter's method was later extended to polynomial phase signals [5, 4] given by (1) and (2).

The least-squares method for IF estimation first requires the phase to be unwrapped. The unwrapped phase $\tilde{\phi}[n]$, $n = 0, \ldots, N-1$, can be written in the matrix form

$$\tilde{\mathbf{\Phi}} = \mathbf{H}\,\mathbf{a} + \boldsymbol{\mathcal{E}}, \qquad (5)$$

where $\tilde{\mathbf{\Phi}} = (\tilde{\phi}[0], \ldots, \tilde{\phi}[N-1])^T$, $\boldsymbol{\mathcal{E}} = (\varepsilon[0], \ldots, \varepsilon[N-1])^T$ and

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & N-1 & (N-1)^2 & \cdots & (N-1)^p \end{pmatrix}.$$

The least-squares estimate for $\mathbf{a}$ is given by

$$\hat{\mathbf{a}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\tilde{\mathbf{\Phi}}, \qquad (6)$$

which is then used to calculate the IF by substituting $\hat{a}_k$, $k = 1, \ldots, p$, into (3). It has been shown [5] that the LS estimates in (6) are efficient in the sense that they reach the Cramér-Rao (CR) bound, assuming $u[n]$ to be Gaussian. The statistical performance of the least-squares IF estimator is shown in Figure 1.

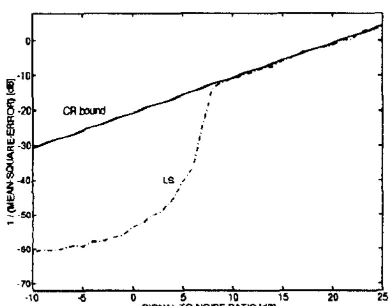

Figure 1. Statistical performance of the least-squares IF estimator.

3. BOOTSTRAP CONFIDENCE BANDS

A confidence interval for a parameter is often more useful than just an estimate. Taken together, the estimate and the interval estimate indicate what is the best guess for the parameter and how far in error that guess might reasonably be [7]. This is of particular interest for the instantaneous frequency of an FM signal in noise.

3.1. The Bootstrap

In the case of a linear estimator such as the least-squares estimator one could easily derive confidence bands analytically, provided $\boldsymbol{\mathcal{E}}$ is Gaussian. Otherwise, the problem is much more difficult and alternatives are sought.

Efron [6] introduced the bootstrap as a general tool for estimating the unknown sampling distribution of a statistic. When the observations are independent and identically distributed (i.i.d.), the bootstrap provides very accurate approximations to the distributions of many commonly used statistics [17, 1, 10, 11]. However, such a statement does not hold for dependent observations. For dependent data, modifications of the bootstrap have been suggested in the literature. The most frequently used methods are those which consider resampling the estimated residuals of specific dependent models such as autoregressive models [8]. An example of estimating the distribution of the parameter estimate of a first order autoregressive model was given earlier [19]. Despite the very good performance of these methods [7], they are restricted to relatively simple contexts where structural models are both plausible and tractable.

Another approach, called the moving-block bootstrap, relies on dividing an observed dependent data set into parts, or blocks and constructing a dependent version of the original pattern by resampling the blocks at random, with replacement [13]. The validity of this method in more general problems has been established by Politis and Romano [16] among others [14]. In this contribution, we shall focus on methods based on resampling residuals rather than the original data or blocks.

3.2. Bootstrap Bands Based on the Least Squares IF Estimator

Based on the observed $\tilde{\phi}[n]$, $n = 0, \ldots, N-1$, the LS estimate of the IF is given in (3) where the coefficients $\hat{a}_k$, $k = 1, \ldots, p$, are obtained through (6). The idea behind the bootstrap is to replace the true distribution F, in a formula, by its empirical distribution $\hat{F}$ [11, 19]. Under the regression, and high SNR where approximation (4) is valid,

$$\tilde{\phi}[n] = \sum_{k=0}^{p} a_i n^k + \varepsilon[n].$$

Let F be the distribution function of i.i.d. errors $\varepsilon[n]$, and $\hat{F}$ is the empirical distribution function of the sequence of the residuals

$$\hat{\varepsilon}[n] = \tilde{\phi}[n] - \sum_{k=0}^{p} \hat{a}_k n^k, \qquad 0 \le n \le N-1,$$

where $\hat{a}_k$, $k = 0, \ldots, p$, are the least-squares estimates of a_k, $k = 0, \ldots, p$. In the bootstrap procedure we resample the residuals $\hat{\varepsilon}[n]$, $n = 0, \ldots, N-1$, which are centred in the sense that $\sum_{n=0}^{N-1} \hat{\varepsilon}[n] = 0$.

Throughout this work we employ the nonparametric bootstrap, for which resampling involves sampling with replacement from the set of residuals $\mathcal{X} = \{\hat{\varepsilon}[0], \ldots, \hat{\varepsilon}[N-1]\}$.

We take resamples $\mathcal{X}^* = \{\hat{\varepsilon}^*[0], \ldots, \hat{\varepsilon}^*[N-1]\}$ and define

$$\tilde{\phi}^*[n] = \sum_{k=0}^{p} \hat{a}_k n^k + \hat{\varepsilon}^*[n], \qquad 0 \leq n \leq N-1.$$

Then, we calculate $\hat{a}_k^*$, $k = 0, \ldots, p$, having the same formulae as $\hat{a}_k$, $k = 0, \ldots, p$, except that $\tilde{\phi}[n]$ is replaced by $\tilde{\phi}^*[n]$ at each appearance of the former. In this way, we get a new IF estimator $\hat{f}^*[n]$ whose distribution approximates the distribution of $\hat{f}[n]$, or more precisely a pivotal version of $\hat{f}[n]$ is approximated by its bootstrap counterpart [11, 19].

Note also that the residuals need to be centred in order to avoid an additional bias at the resampling stage [9]. The procedure is detailed in Table 1.

1. Based on $\tilde{\phi}[n]$, $n = 0, \ldots, N-1$, get the LS estimates $\hat{a}_k$ of a_k, $k = 0, \ldots, p$ and $\hat{f}[n]$.
2. With $\hat{a}_k$, $k = 0, \ldots, p$, compute the residuals, $\hat{\varepsilon}[n]$
$$\hat{\varepsilon}[n] = \tilde{\phi}[n] - \sum_{k=0}^{p} \hat{a}_k n^k, \qquad n = 0, \ldots, N-1.$$
3. Using a pseudo-random number generator, draw a random sample $\mathcal{X}^* = \{\hat{\varepsilon}^*[0], \ldots, \hat{\varepsilon}^*[N-1]\}$ of size N, with replacement, from $\mathcal{X} = \{\hat{\varepsilon}[0], \ldots, \hat{\varepsilon}[N-1]\}$ and calculate the bootstrap phase component (after centring)
$$\tilde{\phi}^*[n] = \sum_{k=0}^{p} \hat{a}_k n^k + \hat{\varepsilon}^*[n], \qquad n = 0, \ldots, N-1.$$
4. With $\tilde{\phi}^*[n]$, get new LS estimates $\hat{a}_k^*$, $k = 0, \ldots, p$.
5. Repeat step 3 and 4 a large number of times (e.g. 999), each time computing
$$\hat{f}^*[n] = \frac{1}{2\pi} \sum_{k=1}^{p} k \hat{a}_k^* n^{k-1}.$$
6. Using pivotal statistics find, the $100(1-\alpha)\%$ confidence interval.

Table 1. Bootstrap procedure based on the LS estimator of the IF.

The last step of the algorithm approximates

$$\mathrm{P}\left(\frac{\hat{f}[n] - f[n]}{\hat{\sigma}[n]} \leq u_\alpha\right) = \frac{\alpha}{2} = \mathrm{P}\left(\frac{\hat{f}[n] - f[n]}{\hat{\sigma}[n]} \geq v_\alpha\right),$$

where u_α and v_α are quantiles of the distribution of $(\hat{f}[n] - f[n])/\hat{\sigma}[n]$, by

$$\mathrm{P}\left(\frac{\hat{f}^*[n] - \hat{f}[n]}{\hat{\sigma}^*[n]} \leq \hat{u}_\alpha \mid \mathcal{X}\right) = \mathrm{P}\left(\frac{\hat{f}^*[n] - \hat{f}[n]}{\hat{\sigma}^*[n]} \geq \hat{v}_\alpha \mid \mathcal{X}\right),$$

to yield the $100(1-\alpha)\%$ confidence interval

$$\hat{I}(\mathcal{X}; n) = (\hat{f}[n] - \hat{\sigma}[n]\hat{v}_\alpha, \hat{f}[n] - \hat{\sigma}[n]\hat{u}_\alpha).$$

It remains to specify variance estimates $\hat{\sigma}[n]$ of $\hat{f}[n]$ and $\hat{f}^*[n]$ needed in the algorithm. These are constructed as follows. We draw a small number B of independent resamples $\mathcal{X}^*$ from $\mathcal{X}$ and $\mathcal{X}^{**}$ from $\mathcal{X}^*$, and compute for $b = 1, \ldots, B$ estimates $\hat{f}_b^*[n]$ and $\hat{f}_b^{**}[n]$, respectively. Then we estimate the variance of $\hat{f}[n]$ by

$$\hat{\sigma}^2[n] = \frac{1}{B-1} \sum_{b=1}^{B} \left(\hat{f}_b^*[n] - \frac{1}{B} \sum_{b=1}^{B} \hat{f}_b^*[n]\right)^2. \qquad (7)$$

A variance estimate of $\hat{f}^*[n]$, $\hat{\sigma}^{*2}[n]$ is computed in the same manner as $\hat{\sigma}^2[n]$ when $\hat{f}_b^*[n]$ in (7) is replaced by $\hat{f}_b^{**}[n]$. Alternatively, we can estimate this variance using the jackknife [19]. As a comparison to the bootstrap, the jackknife method can be thought of as drawing samples of size $N-1$ without replacement [15].

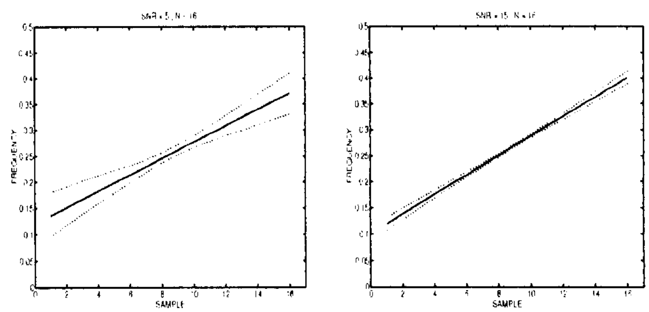

Figure 2. 95% confidence intervals for the IF of a linear FM signal embedded in Gaussian noise with SNR= 5 and 15dB for $N = 16$, based on the LS estimator.

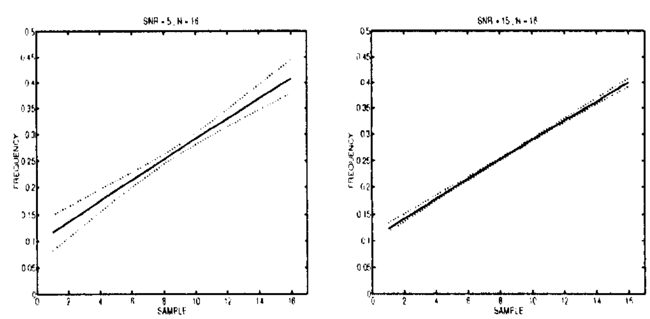

Figure 3. 95% confidence intervals for the IF of a linear FM signal embedded in double exponentially distributed noise with SNR= 5 and 15dB for $N = 16$, based on the LS estimator.

4. SIMULATION RESULTS

Two different scenarios are considered in simulation. First, confidence intervals for the IF of a linear FM signal with unit amplitude were obtained. In accordance with (1) and (3), the IF in this case is given by $f[n] = \frac{1}{2\pi}(a_1 + 2a_2 n)$, with $a_1/(2\pi)$ and a_2/π being the unknown frequency and the frequency rate, respectively. The linear FM signal was embedded in white Gaussian noise with variances corresponding to SNR = 5 and 15 dB.

Next, confidence bands for the IF of a nonlinear FM signal assuming the order p to be known, were evaluated. As an example, a cubic FM signal whose IF is given by

$$f[n] = \frac{1}{2\pi}(a_1 + 2a_2 n + 3a_3 n^2 + 4a_4 n^3) \qquad (8)$$

was chosen.

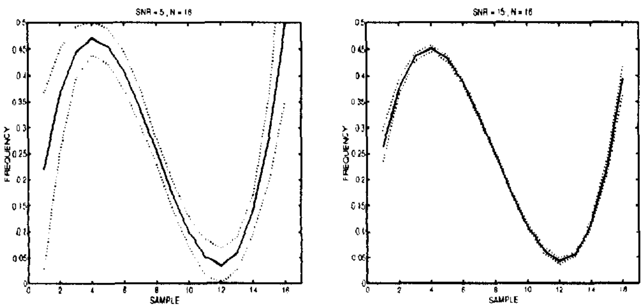

Figure 4. 95% confidence intervals for the IF of a cubic FM signal embedded in Gaussian noise with SNR= 5 and 15dB for $N = 16$, based on the LS estimator.

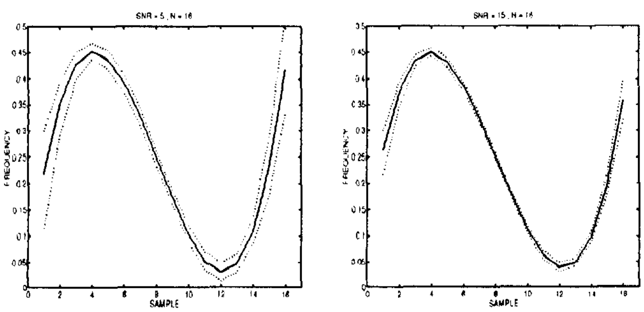

Figure 5. 95% confidence intervals for the IF of a cubic FM signal embedded in double exponentially distributed noise with SNR= 5 and 15dB for $N = 16$, based on the LS estimator.

Results for linear FM signals embedded in Gaussian and double exponentially distributed noise are presented in Figure 2 and Figure 3, respectively. Figure 4 and Figure 5 illustrate the confidence intervals for the IF of a cubic FM signal in Gaussian and double exponentially distributed noise, respectively. The solid lines represent the LS estimates and the dotted lines indicate the lower and upper bounds of a 95% confidence interval. Note that the confidence interval length is getting large at the endpoints of the IF.

A performance analysis of the bootstrap algorithm was carried out using a coverage analysis as follows. A large number of realisations of a noisy FM signal of fixed length N was generated. For each realisation, the 95% confidence interval for the IF was estimated. If the true value of the IF, lies inside the originally calculated confidence bands at a particular time instant, then a coverage was declared. Table 2 and Table 3 show the coverage percentage at 5 different time instants and 300 replications for linear and cubic FM signals respectively. If the estimated confidence interval is truly 95%, then the coverage percentage analysis should confirm it. The obtained results indicate the accuracy of the proposed bootstrap technique.

n	2	8	16	22	32
Cov. [%]	95.33	95.00	95.00	95.00	94.67
Nominal $f[n]$	0.1188	0.1750	0.25	0.3063	0.4
Upper bound	0.1215	0.1767	0.2508	0.3076	0.4032
Lower bound	0.1157	0.1731	0.2492	0.3049	0.3070
Conf. Length	0.0059	0.0037	0.0016	0.0027	0.0062
Std. dev.	0.0010	0.0006	0.0002	0.0004	0.0010

Table 2. Performance results for linear FM signals in Gaussian noise for $N = 32$ evaluated over 300 replications.

n	2	8	16	22	32
Cov. [%]	95.67	94.67	95.00	94.67	95.67
Nominal $f[n]$	0.2726	0.4562	0.25	0.0646	0.4
Upper bound	0.2841	0.4591	0.2519	0.0675	0.4145
Lower bound	0.2617	0.4537	0.2481	0.0615	0.3855
Conf. Length	0.0219	0.0054	0.0038	0.0059	0.0290
Std. dev.	0.0036	0.0009	0.0007	0.0010	0.0050

Table 3. Performance results for cubic FM signals in Gaussian noise for $N = 32$ evaluated over 300 replications.

4.1. Verification of the Bootstrap Confidence Bands

In this subsection the bootstrap confidence bands are compared with theory for the case of a linear FM signal ($p = 2$) and a cubic FM signal ($p = 4$), embedded in white Gaussian noise.

The CR bound of the IF at the central point $n_0 = (N - 1)/2$, for a linear FM signal ($p = 2$) is given asymptotically for large N by [12]

$$\mathrm{CR}(\hat{f}[n_0]) = \frac{6\sigma_u^2}{(2\pi)^2 A^2 N(N^2 - 1)} \qquad (9)$$

while for a cubic FM signal it approaches

$$\mathrm{CR}(\hat{f}[n_0]) = \frac{75\sigma_u^2}{(2\pi)^2 A^2 2N(N^2 - 1)} \qquad (p = 4). \qquad (10)$$

The variance of the LS estimator reaches the CR bound at high SNR, and we have

$$\sigma_{\hat{f}_0}^2 = \mathrm{var}\{\hat{f}[n_0]\} = \mathrm{CR}(\hat{f}[n_0])$$

The distribution of $\hat{f}[n_0]$ is Gaussian and the asymptotic 95% confidence interval for $f[n_0]$ is hence given by

$$I[n_0] = (\hat{f}[n_0] - 1.96\,\sigma_{\hat{f}_0}, \hat{f}[n_0] + 1.96\,\sigma_{\hat{f}_0}). \qquad (11)$$

A comparison between the asymptotic interval (11) and the bootstrap confidence interval for the LS estimator in the case of a linear and a cubic FM signal, is given in Table 4.

	Linear FM		*Cubic FM*	
SNR	Theor. int.	Boot. int.	Theor. int.	Boot. int.
25	0.00048	0.00049	0.00118	0.00122
20	0.00085	0.00088	0.00211	0.00223
15	0.00150	0.00164	0.00375	0.00391
10	0.00267	0.00293	0.00667	0.00821

Table 4. Comparison between the theoretical confidence intervals and the Bootstrap confidence intervals using the LS based method for a linear and a cubic FM signals embedded in zero-mean white Gaussian noise, $N = 32$.

The results obtained show that the bootstrap confidence bands are very close to the asymptotic ones for the case of a linear and a cubic FM signal in white Gaussian noise at high SNR. The bootstrap confidence bands differ from the asymptotic bands at lower SNR since the validity of the asymptotic bound is questionable and the quality of the IF estimator is degraded. The advantage of bootstrap confidence bands is that they provide this information without a priori assumption concerning the distribution of the noise process.

5. SUMMARY

A bootstrap procedure for setting confidence bands for the instantaneous frequency has been presented. The method is nonparametric and requires only two assumptions: 1) a constant amplitude for a frequency modulated signal and 2) independent and identical distribution of the noise. The advantage of using the bootstrap to set the confidence bands for the IF is seen especially in situations where it is difficult to carry out the theoretical analysis or where the distribution of additive noise is unknown. Discussion of other IF estimators is given in [20].

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